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ON θ-β-GENERALIZED CLOSED SETS AND θ-β-GENERALIZED CONTINUITY IN TOPOLOGICAL SPACES

MIGUEL CALDAS

Abstract. Levine [13] introduced and investigated the notion of generalized closed set. It is the object of the present paper to offer a new notion of generalized closed set called θ-β-generalized closed set and study its fundamental properties.

1. INTRODUCTION

In 1970, N. Levine [13] introduced the notion of generalized closed sets. By definition, a subset A of a topological space (X, τ) is called generalized closed set (briefly g-closed) if $\mathrm{Cl}(A) \subset U$ whenever $A \subset U$ and U is open. Noiri [15] and Dontchev and Maki [7] have given independently another new generalization of Levine's g-closed set by utilizing the θ-closure operator. Recall that a subset A of a topological space (X, τ) is θ-generalized closed (briefly θ-g-closed) if $\mathrm{Cl}_\theta(A) \subset U$, whenever $A \subset U$ and U is open in (X, τ). The concept of θ-g-closed sets was applied to the digital line [8]. Note that, in [15], Noiri used the term generalized θ-closed sets (briefly gθ-closed sets). We introduce a new form of generalized closed set called θ-β generalized closed set by utilizing β-θ-closure operator. We define the notion of θ-generalized Λ_β-sets and characterize their properties. We also introduce the notion θ-βg-continuity and θ-βg-irresoluteness by using θ-βg-closed sets and study some of their fundamental properties.

2. PRELIMINARIES

Throughout the present paper, spaces (X, τ) and (Y, σ) (or simply, X and Y) denote topological spaces on which no separation axioms are assumed unless explicitly stated. Abd El Monsef et al. [1] and Andrijević [2] introduced the notion of β-open set, which Andrijević called semipreopen, completely independent of each other. In this paper, we adopt the word β-open for the sake of clarity. They characterized the most important properties of β-open sets. A subset A of a topological space (X, τ) is called β-open if $A \subseteq \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$, where $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$ denote the closure and

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the interior of A, respectively. The complement of a β-open set is called β-closed. The family of all β-open sets of a space X is denoted by $\beta O(X,\tau)$ or $\beta O(X)$. We set $\beta O(X,x) = \{U : x \in U \in \beta O(X)\}$. The intersection of all β-closed sets containing A is called the β-closure of A and is denoted by $\beta\mathrm{Cl}(A)$. The β-interior ($\beta\mathrm{Int}(A)$) of a subset $A \subset X$ is the union of all β-open sets contained in A.

Lemma 2.1 ([1], [2])**.** *The following hold for a subset A of a topological space X:*
(1) $\beta\mathrm{Int}(A) = A \cap \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$*;*
(2) $\beta\mathrm{Cl}(A) = A \cup \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$*;*
(3) $x \in \beta\mathrm{Cl}(A)$ *if and only if* $O \cap A \neq \emptyset$ *for every* $O \in \beta O(X,x)$*;*
(4) $\beta\mathrm{Cl}(X \setminus A) = X \setminus \beta\mathrm{Int}(A)$*;*
(5) *A is β-closed if and only if* $A = \beta\mathrm{Cl}(A)$*.*

Now we begin to recall some known notions which will be used in the sequel.

Definition 2.1. [17] Let A be a subset of X. The *β-θ-closure* of A, $\beta\mathrm{Cl}_\theta(A)$, is the set of all $x \in X$ such that $\beta\mathrm{Cl}(O) \cap A \neq \emptyset$ for every $O \in \beta O(X,x)$. A is called *β-θ-closed* if $A = \beta\mathrm{Cl}_\theta(A)$. The set $\{x \in X \mid \beta\mathrm{Cl}_\theta(O) \subset A$ for some $O \in \beta O(X,x)\}$ is called the *β θ-interior* of A and is denoted by $\beta\mathrm{Int}_\theta(A)$. A subset A is called *β-θ-open* if $A = \beta\mathrm{Int}_\theta(A)$.

The following theorem is known and given by Noiri [17].

Theorem 2.1. *For any subset A of X:*
(1) $\beta\mathrm{Cl}_\theta(\beta\mathrm{Cl}_\theta(A)) = \beta\mathrm{Cl}_\theta(A)$*;*
(2) *$\beta\mathrm{Cl}_\theta(A)$ is β-θ-closed;*
(3) *Intersection of arbitrary collection of β-θ-closed set in X is β-θ-closed;*
(4) *$\beta\mathrm{Cl}_\theta(A)$ is the intersection of all β-θ-closed sets each containing A;*
(5) *If $A \in \beta O(X)$, then $\beta\mathrm{Cl}(A) = \beta\mathrm{Cl}_\theta(A)$.*

For other advances on topological spaces obtained by our research group we recommend [4, 5, 11].

3. θ-β-GENERALIZED CLOSED SETS

Definition 3.1. A subset A of a topological space (X,τ) is called *θ-β-generalized closed* (briefly θ-βg-closed) if $\beta\mathrm{Cl}_\theta(A) \subset O$ whenever $A \subset O$ and $O \in \beta O(X,\tau)$.

The complement of a θ-β-generalized closed set is called θ-β-generalized open (briefly θ-βg-open).

