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NONCONVEX WIENER-HOPF EQUATIONS AND VARIATIONAL INEQUALITIES

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Abstract. We consider the problem of solving the nonconvex Wiener-Hopf equations on the prox-regular sets. Using the projection technique, we show that the nonconvex Wiener-Hopf equations are equivalent to the strongly nonlinear nonconvex variational inequalities. This equivalence is used to suggest several iterative methods for solving the strongly nonlinear nonconvex variational inequalities. We also discuss the convergence of these new iterative methods under suitable conditions. Several special cases are also considered.

1. INTRODUCTION

Variational inequalities theory, the origin of which can be traced back to Stampacchia [34], provides us with a simple, natural, general and unified framework to study a wide class of problems arising in pure and applied sciences. Variational inequalities have been generalized and extended in several direction to tack different problems. Noor [8], [9] introduced and studied a class of variational inequalities involving two different operator, which is known as strongly nonlinear variational inequality. For the applications, formulation, applications and other aspects of strongly nonlinear variational inequalities, see [8]-[11] and the references e therein. These classes of variational inequalities are being studied and investigated in the setting of classical convexity. The main reason of this fact is that one can use the projection technique for establishing the equivalence between the fixed point problems and the variational inequalities. We note that in many important applications of the variational inequalities, the under lying set may not be convex one. This fact has motivated Noor [21] and Bounkhel et al [1] to investigate these variational inequalities for the prox-regular sets, which are nonconvex set. Noor [21]-[29] has shown that one can establish the equivalence between the fixed-point problem and variational inequalities under some suitable conditions. This alternative equivalence

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is being used to explore the new and novel applications of the variational inequalities in different branches of pure and applied sciences. Noor [26] has considered the strongly nonlinear variational inequalities involving two different operators in the setting of nonconvex sets (prox-regular). It has been shown that strongly nonlinear nonconvex variational inequalities are equivalent to the fixed point problems. This equivalence has been used to discuss the existence of a solution of the strongly nonlinear nonconvex variational inequalities as well as the iterative methods, see [1]-[34].

In this paper, we first introduce a new class of Wiener-Hopf equations involving the projection of the real Hilbert space on the nonconvex set. Using the projection results of Noor [26], we show that the solving the strongly nonlinear nonconvex variational inequalities are equivalent to the Wiener-Hopf equations. This approach is more flexible and can be used to suggest some iterative methods for solving the strongly nonlinear nonconvex variational inequalities. This paper may be viewed as the continuation of Noor [26]. Some special cases are also discussed. Results obtained in this paper can be viewed as refinement and improvement of the previously known results for the variational inequalities and related optimization problems.

2. PRELIMINARIES

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ respectively. Let K be a nonempty and convex set in H. We, first of all, recall the following well-known concepts from nonlinear convex analysis and nonsmooth analysis [2], [32].

Definition 2.1. The proximal normal cone of K at $u \in H$ is given by

$$N_K^P(u) := \{\xi \in H : u \in P_K[u + \alpha\xi]\},$$

where $\alpha > 0$ is a constant and

$$P_K[u] = \{u^* \in K : d_K(u) = \|u - u^*\|\}.$$

Here $d_K(.)$ is the usual distance function to the subset K, that is

$$d_K(u) = \inf_{v\in K} \|v - u\|.$$

The proximal normal cone $N_K^P(u)$ has the following characterization.

Lemma 2.1. *Let K be a nonempty, closed and convex subset in H. Then $\zeta \in N_K^P(u)$, if and only if, there exists a constant $\alpha > 0$ such that*

$$\langle\zeta, v - u\rangle \leq \alpha\|v - u\|^2, \quad \forall v \in K.$$

Poliquin et al. [32] and Clarke et al [2] have introduced and studied a new class of nonconvex sets, which are called uniformly prox-regular sets. This class of uniformly prox-regular sets has played an important part in many nonconvex applications such as optimization, dynamic systems and differential inclusions.

Definition 2.2. For a given $r \in (0, \infty]$, a subset K_r is said to be normalized uniformly r-prox-regular if and only if every nonzero proximal normal to K_r can be realized by an r-ball, that is, $\forall u \in K_r$ and $0 \neq \xi \in N^P_{K_r}(u)$, one has

$$\langle (\xi)/\|\xi\|, v-u\rangle \leq (1/2r)\|v-u\|^2, \quad \forall v \in K.$$

It is clear that the class of normalized uniformly prox-regular sets is sufficiently large to include the class of convex sets, p-convex sets, $C^{1,1}$ submanifolds (possibly with boundary) of H, the images under a $C^{1,1}$ diffeomorphism of convex sets and many other nonconvex sets; see [2], [32]. It is clear that if $r = \infty$, then uniformly prox-regularity of K_r is equivalent to the convexity of K. For the sake of simplicity, we take $\gamma = \dfrac{1}{r}$. Clearly if $r = \infty$, then $\gamma = 0$.

