

ON THE FURTHER UNIFIED THEORY OF IDEAL GENERALIZED CLOSED SETS

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ABSTRACT. In this paper we introduce and study two new notions of sets called \mathcal{J} - mng -closed sets and mn - $\mathcal{J}g$ -closed sets, which are defined on a nonempty set with two minimal structures and an ideal. These sets enables us to obtain unified properties for certain notions of generalized closed sets with respect to an ideal. Moreover, we establish a new separation property called the class mn - $T_{\mathcal{J}}$, which generalizes the class mn - $T_{1/2}$ [17].

1. INTRODUCTION

After Levine's work [12], where he introduced the notions of generalized closed sets (briefly g -closed), many mathematician have been published their articles using notions related with generalization of closed sets (see [1]-[2], [4]-[8], [10], [12]-[14], [16], [18], [21], [23], [26], [29] and [32]). A particular attention is the article *Unified operation approach of generalized closed sets* by J. Dontchev, M. Ganster and T. Noiri [6], where they introduced and studied the notion of \mathcal{J} -generalized closed sets (briefly \mathcal{J} - g -closed) using the notion of ideals on a topological space. Another mathematician using the notion of minimal structures have been studied some unified approach of generalized closed sets, as we mention some of them: m_X - g -closed [27], mg^* -closed [19], mng -closed [17], m - $\mathcal{J}g$ -closed [22] and \mathcal{J} - mng -closed [20] (or $m\mathcal{J}g^*$ -closed in [31]). In this work, in order to unify modifications of g -closed sets, we define and study two new notions of sets called \mathcal{J} - mng -closed sets and mn - $\mathcal{J}g$ -closed sets, which are defined on a nonempty set with two minimal structures and an ideal. Also, we introduce a new separation property called the class mn - $T_{\mathcal{J}}$, which generalizes the class mn - $T_{1/2}$ [17].

2. PRELIMINARIES

On a topological space (X, τ) the following variants of the notions of open sets are very well-known (see [1],[2], [4]-[23] and [26]-[32]). Semi-open sets, $SO(X, \tau)$, preopen sets, $PO(X, \tau)$, α -open sets, $\alpha(X, \tau)$ or τ^α , semi-preopen sets or β -open sets, $SPO(X, \tau)$ or $\beta(X, \tau)$, b -open sets, $BO(X, \tau)$, δ -open sets, $\delta(X, \tau)$ or τ_δ , regular

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open sets, $\text{RO}(X, \tau)$. In the case that $\alpha : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ is an operator associated to a topology τ , that is, for all $U \in \tau$, $U \subseteq \alpha(U)$, we obtain the notion of α -semi open sets: $A \subset X$ is said to be α -semi open set if there exists $U \in \tau$ such that $U \subseteq A \subseteq \alpha(U)$. Now if α, β are operators associated to a topology τ , $A \subset X$ is said to be (α, β) -semi open set (see [27]) if for each $x \in A$, there exists a β -semi open set V such that $x \in V$ and $\alpha(V) \subseteq A$. The family of (α, β) -semi open set is denoted by $(\alpha, \beta)\text{-SO}(X, \tau)$.

The complements of semi-open (resp. preopen, α -open, semi-preopen, b -open, δ -open, (α, β) -semi open, regular open) sets are called semi-closed (resp. preclosed, α -closed, semi-preclosed, b -closed, δ -closed, (α, β) -semi closed, regular closed) sets. For a subset A of (X, τ) , the closure (resp. semi-closure, preclosure, α -closure, semi-preclosure or β -closure, b -closure, δ -closure, (α, β) -semi closure), is denoted by $\text{Cl}(A)$ (resp. $\text{sCl}(A)$, $\text{pCl}(A)$, $\alpha\text{Cl}(A)$, $\text{spCl}(A)$ or $\beta\text{Cl}(A)$, $\text{bCl}(A)$, $\text{Cl}_\delta(A)$, $(\alpha, \beta)\text{-sCl}(A)$), is defined as the intersection of all closed (resp. semi-closed, preclosed, α -closed, semi-preclosed or β -closed, b -closed, δ -closed, (α, β) -semi closed) sets of X containing A .

The family of closed sets above mentioned are extended to a general class of sets called a generalized closed sets. The following definition give many well known concepts of generalized closed sets.

Definition 2.1. A subset A of a topological space (X, τ) is said to be:

- (1) g -closed [12], if $\text{Cl}(A) \subset U$ whenever $A \subset U$ and $U \in \tau$.
- (2) gs -closed [1], if $\text{sCl}(A) \subset U$ whenever $A \subset U$ and $U \in \tau$.
- (3) gp -closed [18], if $\text{pCl}(A) \subset U$ whenever $A \subset U$ and $U \in \tau$.
- (4) αg -closed [14], if $\alpha\text{Cl}(A) \subset U$ whenever $A \subset U$ and $U \in \tau$.
- (5) gsp -closed [4], if $\text{spCl}(A) \subset U$ whenever $A \subset U$ and $U \in \tau$.
- (6) γg -closed [7], if $\text{bCl}(A) \subset U$ whenever $A \subset U$ and $U \in \tau$.
- (7) δg -closed [5], if $\text{Cl}_\delta(A) \subset U$ whenever $A \subset U$ and $U \in \tau$.
- (8) rg -closed [23], if $\text{Cl}(A) \subset U$ whenever $A \subset U$ and $U \in \text{RO}(X, \tau)$.
- (9) sg^* -closed [16], if $\text{Cl}(A) \subset U$ whenever $A \subset U$ and $U \in \text{SO}(X, \tau)$.
- (10) pg^* -closed [16], if $\text{Cl}(A) \subset U$ whenever $A \subset U$ and $U \in \text{PO}(X, \tau)$.
- (11) αg^* -closed [16], if $\text{Cl}(A) \subset U$ whenever $A \subset U$ and $U \in \alpha(X, \tau)$.
- (12) βg^* -closed [16], if $\text{Cl}(A) \subset U$ whenever $A \subset U$ and $U \in \text{SPO}(X, \tau)$.
- (13) sg -closed [2], if $\text{sCl}(A) \subset U$ whenever $A \subset U$ and $U \in \text{SO}(X, \tau)$.
- (14) pg -closed [18], if $\text{pCl}(A) \subset U$ whenever $A \subset U$ and $U \in \text{PO}(X, \tau)$.
- (15) $g\alpha$ -closed [13], if $\alpha\text{Cl}(A) \subset U$ whenever $A \subset U$ and $U \in \alpha(X, \tau)$.
- (16) pgs -closed [32], if $\text{pCl}(A) \subset U$ whenever $A \subset U$ and $U \in \text{SO}(X, \tau)$.
- (17) gpr -closed [8], if $\text{pCl}(A) \subset U$ whenever $A \subset U$ and $U \in \text{RO}(X, \tau)$.
- (18) αgs -closed [26], if $\alpha\text{Cl}(A) \subset U$ whenever $A \subset U$ and $U \in \text{SO}(X, \tau)$.
- (19) (α, β) - sg -closed [29], if $(\alpha, \beta)\text{-sCl}(A) \subset U$ whenever $A \subset U$ and $U \in (\alpha, \beta)\text{-SO}(X, \tau)$.