Lemma 3.1. *The following statemets hold true (in each case, the converse implications are not true):*
(i) *Every β-θ-closed set is θ-βg-closed.*
(ii) [17] *Every β-θ-closed set is β-closed.*

Proof. (i) Let $A \subset X$ be β-θ-closed set. Then $A = \beta\mathrm{Cl}_\theta(A)$. Let $A \subset O$ and $O \in \beta O(X,\tau)$. It follows that $\beta\mathrm{Cl}_\theta(A) \subset O$. This means that A is θ-βg-closed.
(ii) See Remark 3.2 in [17]. □

Example 3.1. Let $X = \{a, b, c, d\}$ and let $\tau = \{\emptyset, \{c, d\}, X\}$. If $A = \{a, b, d\}$, then $\beta\mathrm{Cl}(A) = X$ and so A is not β-closed. Hence A is not β-θ-closed. Since X is the only β-open set containing A, A is θ-βg-closed.

Example 3.2. Let $X = \{a, b, c\}$ and let $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, X\}$. If consider $A = \{c\}$, then A is a β-closed set which is not β-θ-closed.

A subset A of a topological space (X, τ) is called a β-*generalized closed* set (briefly βg-closed) if $\beta\mathrm{Cl}(A) \subset U$ whenever $A \subset U$ and $U \in \beta O(X, \tau)$. Note that this notion is particular case of generalized $(m1, m2)$-closed set, introduced by Noiri [16]. A subset B is said to be β-generalized open (briefly βg-open) in (X, τ) if its complement $B^c = X \setminus B$ is βg-closed in (X, τ).

Lemma 3.2. *The following statements hold true (in each case, the converse implications are not true):*

(i) *Every θ-βg-closed set is βg-closed.*

(ii) *Every β-closed set is βg-closed.*

Proof. (i) It is true that $\beta\mathrm{Cl}(A) \subset \beta\mathrm{Cl}_\theta(A)$ for every subset A of (X, τ).

(ii) Let $A \subset X$, be β-closed. Then $A = \beta\mathrm{Cl}(A)$. Let $A \subset O$ and O be β-open. Then $\beta\mathrm{Cl}(A) \subset O$. Hence A is βg-closed. □

Example 3.3. Let $X = \{a, b, c\}$ and let $\tau = \{\emptyset, \{a\}, \{a, b\}, X\}$. Set $A = \{c\}$. A is β-closed since it is closed and therefore βg-closed. Set $O = \{a, c\}$. But $\beta\mathrm{Cl}_\theta(A)$ is not a subset of O, where $O \in \beta O(X, \tau)$. Hence A is not θ-βg-closed.

Remark 3.1. Note that β-closedness is independent of θ-βg-closedness as it is showed in the Example 3.1 and Example 3.3.

Lemma 3.3. *Let A be a β-open subset of a topological space (X, τ). The set A is θ-βg-closed if and only if A is βg-closed.*

Proof. It follows from the fact that $\beta\mathrm{Cl}(A) = \beta Cl_\theta(A)$ for every $A \in \beta O(X, \tau)$ (Theorem 2.1, [5]). □

Recall that a subset A of a topological space (X, τ) is called a θ-*generalized closed* set (briefly θ-g-closed set) [7] if $\mathrm{Cl}_\theta(A) \subset U$ whenever $A \subset U$ and U is open in (X, τ).

It should be noted that θ-g-closedness is independent of θ-βg-closedness as it is shown in the following two examples.

Example 3.4. θ-βg-closedness does not imply θ-g-closedness in general. To show this, let us consider the set $X = \{a, b, c\}$ and $\tau = \{X, \emptyset, \{b\}, \{c\}, \{b, c\}\}$. We have $\beta O(X, \tau) = \{X, \emptyset, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$. The β-θ-closed sets of (X, τ) are $\{X, \emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}\}$. The set $\{c\}$ is θ-βg-closed since it is β- θ-closed but it is not θ-g-closed.

Example 3.5. θ-g-closedness does not imply θ-βg-closedness in general. Consider $X = \{a, b, c, d, e\}$ and $\tau = \{X, \emptyset, \{c\}, \{a, b\}, \{a, b, c\}\}$. Let $A = \{a, b, c, e\}$ be a subset of X. Then A is a θ-g-closed set since X is its only open superset. But A is not θ-βg-closed since $\beta\mathrm{Cl}_\theta(A)$ is not a subset of $O = \{a, b, c, e\}$, where $O \in \beta O(X, \tau)$.

Lemma 3.4. *Let A be a βg-closed subset of (X, τ). Then $\beta\mathrm{Cl}(A) \setminus A$ does not contain a nonempty β-closed set.*

Proof. Let F be a β-closed set such that $F \subset \beta\mathrm{Cl}(A) \setminus A$. Clearly $A \subset F^c$, where A is βg-closed and F^c is β-open . Thus $\beta\mathrm{Cl}(A) \subset F^c$, or equivalently $F \subset (\beta\mathrm{Cl}(A))^c$. Since by assumption $F \subset \beta\mathrm{Cl}(A)$, then $F \subset (\beta\mathrm{Cl}(A))^c \cap \beta\mathrm{Cl}(A) = \emptyset$. This shows that F coincides with the void-set. □

Recall (Definition 3.1) that the complement of a θ-β-generalized closed set is called θ-β-generalized open (briefly θ-βg-open).

Theorem 3.1. *A set $A \subset (X, \tau)$ is θ-βg-open if and only if $F \subset \beta\mathrm{Int}_\theta(A)$ whenever F is β-closed in X and $F \subset A$.*

Proof. Necessity. Let A be θ-βg-open and $F \subset A$, where F is β-closed. It is obvious that A^c is contained in F^c. This implies that $\beta\mathrm{Cl}_\theta(A^c) \subset F^c$. Hence $\beta\mathrm{Cl}_\theta(A^c) = (\beta\mathrm{Int}_\theta(A))^c \subset F^c$, i.e. $F \subset \beta\mathrm{Int}_\theta(A)$.