For given nonlinear operators T, A, we consider the problem of finding $u \in K_r$ such that

$$\langle Tu, v-u\rangle + \mu\|v-u\|^2 \geq \langle A(u), v-u\rangle, \quad \forall v \in K_r, \tag{2.1}$$

which is called the *strongly nonlinear nonconvex variational inequality,* introduced and studied in Noor [26]. Here μ is a positive parameter and can be regarded as the regularization factor.

We note that, if $K_r \equiv K$, the convex set in H, then problem (2.1) is equivalent to finding $u \in K$ such that

$$\langle Tu, v-u\rangle \geq \langle A(u), v-u\rangle, \qquad \forall v \in K. \tag{2.2}$$

Inequality of type (2.2) is called the *strongly nonlinear variational inequality*, which was introduced and studied by Noor [5] - [8] For the applications, numerical methods and other aspects of the strongly nonlinear variational inequalities and related optimization problems, see [5] - [30].

If $A(u) \equiv 0$, then problem (2.1) is equivalent to finding $u \in K_r$ such that

$$\langle Tu, v-u\rangle + \mu\|v-u\|^2 \geq 0, \quad \forall v \in K_r, \tag{2.3}$$

which is called the *nonconvex variational inequality* introduced and studied by Noor [21] - [28]. We would like to mention that problem (2.3) is correct formulation of the nonconvex variational inequality and its variant form considered in Noor [21] - [28]. All the results obtained in these papers continue to hold for this problem with suitable modification.

We note that, if $K_r \equiv K$, the convex set in H, then problem (2.3) is equivalent to finding $u \in K$ such that

$$\langle Tu, v-u\rangle \geq 0, \qquad \forall v \in K. \tag{2.4}$$

Problem (2.4) is known as the classical variational inequality, which was introduced and studied by Stampacchia [34]. It turned out that a number of unrelated obstacle, free, moving, unilateral and equilibrium problems arising in various branches of pure and applied sciences can be studied via variational inequalities, see [1] - [34] and the references therein.

If K_r is a nonconvex (uniformly prox-regular) set, then problem (2.1) is equivalent to finding $u \in K_r$ such that

$$0 \in Tu - A(u) + N^P_{K_r}(u), \tag{2.5}$$

where $N^P_{K_r}(u)$ denotes the normal cone of K_r at u in the sense of nonconvex analysis. Problem (2.5) is called the nonconvex variational inclusion problem associated with nonconvex variational inequality (2.1). This implies that the variational inequality (2.1) is equivalent to finding a zero of the sum of two monotone operators (2.5). This equivalent formulation plays a crucial and basic part in this paper. We would like to point out this equivalent formulation allows us to use the projection operator technique for solving the strongly nonlinear nonconvex variational inequality (2.1).

We now recall the well known proposition which summarizes some important properties of the uniform prox-regular sets, see [32].

Lemma 2.2. *Let K be a nonempty closed subset of H, $r \in (0, \infty]$ and set $K_r = \{u \in H : d_K(u) < r\}$. If K_r is uniformly prox-regular, then:*

i) $\forall u \in K_r, P_{K_r}(u) \neq \emptyset$*;*

ii) $\forall r' \in (0, r)$, P_{K_r} *is Lipschitz continuous with constant* $\delta \dfrac{r}{r - r'}$ *on* $K_{r'}$*.*

Definition 2.3. An operator $T : H \to H$ is said to be:

i) *strongly monotone*, if and only if, there exists a constant $\alpha > 0$ such that

$$\langle Tu - Tv, u - v\rangle \geq \alpha||u - v||^2, \quad \forall u, v \in H;$$

ii) *Lipschitz continuous*, if and only if, there exists a constant $\beta > 0$ such that

$$||Tu - Tv|| \leq \beta||u - v||, \quad \forall u, v \in H.$$

3. MAIN RESULTS

We now consider the problem of solving the nonconvex Wiener-Hopf equations. To be more precise, let P_{K_r} be the projection of H onto the nonconvex set K_r and $Q_{K_r} = I - P_{K_r}$, where I is the identity operator. For given nonlinear operators, consider the problem of finding $z \in H$ such that

$$TP_{K_r}z + \rho^{-1}Q_{K_r}z = A(P_{K_r}z), \tag{3.1}$$

which is called the strongly nonlinear nonconvex Wiener-Hopf equation. Note, for $K_r \equiv K$, the convex set, then the nonconvex Wiener-Hopf equation is exactly the same as considered by Noor [16]. For some special value of the operators T, A, one can obtain the original Wiener-Hopf equations, considered by Shi [33]. For the applications of the Wiener-Hopf equations, see [12] - [30] and the references therein.

