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A VECTOR FORM OF ALEKSANDROV'S THEOREM FOR NORMAL TOPOLOGICAL SPACES

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Abstract. Let X be a Hausdorff normal topological space, E a quasi-complete locally convex space, $C(X)$ (resp. $C_b(X)$) the space of all (resp. all, bounded), scalar-valued continuous functions on X, and $\mathcal{F}$ the algebra generated by the closed subset of X. The following form of Aleksandrov's theorem is proved: Suppose $\mu\colon C_b(X) \to E$ a weakly compact linear mapping. Then there exists a unique finitely additive, exhaustive measure $\nu\colon \mathcal{F} \to E$ such that

(i) ν is inner regular by closed sets and outer regular by open sets;

(ii) $\int f d\nu = \mu(f),\ \forall f \in C_b(X)$.

When X is also countably paracompact some additional results are also proved.

1. INTRODUCTION AND NOTATION

In this paper $\mathbb{R}$ stands for the set of real numbers, K will denote the field of real or complex numbers (we will call them scalars), X a Hausdorff normal topological space and E a quasi-complete locally convex space space over K with topology generated by an increasing family of semi-norms $\|.\|_p,\ p \in P$; E' will denote the topological dual of E. For a $p \in P$, $V_p = \{x \in E : \|x\|_p \leq 1\}$; polars will be taken in the duality $< E, E' >$. We denote by $C(X)$ the space of all K-valued continuous functions on X, and by $C_b(X)$ the bounded elements of $C(X)$. $Z \subset X$ will be called a zero-set if $Z = f^{-1}(0)$ for some $f \in C_b(X)$ and the complements of zero-sets will be called positive sets. The elements of the σ-algebra generated by zero-sets are called Baire sets and the elements of the σ-algebra generated by closed sets are called Borel sets; $\mathcal{B}(X)$ and $\mathcal{B}_0(X)$ are the classes of Borel and Baire subsets of X and $M_\sigma(X)$ denotes the class of all scalar-valued, countably additve Baire measures on X. For locally convex spaces, the notation and results of [11] will be used. For a vector space F, F^* will denote its algebraic dual. N will denote the set of natural numbers. For topological measure theory notations and results of ([5], [7], [12], [13], [14]) will be used. All locally convex spaces are assumed to be Hausdorff and over K. $\tilde{X}$ will denote the Stone-Cech compactification of X.

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With sup-norm topology, $C_b(X)$ is a Banach space which is isometric isomomorphic to $C(\tilde{X})$. The dual of $C_b(X)$ is denoted by $M(X)$. For a function $f \in C_b(X)$, $\tilde{f}$ will denote its unique continuous extension to $\tilde{X}$.

Now we come to vector-valued measures; integrability of scalar-valued functions is taken in the sense of [8]. If $\mathcal{A}$ is a σ-algebra of subsets of a set Y, $\mu\colon \mathcal{A} \to E$ a countably additive vector measure and $p \in P$, we denote the p-semi-variation of μ by $\bar{\mu}_p$, $\bar{\mu}_p(A) = \sup\{|g \circ \mu|(A) : g \in V_p^0\}$ (here V_p^0 is the polar of V_p in the duality $< E, E' >$) [8]; we consider the submeasure $\dot{\mu}_p\colon \mathcal{A} \to R^+$, $\dot{\mu}_p(A) = \sup\{\|\mu(B)\|_p : B \in \mathcal{A},\ B \subset A\}$ ([2], [5]). It is easily verified that $\dot{\mu}_p$ is countably sub-additive [2] and $\dot{\mu}_p \leq \bar{\mu}_p \leq 4\dot{\mu}_p$. Also there is a control measure for $\bar{\mu}_p$; this will be denoted by λ_p; this control measure can be chosen to be in the closed convex hull of $\{|g \circ \mu| : g \in V_p^0\}$, with norm topology on measures ([8], p. 20, proof of Theorem 1). This control measure also has the properties that:

(i) $|f \circ \mu| \ll \lambda_p$ for every f in E' with $\|f\|_p \leq 1$;

(note that $\|f\|_p = \sup\{|f(x)| : x \in V_p\}$);

(ii) if $\lambda_p(A) = 0$ then $\bar{\mu}_p(A) = 0$;

(iii) $\lim\limits_{\lambda_p(A) \to 0} \bar{\mu}_p(A) = 0$;

(iv) $\lambda_p \leq \bar{\mu}_p$.