Sufficiency. If F is a β-closed set with $F \subset \beta\mathrm{Int}_\theta(A)$ whenever $F \subset A$, then it follows that $A^c \subset F^c$ and $(\beta\mathrm{Int}_\theta(A))^c \subset F^c$, that is $\beta\mathrm{Cl}_\theta(A^c) \subset F^c$. Therefore A^c is θ-βg-closed and A is θ-βg-open. □

Recall that a topological space (X, τ) is β-$T_{\frac{1}{2}}$ [3] if and only if every βg-closed set is β-closed.

Next we have some new characterizations of β-$T_{1/2}$ spaces.

Theorem 3.2. *For a topological space (X, τ) the following conditions are equivalent:*

(i) *X is β-$T_{1/2}$.*

(ii) *For each $x \in X$, $\{x\}$ is β-closed or β-open.*

Proof. (i)⇒ (ii): Suppose that for some $x \in X$, $\{x\}$ is not β-closed. Then $\{x\}^c$ is not β-open. Since X is the only β-open containing $\{x\}^c$, the set $\{x\}^c$ is βg-closed and so it is β-closed in the β-$T_{1/2}$ space (X, τ). Therefore $\{x\}^c$ is β-closed or equivalently $\{x\}$ is β-open.

(ii)⇒ (i): Let $A \subset X$ be βg-closed. Let $x \in \beta\mathrm{Cl}(\mathrm{A})$. We will show that $x \in A$. By the hypothesis, the singleton $\{x\}$ is either β-closed or β-open. We consider these two cases.

Case 1. $\{x\}$ is β-closed: Then if $x \notin A$, there exists a β-closed set in $\beta\mathrm{Cl}(A) \setminus A$. By Lemma 3.4 $x \in A$.

Case 2. $\{x\}$ is β-open: Since $x \in \beta\mathrm{Cl}(A)$, then $\{x\} \bigcap A \neq \emptyset$. Thus $x \in A$.

Hence in both cases, we have $x \in A$, that is $\beta\mathrm{Cl}(A) \subset A$ or equivalently A is β-closed since the inclusion $A \subset \beta\mathrm{Cl}(A)$ is trivial. □

Remark 3.2. $\beta O(X, \tau) \subset \beta GO(X, \tau)$, where $\beta GO(X, \tau)$ is the family of all βg-open sets of (X, τ).

Let A be β-open. Then A^c is β-closed and βg-closed (Lemma 3.2). This implies A is βg-open. Hence $\beta O(X,\tau) \subset \beta GO(X,\tau)$.

Theorem 3.3. *A topological space (X,τ) is β-$T_{1/2}$ if and only if holds the equality $\beta O(X,\tau) = \beta GO(X,\tau)$.*

Proof. Necessity. Let (X,τ) be β-$T_{1/2}$ topological space. Let $A \in \beta GO(X,\tau)$. Then A^c is βg-closed. By hypothesis, A^c is β-closed and thus $A \in \beta O(X,\tau)$. Hence $\beta O(X,\tau) = \beta GO(X,\tau)$.

Sufficiency. Let $\beta O(X,\tau) = \beta GO(X,\tau)$. Let A be βg-closed. Then A^c is βg-open. Hence $A^c \in \beta O(X,\tau)$. Thus A is β-closed. Therefore (X,τ) is β-$T_{1/2}$. □

Theorem 3.4. *(X,τ) is a β-$T_{\frac{1}{2}}$ topological space if and only if every θ-βg-closed set is β-closed.*

Proof. Necessity. Let $A \subset X$ be a θ-βg-closed. By Lemma 3.2, A is βg-closed. Since X is a β-$T_{\frac{1}{2}}$ space, A is β-closed.

Sufficiency. Let $x \in X$. If $\{x\}$ is not β-closed, then $\{x\}^c$ is not β-open and thus the only superset of $\{x\}^c$ is X. Trivially $\{x\}^c$ is θ-βg-closed. By hypothesis $\{x\}^c$ is β-closed or equivalently $\{x\}$ is β-open. Hence X is a β-$T_{\frac{1}{2}}$ space. □

Lemma 3.5. *Let A be a θ-βg-closed subset of (X,τ). Then,*

(i) *$\beta\mathrm{Cl}_\theta(A) \setminus A$ does not contain a nonempty β-closed set.*

(ii) *$\beta\mathrm{Cl}_\theta(A) \setminus A$ is θ-βg-open.*

Proof. (i) Let F be a β-closed set such that $F \subset \beta\mathrm{Cl}_\theta(A) \setminus A$. Since F^c is β-open and $A \subset F^c$, it follows that $\beta\mathrm{Cl}_\theta(A) \subset F^c$, i.e. $F \subset (\beta\mathrm{Cl}_\theta(A))^c$. This implies that $F \subset (\beta\mathrm{Cl}_\theta(A))^c \cap \beta\mathrm{Cl}_\theta(A) = \emptyset$.

(ii) If A is θ-βg-closed and F is a β-closed set such that $F \subset \beta\mathrm{Cl}_\theta(A) \setminus A$, then by (i), F is empty and therefore $F \subset \beta\mathrm{Int}_\theta(\beta\mathrm{Cl}_\theta(A) \setminus A)$. By Theorem 3.1, $\beta\mathrm{Cl}_\theta(A) \setminus A$ is θ-βg-open. □

Lemma 3.6. *If the set A is θ-βg-closed set of a topological space (X,τ) such that $A \subset B \subset \beta\mathrm{Cl}_\theta(A)$, then B is also a θ-βg-closed set of (X,τ).*

Proof. Let O be a β-open set of the topological space (X,τ) such that $B \subset O$. Then $A \subset O$. Since A is θ-βg-closed, we obtain $\beta\mathrm{Cl}_\theta(A) \subset O$. By using Theorem 2.1 $\beta\mathrm{Cl}_\theta(B) \subset \beta\mathrm{Cl}_\theta(\beta\mathrm{Cl}_\theta(A)) = \beta\mathrm{Cl}_\theta(A) \subset O$. Therefore B is also a θ-βg-closed set of (X,τ). □

Recall that in a topological space (X,τ) a subset A of a topological space (X,τ) is called a *Λ_β-set* if $A = A^{\Lambda_\beta}$, where $A^{\Lambda_\beta} = \cap\{O \mid A \subset O, O \in \beta O(X,\tau)\}$ [6].