We now show that the nonconvex Wiener-Hopf equations are equivalent to the strongly nonlinear nonconvex variational inequalities. For this purpose, we nee the following result, which is mainly due to Noor [26].

Lemma 3.1. *$u \in K_r$ is a solution of the strongly nonlinear nonconvex variational inequality* (2.1) *if and only if $u \in K_r$ satisfies the relation*

$$u = P_{K_r}[u - \rho Tu + \rho A(u)],$$

where P_{K_r} is the projection of H onto the uniformly prox-regular set K_r.

Lemma 3.1 implies that the strongly nonlinear nonconvex variational inequality (2.1) is equivalent to the fixed point problem (3.1). This alternative equivalent formulation is very useful from the numerical and theoretical point of views. Noor has used this equivalent formulation to discuss the existence of a solution of (2.1) and to suggest some iterative methods for solving strongly nonlinear nonconvex variational inequality (2.1).

We now use Lemma 3.1 to establish the equivalence between problems (2.1) and (3.1) and this is the main motivation of our nest result.

Lemma 3.2. *The nonconvex Wiener-Hopf equation* (3.1) *has a solution $z \in H$, if and only if, the strongly nonlinear nonconvex variational inequality* (2.1) *has solution $u \in K_r$, provided*

$$\begin{aligned} u &= P_{K_r} z \\ z &= u - \rho(Tu - A(u)), \end{aligned} \tag{3.2}$$

where P_{K_r} is the projection of H onto the closed nonconvex set K_r.

Proof. Let $u \in K_r$ be a solution of (2.1). Then, form Lemma 3.1, we have

$$u = P_{K_r}[u - \rho(Tu - A(u)].$$

Let

$$z = u - \rho(Tu - A(u)). \tag{3.3}$$

Then

$$u = P_{K_r} z. \tag{3.4}$$

Then, from (3.3) and (3.4), we have

$$z = P_{K_r} z - \rho T P_{K_r} z + \rho A(P_{K_r} z),$$

This shows that $z \in H$ is a solution of (3.1) and the converse is also true. □

I. The nonconvex Wiener-Hopf equations (3.1) can be written as

$$Q_{K_r}z = -\rho T P_{K_r}z + \rho A(P_{K-r}z),$$

which implies that, using (3.2)

$$z = P_{K_r}z - \rho T P_{K_r}z + \rho A(P_{K_r}z) = u - \rho T u + \rho A(u).$$

This fixed point formulation enables us to suggest the following iterative method for solving the strongly nonlinear nonconvex variational inequality (2.1).

Algorithm 3.1. For a given $z_0 \in H$, compute z_{n+1} by the iterative schemes

$$\begin{aligned} u_n &= P_{K_r}z_n \\ z_{n+1} &= (1-\alpha_n)z_n + \alpha_n\{u_n - \rho(Tu_n - A(u_n))\}, \quad n = 0,1,2,\ldots, \end{aligned}$$

where $0 \leq \alpha_n \leq 1$, for all $n \geq 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$.

II. The nonconvex Wiener-Hopf equations (3.1) may be written as

$$\begin{aligned} z &= P_{K_r}z - \rho T P_{K_r}z + \rho A(P_{K_r}z) + (1-\rho^{-1})Q_{K_r}z \\ &= u - \rho(Tu - A(u)) + (1-\rho^{-1})Q_{K_r}z. \end{aligned}$$

Using this fixed point formulation, we suggest the following iterative method.

Algorithm 3.2. For a given $z_0 \in H$, compute z_{n+1} by the iterative schemes

$$\begin{aligned} u_n &= P_{K_r}z_n \\ z_{n+1} &= (1-\alpha_n)z_n + \alpha_n\{u_n - \rho(Tu_n - A(u_n)) + (1-\rho^{-1})Q_{K_r}z_n\}, \quad n = 0,1,2,\ldots, \end{aligned}$$

where $0 \leq \alpha_n \leq 1$, for all $n \geq 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$.

We would like to point out that one can obtain a number of iterative methods for solving the strongly nonlinear nonconvex variational inequality (2.1) for suitable and appropriate choices of the operators T, A and the space H. This shows that iterative methods suggested in this paper are more general and unifying ones.

We now study the convergence analysis of Algorithm 3.1. In a similar way, one can analyze the convergence analysis of other iterative methods.