We also have the result that if $f\colon Y \to K$ is measurable function, $B \in \mathcal{A}$ and $|f| \leq c$ on B, then $\left\| \int_B f d\mu \right\|_p \leq c\bar{\mu}_p(B)$.

$L^1(\mu)$ will denote the space of μ-integrable functions [8]. For any $f \in L^1(\mu)$, we take $\bar{\mu}_p(f) = \sup\{|g \circ \mu|(|f|) : g \in V_p^0\}$ ([8], Lemma 2, p. 23).

If $\mathcal{F}$ is an algebra of subsets of a set Y and $\mu\colon \mathcal{F} \to E$ a finitely additive measure then μ will be called exhaustive if for any disjoint sequence $\{A_n\} \subset \mathcal{F}$, we have $\mu(A_n) \to 0$ ([2]); exhaustive measures are called strongly bounded measures in [1]; for quasi-complete E, a finitely additive μ is exhaustive iff $\mu(\mathcal{F})$ is relatively weakly compact in$E$- for Banach spaces, it is proved in [1] and it easily extends to quasi-complete locally convex spaces.

2. ALEKSANDROV'S THEOREM

In this section, we extend the celebrated Aleksandrov representation theorem, for Hausdorff normal topological spaces, to the vector-valued measures. In scalar case, in a simple form, this theorem says:

Suppose X is a Hausdorff normal topological space, $\mathcal{F}$ the algebra generated by the closed subset of X and $\mu\colon C_b(X) \to K$ a continuous linear mapping. Then there exist a unique, finitely additive measure $\nu\colon \mathcal{F} \to K$ such that:

(i) ν is inner regular by closed sets and outer regular by open sets;

(ii) $\int f d\nu = \mu(f)$, $\forall f \in C_b(X)$. ([3], p.262; [13]). Note $C_b(X)$ is contained in the uniform closure of $\mathcal{F}$-simple functions on X in the space of all bounded functions on X and so each $f \in C_b(X)$ is ν-integrable.

We state and prove the following extension. Our proof is obtained by the regularity properties of the corresponding regular Borel measure on $\tilde{X}$ and is very different from the given in [3] and other known proofs.

We start with a crucial lemma for normal topological spaces:

Lemma 2.1. *Suppose A and B are two closed sets in a Hausdorff normal space X. Then $\overline{A \cap B} = \overline{A} \cap \overline{B}$ (for a subset $C \subset X$, $\overline{C}$ denotes the closure of C in $\tilde{X}$).*

Proof. By normality, the result is true if $A \cap B = \varnothing$. So we assume $A \cap B \neq \varnothing$. Assume that $a \in \overline{A} \cap \overline{B} \setminus \overline{A \cap B}$. So there is a closed neighborhood Q of a, in $\tilde{X}$, such that $Q \cap (A \cap B) = \varnothing$. Putting $P_1 = A \cap Q$ and $P_2 = B \cap Q$, we get $P_1 \cap P_2 = \varnothing$. Since X is normal, $\overline{P_1} \cap \overline{P_2} = \varnothing$. But $a \in \overline{P_1}$ and also $a \in \overline{P_2}$; this is a contradiction. □

Now we come to the main theorem

Theorem 2.1. *Suppose X is a Hausdorff normal topological space and a weakly compact linear mapping $\mu\colon C_b(X) \to E$. Then there exist a unique finitely additive, exhaustive measure $\nu\colon \mathcal{F} \to E$ such that:*

(i) *ν is inner regular by closed sets and outer regular by open sets;*

(ii) $\int f d\nu = \mu(f)$, $\forall f \in C_b(X)$.

Proof. Considering $\tilde{\mu}\colon C(\tilde{X}) \to E$, we shall get an E-valued regular Borel measure $\tilde{\mu}\colon \mathcal{B}(\tilde{X}) \to E$ ([7]) (we denote the measure $\tilde{\mu}$ by the same notation as operator $\tilde{\mu}$ because the operator $\tilde{\mu}$ uniquely extends to all $\tilde{\mu}$-integrable functions). If A is a subset of X or $\tilde{X}$, $\overline{A}$ will denote the closure of A in $\tilde{X}$. Fix $p \in P$.