We now define a generalized Λ_β-set and θ-generalized Λ_β-set as follows:

Definition 3.2. (i) A subset A of a topological space (X,τ) is called a *generalized Λ_β-set* (briefly g.Λ_β-set) if $A^{\Lambda_\beta} \subset F$ whenever $A \subset F$ and F is a β-closed set of X.
(ii) A subset A of a topological space (X,τ) is called θ-generalized Λ_β-set (briefly

θ-$g.\Lambda_\beta$-set) if $A_\theta^{\Lambda_\beta} \subset F$, whenever $A \subset F$ and F is a β-closed set of (X,τ), where $A_\theta^{\Lambda_\beta} = \{x \in X \mid \beta\mathrm{Cl}_\theta(\{x\}) \cap A \neq \emptyset$

Proposition 3.1. *Let* $\beta\mathrm{Ker}_\theta(A) = \cap\{O \mid A \subset O, O$ *is* β- θ*-open*$\}$ *for any subset* $A \subset (X,\tau)$. *Then* $A_\theta^{\Lambda_\beta} = \beta\mathrm{Ker}_\theta(A)$.

Proof. Let $x \in \beta\mathrm{Ker}_\theta(A)$ and suppose that $x \notin A_\theta^{\Lambda_\beta}$. Then $\beta\mathrm{Cl}_\theta(\{x\}) \cap A = \emptyset$. Hence $x \notin X \setminus \beta\mathrm{Cl}_\theta(\{x\})$, where $X \setminus \beta\mathrm{Cl}_\theta(\{x\})$ is β-θ-open set containing A by Theorem 2.1. But this is impossible since $x \in \beta\mathrm{Ker}_\theta(A)$. Consequently $x \in A_\theta^{\Lambda_\beta}$. Now, if $x \in A_\theta^{\Lambda_\beta}$ and $x \notin \beta\mathrm{Ker}_\theta(A)$, then there exists a β-θ-open set O containing A such that $x \notin O$. Assume that $y \in \beta\mathrm{Cl}_\theta(\{x\}) \cap A$. Thus $y \in O$ and $x \notin O$. But this is a contradiction and we obtain the claim. ☐

Proposition 3.2. *The following statements hold true:*

(i) *For any set* $A \subset X$, $A \subset A^{\Lambda_\beta} \subset A_\theta^{\Lambda_\beta} \subset \beta\mathrm{Cl}_\theta(A)$.
(ii) *Every* β-θ*-closed set is a* Λ_β*-set.*
(iii) *Every* βg*-closed* Λ_β*-set is* β*-closed.*
(iv) *Every* θ-$g.\Lambda_\beta$*-set is a* $g.\Lambda_\beta$*-set.*

Proof. (i) Let A be a subset of (X,τ). It is clear that $A \subset A^{\Lambda_\beta}$. Now we prove that $A^{\Lambda_\beta} \subset A_\theta^{\Lambda_\beta}$. Suppose that x is not a point of $A_\theta^{\Lambda_\beta}$. It follows that $A \subset X \setminus \beta\mathrm{Cl}_\theta(\{x\})$ (= O, say). Since $\beta\mathrm{Cl}_\theta(\{x\})$ is β-θ-closed by Theorem 2.1, so O is β-θ-open. Hence by Lemma 3.1 O is β-open. Therefore, there exists a β-open set O containing A but not x, i.e. $x \notin A^{\Lambda_\beta}$. This shows that $A^{\Lambda_\beta} \subset A_\theta^{\Lambda_\beta}$.

Proving $A_\theta^{\Lambda_\beta} \subset \beta\mathrm{Cl}_\theta(A)$, let $x \in A_\theta^{\Lambda_\beta}$. Suppose that $x \notin \beta\mathrm{Cl}_\theta(A)$. Then, there exists a β-open set O containing x such that $\beta\mathrm{Cl}(O) \cap A = \emptyset$. Since O is a β-open subset of X, it follows that $\beta\mathrm{Cl}_\theta(O) = \beta\mathrm{Cl}(O)$. Hence $\beta\mathrm{Cl}_\theta(O) \cap A = \emptyset$. This implies that $A \subset X \setminus \beta\mathrm{Cl}_\theta(O)$. Therefore $x \notin X \setminus \beta\mathrm{Cl}_\theta(O)$, where $X \setminus \beta\mathrm{Cl}_\theta(O)$ is a β-θ-open set containing A. But this is impossible since $x \in A_\theta^{\Lambda_\beta}$ by Proposition 3.1. Consequently $x \in \beta\mathrm{Cl}_\theta(A)$.

(ii) Since $A^{\Lambda_\beta} \subset \beta\mathrm{Cl}_\theta(A)$ and $A = \beta\mathrm{Cl}_\theta(A)$, then $A^{\Lambda_\beta} = A$. Thus A is a Λ_β-set.