Recall that an ideal \mathcal{J} on a nonempty set X is a nonempty collection of subsets of X which satisfies the following two properties:

- (1) $A \in \mathcal{J}$ and $B \subset A$ implies $B \in \mathcal{J}$.
- (2) $A \in \mathcal{J}$ and $B \in \mathcal{J}$ implies $A \cup B \in \mathcal{J}$.

Let \mathcal{J} an ideal on X and \mathcal{G} a nonempty family of subsets of X . For a subset A of X we define the generalized local function of A with respect to \mathcal{J} and \mathcal{G} as follows:

$$A^*(\mathcal{J}, \mathcal{G}) = \{x \in X : U \cap A \notin \mathcal{J} \text{ for every } U \in \mathcal{G} \text{ such that } x \in U\}.$$

A topological space (X, τ) with an ideal \mathcal{J} on X is called an ideal topological space and is denoted by (X, τ, \mathcal{J}) .

Using the concept of local function, we find in the literature the following class of generalized closed sets:

Definition 2.2. A subset A of an ideal topological space (X, τ) is said to be:

- (1) \mathcal{J} - g -closed [6], if $A^*(\mathcal{J}, \tau) \subset U$ whenever $A \subset U$ and $U \in \tau$.
- (2) $g\mathcal{J}$ -closed [10], if $A^*(\mathcal{J}, \text{SO}(X, \tau)) \subset U$ whenever $A \subset U$ and $U \in \tau$.
- (3) \mathcal{J} - sg -closed [20], if $A^*(\mathcal{J}, \tau) \subset U$ whenever $A \subset U$ and $U \in \text{SO}(X, \tau)$.
- (4) \mathcal{J} - pg -closed [20], if $A^*(\mathcal{J}, \tau) \subset U$ whenever $A \subset U$ and $U \in \text{PO}(X, \tau)$.
- (5) \mathcal{J} - αg -closed [20], if $A^*(\mathcal{J}, \tau) \subset U$ whenever $A \subset U$ and $U \in \alpha(X, \tau)$.
- (6) \mathcal{J} - βg -closed [20], if $A^*(\mathcal{J}, \tau) \subset U$ whenever $A \subset U$ and $U \in \beta(X, \tau)$.

3. MINIMAL STRUCTURES

Definition 3.1. Let X be a nonempty set and $\mathcal{P}(X)$ the power set of X . A subfamily m_X of $\mathcal{P}(X)$ is called a minimal structure (briefly m -structure) on X [24] if $\emptyset \in m_X$ and $X \in m_X$.

A set X with an m -structure m_X (resp. two m -structures m_X and n_X) is called an m -space (resp. bi m -space) and is denoted by (X, m_X) (resp. (X, m_X, n_X)). Each member of m_X is said to be m_X -open and the complement of an m_X -open set is said to be m_X -closed. The family of all m_X -closed sets is denoted by m_X^c .

Definition 3.2. Let X be a nonempty set and m_X an m -structure on X . For a subset A of X , the m_X -closure of A and the m_X -interior of A are defined in [15] as follows:

- (1) $m_X\text{-Cl}(A) = \cap\{F : A \subset F, X \setminus F \in m_X\}$,
- (2) $m_X\text{-Int}(A) = \cup\{U : U \subset A, U \in m_X\}$.

Lemma 3.1. [15] *Let X be a nonempty set and m_X a minimal structure on X . For subsets A and B of X , the following properties hold:*

- (1) $m_X\text{-Cl}(X \setminus A) = X \setminus m_X\text{-Int}(A)$ and $m_X\text{-Int}(X \setminus A) = X \setminus m_X\text{-Cl}(A)$,
- (2) *If $(X \setminus A) \in m_X$, then $m_X\text{-Cl}(A) = A$ and if $A \in m_X$, then $m_X\text{-Int}(A) = A$,*
- (3) *If $A \subset B$, then $m_X\text{-Cl}(A) \subset m_X\text{-Cl}(B)$ and $m_X\text{-Int}(A) \subset m_X\text{-Int}(B)$,*
- (4) $m_X\text{-Cl}(m_X\text{-Cl}(A)) = m_X\text{-Cl}(A)$ and $m_X\text{-Int}(m_X\text{-Int}(A)) = m_X\text{-Int}(A)$.

Lemma 3.2. [24] *Let (X, m_X) be an m -space and A a subset of X . Then $x \in m_X\text{-Cl}(A)$ if and only if $U \cap A \neq \emptyset$ for every $U \in m_X$ containing x .*

Definition 3.3. A minimal structure m_X on a nonempty set X is said to have property \mathcal{B} [15] if the union of any family of subsets belonging to m_X belongs to m_X .

Remark 3.1. Let (X, τ) be a topological space. The families $\text{SO}(X, \tau)$, $\text{PO}(X, \tau)$, $\alpha(X, \tau)$, $\beta(X, \tau)$, $\text{BO}(X, \tau)$, τ_δ , $(\alpha, \beta)\text{-SO}(X, \tau)$ and $\text{RO}(X, \tau)$ are all minimal structures on X have property \mathcal{B} . It is well-known that $\alpha(X, \tau)$ and τ_δ are topologies for X and the others are not topologies.

Lemma 3.3. [25] *Let X be a nonempty set and m_X a minimal structure on X satisfying property \mathcal{B} . For a subset A of X , the following properties hold:*

- (1) $A \in m_X$ if and only if $m_X\text{-Int}(A) = A$,
- (2) A is m_X -closed if and only if $m_X\text{-Cl}(A) = A$,

Using the notions of minimal structure, many mathematician working in unify the notions given in Definition 2.1. This unification was obtained by T. Noiri in [17].

Definition 3.4. Let (X, m_X) be an m -space. A subset A of X is said to be m - g -closed [28] if $m_X\text{-Cl}(A) \subset U$ whenever $A \subset U$ and $U \in m_X$.