We prove this theorem in several steps.

I. Let $\overline{\mathcal{C}} = \{\overline{A} : A \text{ a closed set in } X\}$. Then for every $Q \in \overline{\mathcal{C}}$ and $c > 0$, there exists $W \in \overline{\mathcal{C}}$, such that $W \subset \tilde{X} \setminus Q$ and $\bar{\tilde{\mu}}_p(\tilde{X} \setminus Q \setminus W) < c$.

Proof. By regularity of $\bar{\tilde{\mu}}_p$, there is a positive set $V \subset \tilde{X} \setminus Q$ having the property that $\bar{\tilde{\mu}}_p(\tilde{X} \setminus Q \setminus V) < \frac{c}{2}$ (a positive set in the complement of a zero-set). Take a $g \in C(\tilde{X})$, $0 \leq g \leq 1$, such that $V = g^{-1}(0,1]$. Denote by

$$V_n = \left\{x \in \tilde{X} \colon g(x) > \frac{1}{n}\right\}$$

and

$$Z_n = \left\{x \in \tilde{X} \colon g(x) \geq \frac{1}{n}\right\}.$$

Now,

$$Z_{n+1} \supset \overline{(Z_{n+1} \cap X)} \supset \overline{(V_n \cap X)} \supset V_n$$

(note X is dense in $\tilde{X}$). By choosing n sufficiently large, without loss generality, we can assume that $\bar{\tilde{\mu}}_p(V \setminus V_n) < \frac{c}{2}$. Taking $W = \overline{(Z_{n+1} \cap X)}$, we get the result. $\square$

II. Let $\mathcal{A}$ be the algebra, in $\tilde{X}$, generated by $\overline{\mathcal{C}}$ and denote, by $\mathcal{A}_0$, the elements of $\mathcal{A}$ which have the property that these elements and their complements are inner regular by the elements of $\overline{\mathcal{C}}$. Then $\mathcal{A}_0 = \mathcal{A}$.

Proof. We use Step I and Lemma 2.1 to prove it. By Step I, $\mathcal{A}_0 \supset \overline{\mathcal{C}}$. By definition, $\mathcal{A}_0$ is closed under complements. Also, using Lemma 2.1, it is a routine verification that if A and B are in $\mathcal{A}_0$ then $A \cap B$ and $A \cup B$ are also in $\mathcal{A}_0$. This proves the result. $\square$

III. It is a simple verification that $\mathcal{A} \cap X \supset \mathcal{F}$. Also if $A \in \mathcal{A}$ and $A \cap X = \varnothing$, then $\bar{\tilde{\mu}}_p(A) = 0$. To prove this, take any $\overline{C} \in \overline{\mathcal{C}}$, C being a closed set in X, such that $\overline{C} \subset A$. This means C is empty and so $\bar{\tilde{\mu}}_p(A) = 0$.

Now we can define a $\nu\colon \mathcal{F} \to E$, $\nu(B) = \tilde{\mu}(A)$, A being any element in $\mathcal{A}$ with $B = A \cap X$; it is a trivial verification that ν is well-defined, is finitely additive and it is inner regular by closed sets in X and outer regular by open sets in X (this means for a $p \in P$, $F \in \mathcal{F}$ and $c > 0$, there is, in X, a closed set C and an open set V, $C \subset F$ and $V \supset F$, such that for any $B_1 \in \mathcal{F}, B_1 \subset F \setminus C$ and any $B_2 \in \mathcal{F}, B_2 \subset V \setminus F$, we have $\|\nu(B_i)\|_p < c$ for $i = 1, 2$). We also have $\nu(C) = \tilde{\mu}(\overline{C})$, for any closed $C \subset X$. Since $\nu(\mathcal{F})$ is relatively weakly compact in E, ν is exhaustive ($\equiv$ strongly additive) ([1], Corollary 3, p. 28; this is proved for Banach space E but easily extends to quasi-complete locally convex space E). Also for any $B \in \mathcal{F}$, $\bar{\nu}_p(B) \leq \bar{\tilde{\mu}}_p(A)$, where A is any element in $\mathcal{A}$ such that $B = A \cap X$.