(iii) Let A be a Λ_β-set βg-closed subset of X. Suppose that $x \notin A_\theta^{\Lambda_\beta}$. So there exists a β-open subset O such that $A \subseteq O$ with $x \notin O$. This means that $x \notin \beta\mathrm{Cl}(A)$ since A is βg-closed. Then $\beta\mathrm{Cl}(A) \subset A^{\Lambda_\beta} = A$. This means that A is β-closed.

(iv) Let $A \subset F$, where F is β-closed. Then we have $A^{\Lambda_\beta} \subset A_\theta^{\Lambda_\beta} \subset F$. This implies that A is a $g.\Lambda_\beta$-set. ☐

Lemma 3.7. *Let* (X,τ) *be a topological space and* $x \in X$. *The the following two statements are equivalent:*

(1) $y \in \beta\mathrm{Ker}_\theta(\{x\})$;
(2) $x \in \beta\mathrm{Cl}_\theta(\{y\})$.

Proof. Let $y \notin \beta\mathrm{Ker}_\theta(\{x\})$. It follows that there exists a β-θ-open set U containing x such that $y \notin U$. This means that $x \notin \beta\mathrm{Cl}_\theta(\{y\})$. The converse can be proved by the same token. ☐

Remark 3.3. By Proposition 3.1, we get $\beta\mathrm{Ker}_\theta(A) = \{x \in X \mid \beta\mathrm{Cl}_\theta(\{x\}) \cap A \neq \emptyset\}$.

Lemma 3.8. *The following statements are equivalent for any points x and y in a topological space (X, τ):*

(1) $\beta\mathrm{Ker}_\theta(\{x\}) \neq \beta\mathrm{Ker}_\theta(\{y\})$;

(2) $\beta\mathrm{Cl}_\theta(\{x\}) \neq \beta\mathrm{Cl}_\theta(\{y\})$.

Proof. (1) $\Rightarrow$ (2): Let $\beta\mathrm{Ker}_\theta(\{x\}) \neq \beta\mathrm{Ker}_\theta(\{y\})$. Then there exists a point z in X such that $z \in \beta\mathrm{Ker}_\theta(\{x\})$ and $z \notin \beta\mathrm{Ker}_\theta(\{y\})$. By $z \in \beta\mathrm{Ker}_\theta(\{x\})$, it follows that $\{x\} \cap \beta\mathrm{Cl}_\theta(\{z\}) \neq \emptyset$. This implies $x \in \beta\mathrm{Cl}_\theta(\{z\})$. By $z \notin \beta\mathrm{Ker}_\theta(\{y\})$, we obtain $\{y\} \cap \beta\mathrm{Cl}_\theta(\{z\}) = \emptyset$. Since $x \in \beta\mathrm{Cl}_\theta(\{z\})$, $\beta\mathrm{Cl}_\theta(\{x\}) \subset \beta\mathrm{Cl}_\theta(\{z\})$ and $\{y\} \cap \beta\mathrm{Cl}_\theta(\{x\}) = \emptyset$. Hence it follows that $\beta\mathrm{Cl}_\theta(\{x\}) \neq \beta\mathrm{Cl}_\theta(\{y\})$. Now $\beta\mathrm{Ker}_\theta(\{x\}) \neq \beta\mathrm{Ker}_\theta(\{y\})$ implies that $\beta\mathrm{Cl}_\theta(\{x\}) \neq \beta\mathrm{Cl}_\theta(\{y\})$.

(2) $\Rightarrow$ (1): Let $\beta\mathrm{Cl}_\theta(\{x\}) \neq \beta\mathrm{Cl}_\theta(\{y\})$. Then there exists a point z in X such that $z \in \beta\mathrm{Cl}_\theta(\{x\})$ and $z \notin \beta\mathrm{Cl}_\theta(\{y\})$. This means that there exists a β-θ-open set containing z and therefore x but not y, such that $y \notin \beta\mathrm{Ker}_\theta(\{x\})$. Hence $\beta\mathrm{Ker}_\theta(\{x\}) \neq \beta\mathrm{Ker}_\theta(\{y\})$. ☐

Definition 3.3. A topological space (X, τ) is β-θ-R_0 space if every β-θ-open set contains the β-θ-closure of each of its singletons.

The space in Example 3.4 is a β-θ-R_0 space.

Theorem 3.5. *A topological space (X, τ) is a β-θ-R_0 space if and only for any x and y in X, $\beta\mathrm{Cl}_\theta(\{x\}) \neq \beta\mathrm{Cl}_\theta(\{y\}) \Rightarrow \beta\mathrm{Cl}_\theta(\{x\}) \cap \beta\mathrm{Cl}_\theta(\{y\}) = \emptyset$.*

Proof. Suppose that (X, τ) is β-θ-R_0 and $x, y \in X$ such that $\beta\mathrm{Cl}_\theta(\{x\}) \neq \beta\mathrm{Cl}_\theta(\{y\})$. Then, there exists $z \in \beta\mathrm{Cl}_\theta(\{x\})$ such that $z \notin \beta\mathrm{Cl}_\theta(\{y\})$ (or $z \in \beta\mathrm{Cl}_\theta(\{y\})$ such that $z \notin \beta\mathrm{Cl}_\theta(\{x\})$). There exists $V \in \beta O(X, \tau)$ such that $y \notin V$ and $z \in V$; hence $x \in V$. Therefore, we have $x \notin \beta\mathrm{Cl}_\theta(\{y\})$. Thus $x \in X \setminus \beta\mathrm{Cl}_\theta(\{y\})$, which implies $\beta\mathrm{Cl}_\theta(\{x\}) \subset X \setminus \beta\mathrm{Cl}_\theta(\{y\})$ and $\beta\mathrm{Cl}_\theta(\{x\}) \cap \beta\mathrm{Cl}_\theta(\{y\}) = \emptyset$. The proof for the other case is similar.