Definition 3.5. Let (X, τ) be a topological space and m_X an m -structure on X . A subset A of X is said to be m - g^* -closed [19] if $\text{Cl}(A) \subset U$, whenever $A \subset U$ and $U \in m_X$.

Definition 3.6. Let (X, m_X, n_X) be a bi m -space. A subset A of X is said to be m - ng -closed [17] if $n_X\text{-Cl}(A) \subset U$ whenever $A \subset U$ and $U \in m_X$.

Definition 3.7. Let (X, τ, \mathcal{J}) be an ideal topological space and m_X an m -structure on X . A subset A of X is said to be \mathcal{J} - m - g -closed [20], if $A^*(\mathcal{J}, \tau) \subset U$, whenever $A \subset U$ and $U \in m_X$.

The above definition almost unify all the notions given in Definition 2.1. In [22], they introduced a new concept of local function using minimal structure and obtain a generalization of all concepts given in Definition 2.1.

If (X, m_X) (resp. (X, m_X, n_X)) is a m -space (resp. a bi m -space) and \mathcal{J} is an ideal on X , then (X, m_X, \mathcal{J}) (resp. $(X, m_X, n_X, \mathcal{J})$) is called an ideal m -space (resp. ideal bi m -space).

Definition 3.8. Let (X, m_X, \mathcal{J}) be an ideal m -space and A a subset of X . Then $A^*(\mathcal{J}, m_X) = \{x \in X : A \cap U \notin \mathcal{J} \text{ for every } U \in m_X \text{ such that } x \in U\}$ is called the minimal local function [22] of A with respect to \mathcal{J} and m_X .

When there is no ambiguity, we will write $A_m^*(\mathcal{J})$ or simply A_m^* for $A^*(\mathcal{J}, m_X)$. Observe that $A_m^* = m_X\text{-Cl}(A)$ and $A_m^*(\mathcal{P}(X)) = \emptyset$. Furthermore, $\emptyset_m^* = \emptyset$.

Theorem 3.1. [22] *Let (X, m_X) be a m -space with \mathcal{J}, \mathcal{I} ideals on X , and let A, B subsets of X . Then*

- (1) $A \subset B \Rightarrow A_m^* \subset B_m^*$,
- (2) $A_m^* = m_X\text{-Cl}(A_m^*) \subset m_X\text{-Cl}(A)$,
- (3) $(A_m^*)_m^* \subset A_m^*$.

Definition 3.9. A subset A of an ideal m -space (X, m_X, \mathcal{J}) is $m_X\star$ -closed [22] if $A_m^* \subset A$. The complement of an $m_X\star$ -closed is said to be $m_X\star$ -open.

Definition 3.10. A subset A of an ideal m -space (X, m_X, \mathcal{J}) is m - $\mathcal{J}g$ -closed [22] if $A_m^* \subset U$ whenever $A \subset U$ and $U \in m_X$.

4. \mathcal{J} - mng -CLOSED SETS

Definition 4.1. Let (X, m_X, n_X) be a bi m -space and \mathcal{J} be an ideal on X . A subset A of X is said to be \mathcal{J} - mng -closed if $n_X\text{-Cl}(A) \setminus U \in \mathcal{J}$, whenever $A \subset U$ and $U \in m_X$. The complement of an \mathcal{J} - mng -closed set is said to be \mathcal{J} - mng -open.

Observe that in the above definition, we obtain the unification of all the concepts of generalized closed sets given in Definitions 2.1, 3.4, 3.5, 3.6 and 3.7, also it generalize the notions of mng -closed sets.

Remark 4.1. Let (X, τ) be a topological space.

- (1) If $m_X = \tau$, $n_X = \tau$ (resp. $\text{SO}(X, \tau)$, $\text{PO}(X, \tau)$, $\alpha(X, \tau)$, $\text{SPO}(X, \tau)$, $\text{BO}(X, \tau)$, τ_δ) and $\mathcal{J} = \{\emptyset\}$, then an \mathcal{J} - mng -closed set is a g -closed (resp. gs -closed, gp -closed, αg -closed, gsp -closed, γg -closed, δ - g -closed) set.
- (2) If $m_X = n_X = \alpha(X, \tau)$ (resp. $\text{SO}(X, \tau)$, $\text{PO}(X, \tau)$) and $\mathcal{J} = \{\emptyset\}$, then an \mathcal{J} - mng -closed set is a $g\alpha$ -closed (resp. sg -closed, pg -closed) set.
- (3) If $m_X = \text{RO}(X, \tau)$ (resp. $\text{SO}(X, \tau)$, $\text{PO}(X, \tau)$, $\alpha(X, \tau)$, $\beta(X, \tau)$) and $n_X = \tau$ and $\mathcal{J} = \{\emptyset\}$, then an \mathcal{J} - mng -closed set is an rg -closed (resp. sg^* -closed, pg^* -closed, αg^* -closed, βg^* -closed) set.
- (4) If $m_X = \text{SO}(X, \tau)$ (resp. $\text{RO}(X, \tau)$) and $n_X = \text{PO}(X, \tau)$ and $\mathcal{J} = \{\emptyset\}$, then an \mathcal{J} - mng -closed set is a pgs -closed (resp. gpr -closed) set.
- (5) If $m_X = \text{SO}(X, \tau)$ and $n_X = \alpha(X, \tau)$ and $\mathcal{J} = \{\emptyset\}$, then an \mathcal{J} - mng -closed set is an αgs -closed set.
- (6) If α and β are associated operators to the topology τ on X and $m_X = n_X = (\alpha, \beta)\text{-SO}(X, \tau)$ and $\mathcal{J} = \{\emptyset\}$, then an \mathcal{J} - mng -closed set is an (α, β) - sg -closed set.

Remark 4.2. (1) Let (X, m_X, \mathcal{J}) be an ideal m -space. If $n_X = m_X$ and $\mathcal{J} = \{\emptyset\}$, then an \mathcal{J} - mng -closed set is an m_X - g -closed set.

(2) Let (X, τ, \mathcal{J}) be an ideal topological space and m_X an m -structure on X . If $n_X = \tau$ and $\mathcal{J} = \{\emptyset\}$, then an \mathcal{J} - mng -closed set is an mg^* -closed set.

(3) Let $(X, m_X, n_X, \mathcal{J})$ be an ideal bi m -space. If $\mathcal{J} = \{\emptyset\}$, then an \mathcal{J} - mng -closed set is an mng -closed set.

The following example shows that the union and the intersection of two \mathcal{J} - mng -closed sets need not be an \mathcal{J} - mng -closed set.