Theorem 3.1. *Let P_{K_r} be the Lipschitz continuous operator with constant $\delta = \frac{r}{r-r'}$. Let T be strongly monotone with constant $\alpha > 0$ and Lipschitz continuous with constant $\beta > 0$. Let the operator A be Lipschitz continuous with constant $\gamma > 0$. If there exists a constant ρ such that*

$$|\rho - \frac{\delta\alpha - \gamma}{\delta(\beta^2 - \gamma^2)}| < \frac{\sqrt{\delta(\delta\alpha - \gamma)^2 - (\beta^2 - \gamma^2)(\delta^2 - 1)}}{\delta(\beta^2 - \gamma^2)}, \quad \delta\rho\gamma < 1, \tag{3.5}$$

$$\delta\alpha > \gamma + \sqrt{(\beta^2 - \gamma^2)(\delta^1 - 1)},$$

and $\alpha_n \in [0,1], \quad \forall n \geq 0; \quad \sum_{n=0}^{\infty} \alpha_n = \infty$, *then the approximate solution* u_n *obtained from Algorithm* 3.1 *converges to a solution* $z \in H$ *satisfying the nonconvex Wiener-Hopf equation* (3.1)*.*

Proof. Let $z \in H$ be a solution of (3.1). Then, using Lemma 3.2, we have

$$z = (1-\alpha_n)z + \alpha_n\{u - \rho(Tu - A(u))\}, \tag{3.6}$$

where $0 \leq \alpha_n \leq 1$, and $\sum_{n=0}^{\infty} a_n = \infty$.

From (3.5) and (3.6), we have

$$\begin{aligned}\|z_{n+1} - z\| \leq (1-\alpha_n)\|z_n - z\| + \alpha_n\|u_n - u - \rho(Tu_n - Tu)\| \\ +\alpha_n\rho\|A(u_n) - A(u)\|.\end{aligned} \tag{3.7}$$

Since the operator T is strongly monotone with constant $\alpha > 0$ and Lipschitz continuous with constant $\beta > 0$, it follows that

$$\begin{aligned}\|u_n - u - \rho(Tu_n - Tu)\|^2 &\leq \|u_n - u\|^2 - 2\rho\langle Tu_n - Tu, u_n - u\rangle + \rho^2\|Tu_n - Tu\|^2 \\ &\leq (1 - 2\rho\alpha + \rho^2\beta^2)\|u_- u\|^2.\end{aligned} \tag{3.8}$$

From (3.7), (3.8) and using the Lipschitz continuity of the operator A with constant $\gamma > 0$, we have

$$\begin{aligned}\|z_{n+1} - z\| \leq (1-\alpha_n)\|z_n - z\| \\ +\delta\{\gamma\rho + \sqrt{1 - 2\alpha\rho + \beta^2\rho^2)}\}\|z_n - z\| = \theta\|z_n - z\|,\end{aligned} \tag{3.9}$$

where

$$\theta = \delta\{\gamma\rho + \sqrt{1 - 2\alpha\rho + \beta^2\rho^2}\}. \tag{3.10}$$

Also from (3.2), (3.4) and the Lipschitz continuity of the projection operator P_{K_r} with constant δ, we have

$$\|u_n - u\| = \|P_{K_r}z_n - P_{K_r}z\| \leq \delta\|z_n - z\|. \tag{3.11}$$

Combining (3.9), (3.10), and (3.11), we have

$$\begin{aligned}\|z_{n+1} - z\| &\leq (1-\alpha_n)\|z_n - z\| + \alpha_n\delta\{\rho\gamma + \sqrt{1 - 2\rho\alpha + \beta^2\rho^2}\}\|z_n - z\| \\ &= (1-\alpha_n)\|z_n - z\| + \alpha_n\theta\|z_n - z\|.\end{aligned}$$

From (3.5), we see that $\theta < 1$ and consequently

$$\begin{aligned}||z_{n+1} - z|| &\leq (1-\alpha_n)||z_n - z|| + \alpha_n\theta||z_n - z|| \\ &= [1 - (1-\theta)\alpha_n]||z_n - z|| \\ &\leq \prod_{i=0}^{n}[1 - (1-\theta)\alpha_i]||z_0 - z||.\end{aligned}$$

Since $\sum_{n=0}^{\infty}\alpha_n$ diverges and $1-\theta>0$, we have $\lim_{n\to\infty}\prod_{i=0}^{n}[1-(1-\theta)\alpha_i]=0$. Consequently the sequence $\{z_n\}$ convergences strongly to z in H satisfying the nonconvex Wiener-Hopf equation (3.1). □

Remark 3.1. We would like to point out that the ideas and techniques of this paper can be used to suggest and analyze some iterative methods for solving systems of (multivalued) general nonconvex variational inequalities involving several different operators with appropriate modifications. We hope that the results proved in this paper may inspire the interested readers to discover the novel applications of these (multivalued) nonconvex variational inequalities and systems of (multivalued) nonconvex variational inequalities in different branches of pure and applied sciences.

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