Sufficiency. Let V be β-θ-open and let $x \in V$. We will show that $\beta\mathrm{Cl}_\theta(\{x\}) \subset V$. Let $y \notin V$, such that $y \in X \setminus V$. Then $x \neq y$ and $x \notin \beta\mathrm{Cl}_\theta(\{y\})$. This shows that $\beta\mathrm{Cl}_\theta(\{x\}) \neq \beta\mathrm{Cl}_\theta(\{y\}$. By assumption $\beta\mathrm{Cl}_\theta(\{x\}) \cap \beta\mathrm{Cl}_\theta(\{y\}) = \emptyset$. Hence $y \notin \beta\mathrm{Cl}_\theta(\{x\})$. Therefore $\beta\mathrm{Cl}_\theta(\{x\}) \subset V$. ☐

Theorem 3.6. *A topological space (X, τ) is a β-θ-R_0 space if and only if for any points x and y in X, $\beta\mathrm{Ker}_\theta(\{x\}) \neq \beta\mathrm{Ker}_\theta(\{y\}) \Rightarrow \beta\mathrm{Ker}_\theta(\{x\}) \cap \beta\mathrm{Ker}_\theta(\{y\}) = \emptyset$.*

Proof. Suppose that (X, τ) is an β-θ-R_0 space. By Lemma 3.8, we find out that for any points x and y in X, if $\beta\mathrm{Ker}_\theta(\{x\}) \neq \beta\mathrm{Ker}_\theta(\{y\})$, then $\beta\mathrm{Cl}_\theta(\{x\}) \neq \beta\mathrm{Cl}_\theta(\{y\})$. Now we prove that $\beta\mathrm{Ker}_\theta(\{x\}) \cap \beta\mathrm{Ker}_\theta(\{y\}) = \emptyset$. In this respect, we assume that $z \in \beta\mathrm{Ker}_\theta(\{x\}) \cap \beta\mathrm{Ker}_\theta(\{y\})$. By $z \in \beta\mathrm{Ker}_\theta(\{x\})$ and Lemma 3.7, it follows that

$x \in \beta\mathrm{Cl}_\theta(\{z\})$. Since $x \in \beta\mathrm{Cl}_\theta(\{x\})$, by Theorem 3.5 $\beta\mathrm{Cl}_\theta(\{x\}) = \beta\mathrm{Cl}_\theta(\{z\})$. Similarly, we have $\beta\mathrm{Cl}_\theta(\{y\}) = \beta\mathrm{Cl}_\theta(\{z\}) = \beta\mathrm{Cl}_\theta(\{x\})$. This is a contradiction. Therefore, we have $\beta\mathrm{Ker}_\theta(\{x\}) \cap \beta\mathrm{Ker}_\theta(\{y\}) = \emptyset$.

To prove the converse, let (X,τ) be a topological space such that for any points x and y in X, $\beta\mathrm{Ker}_\theta(\{x\}) \neq \beta\mathrm{Ker}_\theta(\{y\})$ implies $\beta\mathrm{Ker}_\theta(\{x\}) \cap \beta\mathrm{Ker}_\theta(\{y\}) = \emptyset$. If $\beta\mathrm{Cl}_\theta(\{x\}) \neq \beta\mathrm{Cl}_\theta(\{y\})$, then by Lemma 3.8, $\beta\mathrm{Ker}_\theta(\{x\}) \neq \beta\mathrm{Ker}_\theta(\{y\})$. Hence $\beta\mathrm{Ker}_\theta(\{x\}) \cap \beta\mathrm{Ker}_\theta(\{y\}) = \emptyset$ which implies $\beta\mathrm{Cl}_\theta(\{x\}) \cap \beta\mathrm{Cl}_\theta(\{y\}) = \emptyset$. Now, because $z \in \beta\mathrm{Cl}_\theta(\{x\})$ implies $x \in \beta\mathrm{Ker}_\theta(\{z\})$, therefore $\beta\mathrm{Ker}_\theta(\{x\}) \cap \beta\mathrm{Ker}_\theta(\{z\}) \neq \emptyset$. By hypothesis, we have $\beta\mathrm{Ker}_\theta(\{x\}) = \beta\mathrm{Ker}_\theta(\{z\})$. Then $z \in \beta\mathrm{Cl}_\theta(\{x\}) \cap \beta\mathrm{Cl}_\theta(\{y\})$ implies $\beta\mathrm{Ker}_\theta(\{x\}) = \beta\mathrm{Ker}_\theta(\{z\}) = \beta\mathrm{Ker}_\theta(\{y\})$ and this is a contradiction. Hence, $\beta\mathrm{Cl}_\theta(\{x\}) \cap \beta\mathrm{Cl}_\theta(\{y\}) = \emptyset$ and by Theorem 3.25 (X,τ) is a β-θ-R_0 space. □

4. θ-β GENERALIZED CONTINUOUS FUNCTIONS

Definition 4.1. A function $f\colon (X,\tau) \to (Y,\sigma)$ is called

(i) *θ-β generalized continuous* (briefly θ-βg-continuous) if $f^{-1}(F)$ is θ-βg-closed in (X,τ) for every β-closed set F of (Y,σ).

(ii) *θ-β generalized irresolute* (briefly θ-βg-irresolute) if $f^{-1}(F)$ is θ-βg-closed in (X,τ) for every θ-βg-closed set F of (Y,σ).