Example 4.1. Let $X = \{a, b, c, d\}$, $m_X = \{\emptyset, X, \{a\}, \{c, d\}, \{a, b, c\}\}$, $n_X = \{\emptyset, X, \{a, d\}, \{b, c\}, \{c, d\}, \{a, b, d\}\}$ and $\mathcal{J} = \{\emptyset, \{a\}, \{c\}, \{a, c\}\}$. Then, $n_X^c = \{\emptyset, X, \{c\}, \{a, b\}, \{a, d\}, \{b, c\}\}$. Let $A = \{a\}$, $B = \{c\}$, $C = \{a, c\}$, $D = \{c, d\}$, $E = \{b, c, d\}$ and $F = \{a, c, d\}$. Observe that,

$$\begin{aligned}
n_X\text{-Cl}(A) &= n_X\text{-Cl}(\{a\}) = \{a\}, \\
n_X\text{-Cl}(B) &= n_X\text{-Cl}(\{c\}) = \{c\}, \\
n_X\text{-Cl}(C) &= n_X\text{-Cl}(\{a, c\}) = X, \\
n_X\text{-Cl}(D) &= n_X\text{-Cl}(\{c, d\}) = X, \\
n_X\text{-Cl}(E) &= n_X\text{-Cl}(\{b, c, d\}) = X, \\
n_X\text{-Cl}(F) &= n_X\text{-Cl}(\{a, c, d\}) = X.
\end{aligned}$$

Furthermore, $n_X\text{-Cl}(A) \setminus U \in \mathcal{J}$ and $n_X\text{-Cl}(B) \setminus V \in \mathcal{J}$ for every $U, V \in m_X$ such that $A \subset U$ and $B \subset V$. Therefore, $A = \{a\}$ and $B = \{c\}$ are \mathcal{J} -*mng*-closed sets. On the other hand, $n_X\text{-Cl}(C) \setminus \{a, b, c\} = X \setminus \{a, b, c\} = \{d\} \notin \mathcal{J}$. Thus, $A \cup B = C = \{a, c\}$ is not \mathcal{J} -*mng*-closed. Finally, $n_X\text{-Cl}(E) \setminus U \in \mathcal{J}$ and $n_X\text{-Cl}(F) \setminus V \in \mathcal{J}$ for every $U, V \in m_X$ such that $E \subset U$ and $F \subset V$. Thus, $E = \{b, c, d\}$ and $F = \{a, c, d\}$ are \mathcal{J} -*mng*-closed sets and $n_X\text{-Cl}(D) \setminus \{c, d\} = X \setminus \{c, d\} = \{a, b\} \notin \mathcal{J}$. Therefore, $E \cap F = D = \{c, d\}$ is not \mathcal{J} -*mng*-closed.

In the sequel, the ideal bi m -space $(X, m_X, n_X, \mathcal{J})$ is simply denoted by X . We give some results related with \mathcal{J} -*mng*-closed sets.

Proposition 4.1. *Every mng -closed set is \mathcal{J} - mng -closed.*

Proof. Suppose that A is mng -closed and let $U \in m_X$ such that $A \subset U$. Then, $n_X\text{-Cl}(A) \subset U$ and hence $n_X\text{-Cl}(A) \setminus U = \emptyset \in \mathcal{J}$ for each ideal \mathcal{J} on X . Therefore, A is \mathcal{J} -*mng*-closed. \square

The following example shows that the converse of the above theorem is in general not true.

Example 4.2. Let $X = \{a, b, c\}$, $m_X = \{\emptyset, X, \{a\}, \{b\}\}$, $n_X = \{\emptyset, X, \{a\}, \{c\}\}$ and $\mathcal{J} = \{\emptyset, \{b\}, \{c\}, \{b, c\}\}$. Then, $\{a\}$ is \mathcal{J} -*mng*-closed but not mng -closed in X , since $n_X\text{-Cl}(\{a\}) = \{a, b\}$ is not contained in $\{a\} \in m_X$.

Proposition 4.2. *If A is \mathcal{J} - mng -closed and $A \subset B \subset n_X\text{-Cl}(A)$, then B is \mathcal{J} - mng -closed.*

Proof. Suppose that A is an \mathcal{J} -*mng*-closed set and $A \subset B \subset n_X\text{-Cl}(A)$. Let $U \in m_X$ such that $B \subset U$. Then $A \subset U$ and hence $n_X\text{-Cl}(A) \setminus U \in \mathcal{J}$. Since $B \subset n_X\text{-Cl}(A)$, we have $n_X\text{-Cl}(B) \subset n_X\text{-Cl}(A)$ and $n_X\text{-Cl}(B) \setminus U \subset n_X\text{-Cl}(A) \setminus U \in \mathcal{J}$. Therefore, B is an \mathcal{J} -*mng*-closed set. \square

Proposition 4.3. *A subset A of X is \mathcal{J} - mng -open if and only if $F \setminus G \subset n_X\text{-Int}(A)$ for some $G \in \mathcal{J}$, whenever $F \subset A$ and F is m_X -closed.*

Proof. Necessity: Suppose that A is \mathcal{J} -*mng*-open. Let $F \subset A$ and F be m_X -closed. Then, $X \setminus A \subset X \setminus F \in m_X$ and $X \setminus A$ is \mathcal{J} -*mng*-closed. Therefore, $n_X\text{-Cl}(X \setminus A) \setminus (X \setminus F) \in \mathcal{J}$ and hence, $n_X\text{-Cl}(X \setminus A) \subset (X \setminus F) \cup G$ for some $G \in \mathcal{J}$. This implies that, $X \setminus ((X \setminus F) \cup G) \subset X \setminus n_X\text{-Cl}(X \setminus A)$ and hence $F \setminus G \subset n_X\text{-Int}(A)$. Sufficiency: Let $X \setminus A \subset U$ and $U \in m_X$. Then, $X \setminus U \subset A$ and $X \setminus U$ is m_X -closed. By the hypothesis, we have $(X \setminus U) \setminus G \subset n_X\text{-Int}(A) = X \setminus n_X\text{-Cl}(X \setminus A)$ for some $G \in \mathcal{J}$. This gives $X \setminus (U \cup G) \subset X \setminus n_X\text{-Cl}(X \setminus A)$ and consequently

$n_X\text{-Cl}(X \setminus A) \subset U \cup G$ for some $G \in \mathcal{J}$. Therefore, $n_X\text{-Cl}(X \setminus A) \setminus U \in \mathcal{J}$ and hence, $X \setminus A$ is \mathcal{J} -mng-closed. \square

Definition 4.2. Let (X, m_X) be an m -space and A a subset of X . The subset $\Lambda_m(A)$ [3] is defined as follows: $\Lambda_m(A) = \bigcap \{U : A \subset U \in m_X\}$.