IV. For any $f \in C_b(X)$, $\mu(f) = \int f d\nu$.

Proof. Assume $\bar{\tilde{\mu}}_p(X) \leq 1$. Fix $c > 0$ and take an $f \in C_b(X)$, $0 \leq f \leq 1$. Then there is a non-negative, $\mathcal{F}$-simple function $\sum_{i=1}^{n} a_i \chi_{B_i}$ such that B_i's are mutually disjoint, their union is X and $\left| f - \sum_{i=1}^{n} a_i \chi_{B_i} \right| < c$ on X.

Take mutually disjoint $\{A_i\} \subset \mathcal{A}$ such that $B_i = A_i \cap X$ for every i. Also consider mutually disjoint closed sets $\{C_i\} \subset X$, such that $\bar{\tilde{\mu}}_p(A_i \setminus \overline{C_i}) < \frac{c}{n}$, for each i. Now

$$\begin{aligned}\left\| \int f d\nu - \sum a_i \nu(C_i) \right\|_p &\leq \left\| \int f d\nu - \sum a_i \nu(B_i) \right\|_p + \left\| \sum a_i \nu(B_i \setminus C_i) \right\|_p \\ &\leq c + \left\| \sum a_i \tilde{\mu}(A_i \setminus \overline{C_i}) \right\|_p \leq c + n \cdot \frac{c}{n} = 2c.\end{aligned}$$

Also $\left| f - \sum_{i=1}^{n} a_i \chi_{B_i} \right| \leq c$ implies that

$$\left| \tilde{f} - \sum_{i=1}^{n} a_i \chi_{\overline{(C_i)}} \right| \leq c,$$

on $\cup(\overline{C_i})$ (note $\overline{C_i}$ are also mutually disjoint by Lemma 2.1).

Therefore,

$$\begin{aligned}\left\| \int \tilde{f} d\tilde{\mu} - \sum a_i \nu(C_i) \right\|_p &= \left\| \int \tilde{f} d\tilde{\mu} - \sum a_i \tilde{\mu}(\overline{C_i}) \right\|_p \\ &\leq c + \left\| \sum a_i \tilde{\mu}(A_i \setminus \overline{C_i}) \right\|_p \\ &\leq c + n \cdot \frac{c}{n} = 2c.\end{aligned}$$

This proves that $\mu(f) = \nu(f)$. □

V. Uniqueness.

Proof. Let $\nu \colon \mathcal{F} \to E$ be a finitely additive regular (inner regular by closed sets in X and outer regular by open sets in X) measure, having relatively weakly compact range, such that $\int f d\nu = 0$, $\forall f \in C_b(X)$. This means ν is exhaustive and so $\bar{\nu}_p(X) < \infty$, $\forall p \in P$. If $\nu \neq 0$, there is a $p \in P$, a closed set $Z \subset X$, and a $c > 0$ such that $\|\nu(Z)\|_p = 2c$. Take a a open set $U \supset Z$ such that $\bar{\nu}_p(U \setminus Z) < c$. Take an $f \in C_b(X)$, $0 \leq f \leq 1$, $f(Z) = 1$, $f(X \setminus U) = 0$. We get

$$0 = \int f d\nu = \int_Z f d\nu + \int_{U \setminus Z} f d\nu.$$

This means $\nu(Z) = -\int_{U\setminus Z} f d\nu$ and so $2c \leq 1$, $\bar{\nu}_p(U \setminus Z) < c$. This contradiction proves the uniqueness. □

These steps prove the result. □

Now we make the additional assumption that X is also countably paracompact. In this case we prove that any Baire measure can be uniquely extended to regular Borel measure.