Note that a function $f\colon (X,\tau) \to (Y,\sigma)$ is said to be *pre β-open* [14] (resp. *pre β-closed* [14]) if $f(U) \in \beta O(Y,\sigma)$ for every $U \in \beta O(X,\tau)$ (resp. if $f(U)$ is β-closed for every β-closed set U).

Theorem 4.1. *If $f\colon (X,\tau) \to (Y,\sigma)$ is a bijective pre β-open and θ-βg-continuous function, then f is θ-βg-irresolute.*

Proof. Let F be a θ-βg-closed set of (Y,σ). Assume that $f^{-1}(F) \subset O$, where $O \in \beta O(X,\tau)$. Clearly $F \subset f(O)$. Since $f(O) \in \beta O(Y,\sigma)$ and F is a θ-βg-closed set, we have $\beta\mathrm{Cl}_\theta(F) \subset f(O)$ and therefore $f^{-1}(\beta\mathrm{Cl}_\theta(F)) \subset O$. By the fact that f is θ-βg-continuous and $\beta\mathrm{Cl}_\theta(F)$ is β-closed (Lemma 3.1), it follows that $\beta\mathrm{Cl}_\theta(f^{-1}(\beta\mathrm{Cl}_\theta(F))) \subset O$ and thus $\beta\mathrm{Cl}_\theta(f^{-1}(F)) \subset O$. This means that $f^{-1}(F)$ is θ-βg-closed in (X,τ) and therefore f is θ-βg-irresolute. □

Corollary 4.1. *If $f\colon (X,\tau) \to (Y,\sigma)$ is a bijective pre β-open θ-βg-continuous function and (X,τ) is β-$T_{\frac{1}{2}}$, then (Y,σ) is β-$T_{\frac{1}{2}}$.*

Proof. It suffices to apply Theorem 3.4 and Theorem 4.1. □

Definition 4.2. A function $f\colon (X,\tau) \to (Y,\sigma)$ is said to be *θ-βg-closed* if for every β-closed set F of (X,τ), $f(F)$ is θ-βg-closed in (Y,σ).

Recall that a function $f\colon (X,\tau) \to (Y,\sigma)$ is said to be *β-irresolute* [14] if $f^{-1}(V) \in \beta O(X,\tau)$ for every $V \in \beta O(Y,\sigma)$.

Theorem 4.2. *If $f\colon (X,\tau) \to (Y,\sigma)$ is β-irresolute and θ-βg-closed, then $f(A)$ is θ-βg-closed in Y for a θ-βg-closed set A of X.*

Proof. Let A be a θ-βg-closed set of X. Assume that $f(A) \subset O$ where $O \in \beta O(Y)$. Clearly $A \subset f^{-1}(O)$. Since $f^{-1}(O) \in \beta O(X)$ and A be a θ-βg-closed set, we have $\beta\mathrm{Cl}_\theta(A) \subset f^{-1}(O)$ and therefore $f(\beta\mathrm{Cl}_\theta(A)) \subset O$. Since f is θ-βg-closed and $\beta\mathrm{Cl}_\theta(A)$ is β-closed, it follows that $\beta\mathrm{Cl}_\theta(f(\beta\mathrm{Cl}_\theta(A))) \subset O$ and thus $\beta\mathrm{Cl}_\theta(f(A)) \subset O$. Thus means that $f(A)$ is θ-βg-closed. □

Definition 4.3. A function $f\colon (X,\tau) \to (Y,\sigma)$ is said to be *quasi-β-irresolute* [17] (resp. *strongly β-irresolute* [17]) if for each $x \in X$ and each $V \in \beta O(Y, f(x))$, there exists $U \in \beta O(X,x)$ such that $f(U) \subset \beta\mathrm{Cl}(V)$ (resp. $f(\beta\mathrm{Cl}(U)) \subset V$).

Note that, in [17], quasi-β-irresolute function is defined as weakly-β-irresolute.

Theorem 4.3. [17] *Let $f\colon (X,\tau) \to (Y,\sigma)$ be a function, then:*

i) *f is quasi-β-irresolute if and only if $f^{-1}(V)$ is β- θ-closed (resp.β-θ-open) in (X,τ) for every β-θ-closed (resp. β- θ-open) set V in (Y,σ)*

ii) *f is quasi-β-irresolute if and only if $f^{-1}(V)$ is β-closed in (X,τ) for every β-θ-closed set V in (Y,σ)*

iii) *f is strongly-β-irresolute if and only if $f^{-1}(V)$ is β- θ-closed (resp. β-θ-open) in (X,τ) for every β-closed (resp. β-open) set V in (Y,σ)*

Theorem 4.4. *If the function $f\colon (X,\tau) \to (Y,\sigma)$ is strongly β-irresolute then f is θ-βg-continuous.*

Theorem 4.5. *Let $f\colon (X,\tau) \to (Y,\sigma)$ be θ-βg-irresolute function. Then f is quasi-β-irresolute if (X,τ) is β-$T_{\frac{1}{2}}$.*

Proof. Suppose that V is a β-θ-closed set in (Y,σ). By Lemma 3.1, V is θ-βg-closed in (Y,σ). Since f is θ-βg-irresolute, $f^{-1}(V)$ is θ-βg-closed in (X,τ). By Theorem 3.4, $f^{-1}(V)$ is β-closed. This shows that f is quasi-β-irresolute (Theorem 4.3). □

Definition 4.4. A function $f\colon (X,\tau) \to (Y,\sigma)$ is called *pre-β-θ-open* (resp. *pre-β-θ-closed*) if $f(U)$ is a β-θ-open (resp. β-θ-closed) set of (Y,σ) for every β- θ-open (resp. β-θ-closed) set U of (X,τ).