Theorem 4.1. A subset A of X is \mathcal{J} -mng-closed if and only if $n_X\text{-Cl}(A) \setminus \Lambda_m(A) \in \mathcal{J}$.

Proof. Necessity: Suppose that A is \mathcal{J} -mng-closed. Let U be any m_X -open set containing A . Then $n_X\text{-Cl}(A) \setminus U \in \mathcal{J}$ and $\Lambda_m(A) \subset U$. Hence, $n_X\text{-Cl}(A) \setminus \Lambda_m(A) \subset n_X\text{-Cl}(A) \setminus U \in \mathcal{J}$.

Sufficiency: Suppose that $n_X\text{-Cl}(A) \setminus \Lambda_m(A) \in \mathcal{J}$ and let V be any m_X -open set containing A . Then,

$$\begin{aligned} n_X\text{-Cl}(A) \setminus V &\subset \bigcup_{U \in m_X} \{n_X\text{-Cl}(A) \setminus U : A \subset U\} \\ &= n_X\text{-Cl}(A) \setminus \bigcap_{U \in m_X} \{U : A \subset U\} \\ &= n_X\text{-Cl}(A) \setminus \Lambda_m(A) \in \mathcal{J}. \end{aligned}$$

Thus, $n_X\text{-Cl}(A) \setminus V \in \mathcal{J}$ and hence A is an \mathcal{J} -mng-closed set. \square

5. mn - $\mathcal{J}g$ -CLOSED SETS AND mn - $T_{\mathcal{J}}$ -SPACES

The Definition 4.1 can not be described using the concept of minimal local functions [22]. In consequence, we define a new class of sets in order to unify all the given concepts in Definitions 2.1, 3.4, 3.5, 3.6, 3.7 and 3.10.

Definition 5.1. Let (X, m_X, n_X) be a bi m -space and \mathcal{J} be an ideal on X . A subset A of X is said to be mn - $\mathcal{J}g$ -closed if $A_n^* \subset U$, whenever $A \subset U$ and $U \in m_X$. The complement of an mn - $\mathcal{J}g$ -closed set is said to be mn - $\mathcal{J}g$ -open.

Remark 5.1. Note that if $\mathcal{J} = \{\emptyset\}$, then

$$mn\text{-}\mathcal{J}g\text{-closed} \Leftrightarrow \mathcal{J}\text{-mng-closed} \Leftrightarrow mng\text{-closed}.$$

Thus, the Remarks 4.1 and 4.2 are also true if we replace \mathcal{J} -mng-closed by mn - $\mathcal{J}g$ -closed.

Remark 5.2. Let (X, τ) be a topological space and \mathcal{J} be an ideal on X .

(1) If $m_X = \tau$, $n_X = \tau$ (resp. $\text{SO}(X)$), then an mn - $\mathcal{J}g$ -closed set is an \mathcal{J} - g -closed (resp. $g\mathcal{J}$ -closed) set.

(2) If $m_X = \text{SO}(X, \tau)$ (resp. $\text{PO}(X, \tau)$, $\alpha(X, \tau)$, $\beta(X, \tau)$) and $n_X = \tau$, then an mn - $\mathcal{J}g$ -closed set is an \mathcal{J} - sg -closed, \mathcal{J} - pg -closed, \mathcal{J} - αg -closed, \mathcal{J} - βg -closed) set.

Remark 5.3. (1) Let (X, τ, \mathcal{J}) be an ideal topological space and m_X an m -structure on X . If $n_X = \tau$, then an mn - $\mathcal{J}g$ -closed set is an \mathcal{J} - mg -closed set.

(2) Let (X, m_X, \mathcal{J}) be an ideal m -space. If $n_X = m_X$, then an mn - $\mathcal{J}g$ -closed set is an m - $\mathcal{J}g$ -closed set.

The following two examples shows that, in general, the notions of \mathcal{J} -*mng*-closed and *mn*- $\mathcal{J}g$ -closed are independent.

Example 5.1. Let $X = \{a, b, c\}$, $m_X = \{\emptyset, X, \{a\}, \{b\}\}$, $n_X = \{\emptyset, X, \{a\}, \{c\}\}$ and $\mathcal{J} = \{\emptyset, \{b\}, \{c\}, \{b, c\}\}$. Then, $n_X^c = \{\emptyset, X, \{b, c\}, \{a, b\}\}$. Let $A = \{a\}$. Observe that $n_X\text{-Cl}(A) = n_X\text{-Cl}(\{a\}) = \{a, b\}$, $A = \{a\} \subset \{a\} \in m_X$ and $A = \{a\} \subset X \in m_X$. Furthermore,

$$\begin{aligned} n_X\text{-Cl}(A) \setminus \{a\} &= \{a, b\} \setminus \{a\} = \{b\} \in \mathcal{J}, \\ n_X\text{-Cl}(A) \setminus X &= \{a, b\} \setminus X = \emptyset \in \mathcal{J}. \end{aligned}$$

Therefore, $A = \{a\}$ is an \mathcal{J} -*mng*-closed set. On the other hand, $A_n^* = \{a, b\}$ is not contained in $\{a\} \in m_X$. Thus, $A = \{a\}$ is not *mn*- $\mathcal{J}g$ -closed.

Example 5.2. Let $X = \{a, b, c, d\}$, $m_X = \{\emptyset, X, \{c\}, \{a, b\}\}$, $n_X = \{\emptyset, X, \{a\}, \{c\}\}$ and $\mathcal{J} = \{\emptyset, \{b\}, \{c\}, \{b, c\}\}$. Then, $n_X^c = \{\emptyset, X, \{b, c, d\}, \{a, b, d\}\}$. Let $A = \{c\}$. Observe that $n_X\text{-Cl}(A) = n_X\text{-Cl}(\{c\}) = \{b, c, d\}$, $A = \{c\} \subset \{c\} \in m_X$, but $n_X\text{-Cl}(A) \setminus \{c\} = \{b, c, d\} \setminus \{c\} = \{b, d\} \notin \mathcal{J}$. Therefore, $A = \{c\}$ is not an \mathcal{J} -*mng*-closed set. On the other hand, $A_n^* = \emptyset$ and so $A_n^* \subset U$, whenever $A \subset U$ and $U \in m_X$. Therefore, $A = \{c\}$ is an *mn*- $\mathcal{J}g$ -closed.

Now, we present some results related with *mn*- $\mathcal{J}g$ -closed sets.