Theorem 2.2. *Suppose X is a Hausdorff normal and countably paracompact topological space and $\mu \colon \mathcal{B}_0(X) \to E$ a Baire measure. Then it has a unique extension to a countably additive Borel measure $\mu \colon \mathcal{B}(X) \to E$ which is inner regular by closed sets and outer regular by open sets.*

Proof. By ([7], Therem 2), we get a unique linear, continuous, and weakly compact mapping $\mu\colon C_b(X) \to E$ which is continuous in the strict topology β_σ (which is weaker than norm topology) on $C_b(X)$. We fix a $p \in P$ and use the notations of Theorem 2.1. As in Theorem 2.1, we get an E-valued regular Borel measure $\tilde{\mu}\colon \mathcal{B}(\tilde{X}) \to E$.

We do the proof in several steps:

I. Let $\{C_n\}$ be a decreasing sequence of closed subsets of X such that $\cap C_n = C$. Then $\bar{\tilde{\mu}}_p(\cap(\overline{C_n}) \setminus \overline{C}) = 0$.

Proof. We first consider the case when $C = \varnothing$. In this case $\{V_n = C_n' : 1 \leq n < \infty\}$ is an opening covering X. Take an open neighborhood finite refinement covering and a partition of unity $\{g_\alpha : \alpha \in I\}$ subordinated to this refinement. Let $f_n = \sum_{\alpha \in I_n} g_\alpha$ where $I_n = \{\alpha :$ support of $g_\alpha \subset V_n\}$. We get $f_n \uparrow 1$ and so $Z_n = f_n^{-1}(0) \supset C_n$ and $\cap Z_n = \varnothing$. Note Z_n are zero-sets. Since μ is a Baire measure, we have $0 = \bar{\tilde{\mu}}_p(\cap(\overline{Z_n}))$. Since $\cap(\overline{C_n}) \subset \cap(\overline{Z_n})$, we get $\bar{\tilde{\mu}}_p(\cap(\overline{C_n})) = 0$. Now we consider the case when $C \neq \varnothing$. Fix a $c > 0$. By Theorem 2.1 (I), there is a closed set $Q \subset X \setminus C$ such that $\bar{\tilde{\mu}}_p(\tilde{X} \setminus \overline{C} \setminus \overline{Q}) < c$. Since $\cap(C_n \cap Q) = \varnothing$, by using which is just proved above and Lemma 2.1, we get $\bar{\tilde{\mu}}_p(\cap(\overline{C_n} \cap \overline{Q})) = 0$.

Now

$$\begin{aligned}\bar{\tilde{\mu}}_p(\overline{C_n} \setminus \overline{C}) &= \bar{\tilde{\mu}}_p((\overline{C_n} \cap \overline{Q}) \cup (\overline{C_n} \setminus \overline{Q} \setminus \overline{C})) \\ &\leq \bar{\tilde{\mu}}_p(\overline{C_n} \cap \overline{Q}) + \bar{\tilde{\mu}}_p(\tilde{X} \setminus \overline{Q} \setminus \overline{C}).\end{aligned}$$

Taking limits, we get $\bar{\tilde{\mu}}_p(\cap(\overline{C_n}) \setminus \overline{C}) < c$. This proves the result. □

II. In the notations of Theorem 2.1(I, II), let, this time, $\mathcal{A}$ be the σ-algebra, in $\tilde{X}$, generated by $\overline{\mathcal{C}}$ and denote, by $\mathcal{A}_0$, the elements of $\mathcal{A}$ which have the property that these elements and their complements are inner regular by the elements of $\overline{\mathcal{C}}$. Then $\mathcal{A}_0 = \mathcal{A}$.

Proof. As done in Theorem 2.1(II), $\mathcal{A}_0 \supset \overline{\mathcal{C}}$ and it is an algebra. Take a sequence $\{A_n\} \subset \mathcal{A}_0$. Fix a $c > 0$ and take a sequence $\{\overline{C_n}\} \subset \overline{\mathcal{C}}$ such that $\bar{\tilde{\mu}}_p(A_n \setminus \overline{C_n}) < \frac{c}{4^n}$ $\forall n$. From this it easily follows that $\bar{\tilde{\mu}}_p(\cup A_n \setminus (\bigcup_{1 \leq i \leq m} \overline{C_i})) < c$ for some m and $\bar{\tilde{\mu}}_p(\cap A_n \setminus \cap \overline{C_n})) < c$. Putting $C = \bigcup_{1 \leq i \leq m} C_i$, we get $\bar{\tilde{\mu}}_p(\cup A_n \setminus \overline{C}) < c$. Let $D_n = \bigcap_{1 \leq i \leq n} C_i$; then $D_n \downarrow$ and $\cap C_n = \cap D_n$ and so $\cap \overline{C_n} = \cap \overline{D_n}$. Put $D = \cap D_n$. By I, $\bar{\tilde{\mu}}_p(\cap(\overline{D_n}) \setminus \overline{D}) = 0$ and we also have $\bar{\tilde{\mu}}_p(\cap A_n \setminus \cap \overline{C_n})) < c$. This gives