Proposition 5.1. *Every mng-closed set is mn- $\mathcal{J}g$ -closed.*

Proof. Suppose that A is *mng*-closed and let $U \in m_X$ such that $A \subset U$. Since $A_n^* \subset n_X\text{-Cl}(A)$, then $A_n^* \subset U$. Thus, A is also *mn*- $\mathcal{J}g$ -closed. \square

The following example shows that the converse of the above theorem is not true.

Example 5.3. Consider the ideal bi m -space $(X, m_X, n_X, \mathcal{J})$ given in Example 5.2. Then, $A = \{c\}$ is an *mn*- $\mathcal{J}g$ -closed but not *mng*-closed in X , since $n_X\text{-Cl}(A) = \{b, c, d\}$ is not contained in $\{c\} \in m_X$.

Definition 5.2. A minimal structure m_X on a nonempty set X is said to have property \mathcal{F} if any finite intersection of subsets belonging to m_X belongs to m_X .

Definition 5.3. Let (X, m_X) be an m -space. A family $\mathcal{A} = \{A_\lambda : \lambda \in \Lambda\}$ of subsets of X is said to be m_X -locally finite if for each $x \in X$, there exists an m_X -open set U containing x and U intersects A_λ at most for finitely many λ .

Note that every finite family of subsets of (X, m_X) is m_X -locally finite.

Proposition 5.2. *Let (X, τ) be a topological space and m_X an m -structure on X such that $\tau \subset m_X$. Then, every locally finite family of subsets of X is m_X -locally finite.*

The following example shows that the converse of the above proposition need not be true in general.

Example 5.4. Let $X = \mathbb{R}$ be the real numbers and $\tau = \tau_u$ the usual topology. Let $\mathcal{A} = \{\{1/n\} : n = 1, 2, \dots\}$ and $m_X = \text{SO}(X, \tau)$. Then the family \mathcal{A} is m_X -locally finite since for every $a \in X$ we have $(a - 1, 0] \in m_X$ but \mathcal{A} is not locally finite.

Lemma 5.1. *Let (X, m_X, \mathcal{J}) be an ideal m -space and m_X has property \mathcal{F} . If $\{A_\lambda : \lambda \in \Lambda\}$ is an m_X -locally finite family of subsets of (X, m_X, \mathcal{J}) , then*

$$\bigcup_{\lambda \in \Lambda} (A_\lambda)_m^* = \left(\bigcup_{\lambda \in \Lambda} A_\lambda \right)_m^*.$$

Proof. For every $\lambda \in \Lambda$ we have $A_\lambda \subset \bigcup_{\lambda \in \Lambda} A_\lambda$ and this implies $(A_\lambda)_m^* \subset \left(\bigcup_{\lambda \in \Lambda} A_\lambda \right)_m^*$.

Hence we have $\bigcup_{\lambda \in \Lambda} (A_\lambda)_m^* \subset \left(\bigcup_{\lambda \in \Lambda} A_\lambda \right)_m^*$.

Conversely, let $x \in \left(\bigcup_{\lambda \in \Lambda} A_\lambda \right)_m^*$ and U an m_X -open set such that $x \in U$. Since $\{A_\lambda : \lambda \in \Lambda\}$ is m_X -locally finite, there exists an m_X -open set V containing x and V intersects A_λ at most for finitely many λ , says $A_{\lambda_1}, A_{\lambda_2}, \dots, A_{\lambda_k}$. Note that $U \cap V$ is an m_X -open set containing x . Since $x \in \left(\bigcup_{\lambda \in \Lambda} A_\lambda \right)_m^*$, then $(U \cap V) \cap \left(\bigcup_{\lambda \in \Lambda} A_\lambda \right) \notin \mathcal{J}$ for every $U \in m_X$ such that $x \in U$. It follows that

$\left(\bigcup_{\lambda \in \Lambda} A_\lambda \right) \cap \left(\bigcup_{i=1}^k (U \cap V) \cap A_{\lambda_i} \right) \notin \mathcal{J}$ for every $U \in m_X$ such that $x \in U$. Therefore, there exists

$A_{\lambda_j} \in \{A_{\lambda_1}, A_{\lambda_2}, \dots, A_{\lambda_k}\}$ such that $(U \cap V) \cap A_{\lambda_j} \notin \mathcal{J}$, and hence $U \cap A_{\lambda_j} \notin \mathcal{J}$.

Thus, we obtain $x \in (A_{\lambda_j})_m^*$ which implies $x \in \bigcup_{i=1}^k (A_{\lambda_i})_m^* \subset \bigcup_{\lambda \in \Lambda} (A_\lambda)_m^*$. This shows

that $\left(\bigcup_{\lambda \in \Lambda} A_\lambda \right)_m^* \subset \bigcup_{\lambda \in \Lambda} (A_\lambda)_m^*$. This completes the proof. \square

Theorem 5.1. *Let $(X, m_X, n_X, \mathcal{J})$ be an ideal bi m -space where n_X has the property \mathcal{F} . If $\mathcal{A} = \{A_\lambda : \lambda \in \Lambda\}$ is an n_X -locally finite family of subsets of X and A_λ is mn - $\mathcal{J}g$ -closed for each $\lambda \in \Lambda$, then $\bigcup_{\lambda \in \Lambda} A_\lambda$ is mn - $\mathcal{J}g$ -closed.*

Proof. Let $U \in m_X$ such that $\bigcup_{\lambda \in \Lambda} A_\lambda \subset U$. Then, $A_\lambda \subset U$ for each $\lambda \in \Lambda$. Since A_λ

is mn - $\mathcal{J}g$ -closed for each $\lambda \in \Lambda$, we have $(A_\lambda)_n^* \subset U$ and hence, $\bigcup_{\lambda \in \Lambda} (A_\lambda)_n^* \subset U$. By

Lemma 5.1, we have $\left(\bigcup_{\lambda \in \Lambda} A_\lambda \right)_n^* \subset U$. Therefore, $\bigcup_{\lambda \in \Lambda} A_\lambda$ is mn - $\mathcal{J}g$ -closed. \square

Corollary 5.1. *Let $(X, m_X, n_X, \mathcal{J})$ be an ideal bi m -space where n_X has the property \mathcal{F} . If A and B are mn - $\mathcal{J}g$ -closed sets of X , then $A \cup B$ is also an mn - $\mathcal{J}g$ -closed set.*

The following example shows that the condition that n_X has the property \mathcal{F} is necessary in the above corollary.

Example 5.5. Let $X = \{a, b, c, d\}$, $m_X = \{\emptyset, X, \{a, b\}, \{a, c, d\}\}$, $n_X = \{\emptyset, X, \{a, b\}, \{b, c\}, \{c, d\}\}$ and $\mathcal{J} = \{\emptyset, \{b\}, \{d\}, \{b, d\}\}$. Then, $A = \{a\}$ and $B = \{c, d\}$ are mn - $\mathcal{J}g$ -closed sets but $A \cup B = \{a, c, d\}$ is not mn - $\mathcal{J}g$ -closed in X , since $(A \cup B)_n^* = X$ is not contained in $\{a, c, d\} \in m_X$.

Definition 5.4. Two minimal structures m_X and n_X on a nonempty set X are said to have property \mathcal{D} if $M \in m_X$ and $N \in n_X$ imply $M \cap N \in n_X$.

Lemma 5.2. Let $(X, m_X, n_X, \mathcal{J})$ be an ideal bi m -space and m_X and n_X have property \mathcal{D} . If $\{A_\lambda : \lambda \in \Lambda\}$ is an m_X -locally finite family of subsets of $(X, m_X, n_X, \mathcal{J})$, then

$$\bigcup_{\lambda \in \Lambda} (A_\lambda)_n^* = \left(\bigcup_{\lambda \in \Lambda} A_\lambda \right)_n^*.$$

Proof. Arguin as in the proof of Lemma 5.1. Just use the fact that m_X and n_X have property \mathcal{D} instead of m_X has property \mathcal{F} . \square

Theorem 5.2. Let $(X, m_X, n_X, \mathcal{J})$ be an ideal bi m -space where m_X and n_X have property \mathcal{D} . If $\mathcal{A} = \{A_\lambda : \lambda \in \Lambda\}$ is an m_X -locally finite family of subsets of X and A_λ is mn - $\mathcal{J}g$ -closed for each $\lambda \in \Lambda$, then $\bigcup_{\lambda \in \Lambda} A_\lambda$ is mn - $\mathcal{J}g$ -closed.

Proof. Follows from Lemma 5.2, using a similar argument as in the proof of Theorem 5.1. \square

Corollary 5.2. Let $(X, m_X, n_X, \mathcal{J})$ be an ideal bi m -space where m_X and n_X have property \mathcal{D} . If A and B are mn - $\mathcal{J}g$ -closed sets of X , then $A \cup B$ is also an mn - $\mathcal{J}g$ -closed set.

Proposition 5.3. If A is mn - $\mathcal{J}g$ -closed and $A \subset B \subset A_n^*$, then B is mn - $\mathcal{J}g$ -closed.

Proof. Suppose that A is an mn - $\mathcal{J}g$ -closed set and $A \subset B \subset A_n^*$. Let $U \in m_X$ such that $B \subset U$. Then $A \subset U$ and hence $A_m^* \subset U$. Since $B \subset A_n^*$, we have $B_n^* \subset (A_n^*)_n^* \subset A_m^* \subset U$. Therefore, B is an mn - $\mathcal{J}g$ -closed set. \square

Theorem 5.3. A subset A of X is mn - $\mathcal{J}g$ -closed if and only if $A_n^* \subset \Lambda_m(A)$.

Proof. Necessity: Suppose that A is mn - $\mathcal{J}g$ -closed. Let U be any m_X -open set containing A . Then $A_n^* \subset U$ and hence, we obtain $A_n^* \subset \Lambda_m(A)$.

Sufficiency: Suppose that $A_n^* \subset \Lambda_m(A)$ and let U be any m_X -open set containing A . Then, $A_n^* \subset \Lambda_m(A) \subset U$ and hence A is an mn - $\mathcal{J}g$ -closed set. \square

Proposition 5.4. If m_X has property \mathcal{B} and A is a subset of X , then the following properties are equivalent:

- (1) A is mn - $\mathcal{J}g$ -closed.
- (2) $n_X\text{-Cl}(A_n^*) \subset U$ for every m_X -open set U containing A .
- (3) $m_X\text{-Cl}(\{x\}) \cap A \neq \emptyset$ for every $x \in n_X\text{-Cl}(A_n^*)$.

Proof. (1) \Rightarrow (2). Suppose that A is mn - $\mathcal{J}g$ -closed. Let $U \in m_X$ such that $A \subset U$. Then, $A_n^* \subset U$ and so by Theorem 3.1, $n_X\text{-Cl}(A_n^*) \subset U$ whenever $A \subset U$ and $U \in m_X$.

(2) \Rightarrow (3). Let $x \in n_X\text{-Cl}(A_n^*)$ and suppose that $m_X\text{-Cl}(\{x\}) \cap A = \emptyset$. Then, $A \subset X \setminus m_X\text{-Cl}(\{x\})$ and $X \setminus m_X\text{-Cl}(\{x\})$ is m_X -open, since that m_X has property \mathcal{B} . By (2), we have $n_X\text{-Cl}(A_n^*) \subset X \setminus m_X\text{-Cl}(\{x\}) \subset X \setminus \{x\}$. This is a contradiction. Therefore, $m_X\text{-Cl}(\{x\}) \cap A \neq \emptyset$.

(3) \Rightarrow (1). Suppose that A is not mn - $\mathcal{J}g$ -closed. Then, there exists an m_X -open set U such that $A \subset U$ and A_n^* is not contained in U . Therefore, there exists a point $x \in A_n^*$ such that $x \notin U$. Thus, we have $\{x\} \cap U = \emptyset$ and hence, $m_X\text{-Cl}(\{x\}) \cap U = \emptyset$. Since $A \subset U$, we obtain $m_X\text{-Cl}(\{x\}) \cap A = \emptyset$. By Theorem 3.1, we have $x \in A_n^* = n_X\text{-Cl}(A_n^*)$ and it follows that (3) does not hold. This complete the proof. \square

The following example shows that the implication (2) \Rightarrow (3) of Theorem 5.4 is not satisfied when m_X does not have property \mathcal{B} .

Example 5.6. Let $X = \{a, b, c, d\}$, $m_X = \{\emptyset, X, \{a, b\}, \{b, c\}\}$, $n_X = \{\emptyset, X, \{a\}, \{a, c\}, \{c, d\}\}$ and $\mathcal{J} = \{\emptyset, \{b\}\}$. Let $A = \{a, c\}$, then $n_X\text{-Cl}(A_n^*) = A_n^* = X$ and X is the only m_X -open set containing A . Thus, (2) is satisfied. On the other hand, $d \in n_X\text{-Cl}(A_n^*) = X$ and $m_X\text{-Cl}(\{d\}) = \{d\}$. It follows that $m_X\text{-Cl}(\{d\}) \cap A = \emptyset$ and hence (3) is not satisfied.

Definition 5.5. Minimal structures m_X and n_X on a nonempty set X are said to have property \mathcal{C} [17] if $M \in m_X$ and $N \in n_X$ imply $M \cup N \in m_X$.