$$\begin{aligned}\bar{\tilde{\mu}}_p(\cap A_n \setminus \overline{D}) &= \bar{\tilde{\mu}}_p((\cap A_n \setminus \cap \overline{D_n}) \cup (\cap \overline{D_n} \setminus \overline{D})) = \bar{\tilde{\mu}}_p((\cap A_n \setminus \cap \overline{C_n}) \cup (\cap \overline{D_n} \setminus \overline{D})) \\ &\leq \bar{\tilde{\mu}}_p(\cap A_n \setminus \cap \overline{C_n}) + \bar{\tilde{\mu}}_p(\cap \overline{D_n} \setminus \overline{D}) \leq c,\end{aligned}$$

(using again I).

This proves $\mathcal{A} = \mathcal{A}_0$. This proves the result. □

III. $\mathcal{A} \cap X \supset \mathcal{B}(X)$

Proof. From $\mathcal{A} \supset \overline{\mathcal{C}}$, it follow that $\mathcal{A} \cap X$ contains all closed subset of X. Using this with the easily verified fact that $\mathcal{A} \cap X$ is a σ-algebra, the result follows. □

IV. For any $A \in \mathcal{A}$ with $A \cap X = \varnothing$, $\bar{\tilde{\mu}}_p(A) = 0$

Proof. By II, A is inner regular by the elements of $\overline{\mathcal{C}}$). Since $A \cap X = \varnothing$, the result follows. □

Now we can define a $\mu\colon \mathcal{B}(X) \to E$. Take any $B \in \mathcal{B}(X)$ and select any $\tilde{B} \in \mathcal{A}$ such that $\tilde{B} \cap X = B$; define $\mu(B) = \tilde{\mu}(\tilde{B})$; it is a trivial verification that μ is well-defined, is countably additive and it is inner regular by closed sets in X and outer regular by open sets in X. □

Remark 2.1. For scalar valued measure, this result is prove in [10].

3. REPRESENTATION THEOREM FOR $C(X)$, WITH X COMPLETELY REGULAR

In this section we assume that $K = \mathbb{R}$. A subset $B \subset C(X)$ will be called order-bounded if there are elements f and g in $C(X)$ such that $f \leq b \leq g$, $\forall b \in B$. It is well-known that a linear map $\mu\colon C(X) \to \mathbb{R}$, which maps order-bounded sets into bounded sets, gives a unique $\nu \in M_\sigma(X)$ such that $C(X) \subset L^1(\nu)$ and $\mu(f) = \int f d\nu$ ([4]; [14], Theorem 23).

We will extend to the vector case.

Theorem 3.1. *Suppose X is a Hausdorff normal and countably paracompact topological space and $\mu\colon C(X) \to E$ be a linear map such that order-bounded subsets are mapped into relatively weakly compact subsets of E. Then:*

(i) *there is a unique Borel measure $\nu : \mathcal{B}(X) \to E$, which is inner regular by closed subsets of X and outer regular by open subsets of X, such that $C(X) \subset L^1(\nu)$ and $\mu(f) = \displaystyle\int f d\nu$;*

(ii) *for every $p \in P$, there is compact $C \subset \upsilon X$ (the real-compactification of X), depending on p, such that $\bar{\tilde{\nu}}_p(\tilde{X} \setminus C) = 0$ ([4]).*

Proof. By [7], Theorem 7, p. 695, there is $\nu : \mathcal{B}_0(X) \to E$, a unique Baire measure, satisfying all the conditions of the above theorem, except Borel extension. But the regular Borel extension follows form Theorem 2.2. □

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