Theorem 5.4. *If a subset A of X is mn - $\mathcal{J}g$ -closed, then $n_X\text{-Cl}(A_n^*) \setminus A = A_n^* \setminus A$ does not contain any nonempty m_X -closed. The converse holds if n_X has property \mathcal{B} and m_X and n_X have property \mathcal{C} .*

Proof. Suppose that A is mn - $\mathcal{J}g$ -closed. Assume that $F \subset A_n^* \setminus A$ and F is m_X -closed. Observe that $A \subset X \setminus F$ and $X \setminus F$ is m_X -open. Then, $A_n^* \subset X \setminus F$ and $F \subset X \setminus A_n^*$. Since $F \subset A_n^*$, we have $F \subset (X \setminus A_n^*) \cap A_n^* = \emptyset$. Thus, $n_X\text{-Cl}(A_n^*) \setminus A = A_n^* \setminus A$ does not contain any nonempty m_X -closed.

Conversely, suppose that n_X has property \mathcal{B} and m_X and n_X has property \mathcal{C} . Let $A \subset U$ and $U \in m_X$. Since $n_X\text{-Cl}(A_n^*) = A_n^*$ and n_X has property \mathcal{B} , then A_n^* is n_X -closed. Note that $A_n^* \cap (X \setminus U) \subset A_n^* \setminus A$ and $(X \setminus U)$ is m_X -closed. Since m_X and n_X have property \mathcal{C} , then $A_n^* \cap (X \setminus U)$ is an m_X -closed set contained in $A_n^* \setminus A$. By assumption, $A_n^* \cap (X \setminus U) = \emptyset$ and hence $A_n^* \subset U$. Thus, A is mn - $\mathcal{J}g$ -closed. \square

The following example shows that, in the theorem above, the conditions that n_X has property \mathcal{B} and m_X and n_X have property \mathcal{C} cannot be dropped.

Example 5.7. Consider the ideal bi m -space $(X, m_X, n_X, \mathcal{J})$ given in Example 5.2. If $B = \{a\}$, then $B_n^* = \{a, b, d\}$ and $B = \{a\} \subset \{a, b\} \in m_X$ but $B_n^* = \{a, b, d\}$ is not contained in $\{a, b\}$. Therefore, $B = \{a\}$ is not an mn - $\mathcal{J}g$ -closed set. On the other hand, $m_X^c = \{\emptyset, X, \{c, d\}, \{a, b, d\}\}$. Thus, $B_n^* \setminus B = \{b, d\}$ does not contain any nonempty m_X -closed.

Proposition 5.5. *A subset A of X is mn - $\mathcal{J}g$ -closed if and only if $A_n^* \cap F = \emptyset$ whenever $A \cap F = \emptyset$ and F is m_X -closed.*

Proof. Necessity: Suppose that A is a mn - $\mathcal{J}g$ -closed set. Let $A \cap F = \emptyset$ and F be m_X -closed. Then $A \subset X \setminus F$ and $X \setminus F$ is m_X -open. Therefore, $A_n^* \subset X \setminus F$ and $A_n^* \cap F = \emptyset$.

Sufficiency: Let $A \subset U$ and U is m_X -open. Then, $A \cap (X \setminus U) = \emptyset$ and $X \setminus U$ is m_X -closed. By the hypothesis we have $A_n^* \cap (X \setminus U) = \emptyset$ and hence, $A_n^* \subset U$. This shows that A is a mn - $\mathcal{J}g$ -closed set. \square

Definition 5.6. An ideal bi m -space $(X, m_X, n_X, \mathcal{J})$, is said to be mn - $T_{\mathcal{J}}$ if every mn - $\mathcal{J}g$ -closed set of X is n_X -closed.

Theorem 5.5. *For an ideal bi m -space $(X, m_X, n_X, \mathcal{J})$, the following properties are equivalent:*

- (1) X is mn - $T_{\mathcal{J}}$,
- (2) For each $x \in X$, $\{x\}$ is an m_X -closed set or $\{x\}$ is an n_X -open set.

Proof. (1) \Rightarrow (2). Let $x \in X$. If $\{x\}$ is not m_X -closed, then $A = X \setminus \{x\}$ is not m_X -open and hence, A is trivially mn - $\mathcal{J}g$ -closed. By (1), A is n_X -closed and hence, $\{x\}$ is n_X -open.

(2) \Rightarrow (1). Let $A \subset X$ be mn - $\mathcal{J}g$ -closed and let $x \in A_n^*$. We have the following two cases:

Case 1. $\{x\}$ is m_X -closed. By Theorem 5.4, $A_n^* \setminus A$ does not contain any nonempty m_X -closed. Since $x \in A_n^*$, then we obtain $x \in A$.

Case 2. $\{x\}$ is n_X -open. We have $X \setminus \{x\}$ is n_X -closed and hence:

$$(X \setminus \{x\})_n^* \subset X \setminus \{x\}$$

or equivalently

$$\{x\} \subset X \setminus (X \setminus \{x\})_n^*.$$

Since $x \in \{x\}$, then $x \notin (X \setminus \{x\})_n^*$ and hence, there exists an n_X -open set U such that $x \in U$ and $U \cap (X \setminus \{x\}) \in \mathcal{J}$. On the other hand, $x \in A_n^*$ entails that $V \cap A \notin \mathcal{J}$ for each n_X -open set such that $x \in V$. In particular, there exists $U \in n_X$ such that $x \in U$, $U \cap A \notin \mathcal{J}$ and $U \cap (X \setminus \{x\}) \in \mathcal{J}$. We claim that $\{x\} \cap A \neq \emptyset$. If $\{x\} \cap A = \emptyset$, then $A \subset X \setminus \{x\}$ and hence $U \cap A \subset U \cap (X \setminus \{x\}) \in \mathcal{J}$. It follows that $U \cap A \in \mathcal{J}$. This is a contradiction. Therefore, $\{x\} \cap A \neq \emptyset$ and so $x \in A$.

Thus in both cases x is in A and hence $A_n^* \subset A$. Thus, A is n_X -closed, which shows that X is mn - $T_{\mathcal{J}}$. \square

Definition 5.7. [17] A bi m -space (X, m_X, n_X) , where n_X has property \mathcal{B} , is said to be mn - $T_{1/2}$ if and only if every singleton $\{x\}$ of X is n_X -open or m_X -closed.

Corollary 5.3. *Let $(X, m_X, n_X, \mathcal{J})$ an ideal bi m -space. If X is mn - $T_{1/2}$, then X is mn - $T_{\mathcal{J}}$.*

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