Theorem 4.6. *If a function $f\colon (X,\tau) \to (Y,\sigma)$ is quasi-β-irresolute and pre-β-closed, then for every θ-βg-closed set F of (Y,σ), $f^{-1}(F)$ is θ-βg-closed set of (X,τ) (i.e., f is θ-βg-irresolute).*

Proof. Suppose the set F is a θ-βg-closed set of (Y,σ). Assume $f^{-1}(F) \subset O$, where $O \in \beta O(X,\tau)$. By hypothesis, the function f is quasi-β-irresolute and therefore $f(\beta\mathrm{Cl}_\theta(f^{-1}(F)) \cap O^c) \subset \beta\mathrm{Cl}_\theta(f(f^{-1}(F))) \cap f(f^{-1}(F^c)) \subset \beta\mathrm{Cl}_\theta(F) \setminus F$. By the fact that f is pre-β-closed, it follows that $\beta\mathrm{Cl}_\theta(F) \setminus F$ contains a β-closed subset $f(\beta\mathrm{Cl}_\theta(f^{-1}(F)) \cap O^c)$. Now we have $f(\beta\mathrm{Cl}_\theta(f^{-1}(F)) \cap O^c) = \emptyset$ by Lemma 3.5. This implies that $\beta\mathrm{Cl}_\theta(f^{-1}(F)) \subset O$. Therefore, this shows that $f^{-1}(F)$ is θ-βg-closed set of (X,τ). □

Theorem 4.7. *If a function $f\colon (X,\tau) \to (Y,\sigma)$ is β-irresolute and pre-β-θ-closed, then for every θ-βg-closed set D of (X,τ), $f(D)$ is θ-βg-closed set of (Y,σ).*

Proof. Suppose that D is a θ-βg-closed set of (X,τ). Assume $f(D)\subset O$, where $O\in\beta O(Y,\sigma)$. Now $D\subset f^{-1}(O)$ and since f is β-irresolute $f^{-1}(O)\in\beta O(X,\tau)$. But D is θ-βg-closed and therefore $\beta\mathrm{Cl}_\theta(D)\subset f^{-1}(O)$. Thus $f(\beta\mathrm{Cl}_\theta(D)\subset O$. Now we have $\beta\mathrm{Cl}_\theta(f(D))\subset\beta\mathrm{Cl}_\theta(f(\beta\mathrm{Cl}_\theta(D)))=f(\beta\mathrm{Cl}_\theta(D))\subset O$. This shows that $f(D)$ is a θ-βg-closed set of (Y,σ). □

Theorem 4.8. *If a topological space (X,τ) is β-$T_{\frac{1}{2}}$ and $f\colon(X,\tau)\to(Y,\sigma)$ is surjective, quasi-β-irresolute and pre-β-closed, then (Y,σ) is β-$T_{\frac{1}{2}}$.*

Proof. Assume that A is a θ-βg-closed subset of (Y,σ). Then by Theorem 4.6 we have $f^{-1}(A)$ is a θ-βg-closed subset of (X,τ). By Theorem 3.4, $f^{-1}(A)$ is β-closed and hence, A is β-closed. It follows that (Y,σ) is β-$T_{\frac{1}{2}}$. □

Underline that:

i) If $f\colon(X,\tau)\to(Y,\sigma)$ is θ-βg-irresolute function and $g\colon(Y,\sigma)\to(Z,\mu)$ is θ-βg-continuous, then $g\circ f$ is θ-βg-continuous.

ii) If $f\colon(X,\tau)\to(Y,\sigma)$ and $g\colon(Y,\sigma)\to(Z,\mu)$ are θ-βg-irresolute, then $g\circ f$ is θ-βg-irresolute.

Recall that a topological space X is said to be β-connected [18] if X can not be written as a disjoint union of two non-empty β-open sets. For other properties of β-connected spaces see [12].

We now have the following,

Definition 4.5. A topological space X is said to be *$T\beta GO$-connected* (resp. *βGO-connected*) if X can not be written as a disjoint union of two non-empty θ-βg-open (resp. βg-open) sets.

Theorem 4.9. *For a topological space (X,τ) the following conditions are equivalent:*

(1) *X is $T\beta GO$-connected;*

(2) *the only subsets of X both θ-βg-closed and θ-βg-open are $\emptyset$ and X.*

Proof. (1) ⇒ (2): Suppose that A is a θ-βg-closed and θ-βg-open subset of X. Then the complement of A, i.e. $X\setminus A$ is θ-βg-closed and θ-βg-open. By the fact that $X=A\cup(X\setminus A)$ and $A\cap(X\setminus A)=\emptyset$, either A is empty or A is X.

(2) ⇒ (1): Let $X=A\cup B$, where A and B are θ-βg-open sets and $A\cap B=\emptyset$. Then A is θ-βg-open and θ-βg-closed. By hypothesis either A is empty or $A=X$. This implies that X is $T\beta GO$-connected. □

Theorem 4.10. *If (X,τ) is β-$T_{\frac{1}{2}}$, then the following conditions are equivalent:*

(1) *X is βGO-connected;*

(2) *X is $T\beta GO$-connected;*

(3) *X is β-connected.*

Example 4.1. Let us remark that the space $X=\{a,b,c\}$ with the topology $\tau=\{X,\emptyset,\{a\},\{b\},\{a,b\}\}$ is connected. Since $\{b\}$ is both θ-βg-open and θ-βg-closed, (X,τ) is neither $T\beta GO$-connected nor βGO-connected nor β-connected.

Theorem 4.11. *The following statements hold true:*

(1) If $f\colon (X,\tau) \to (Y,\sigma)$ is a surjective θ-βg-continuous function and X is $T\beta GO$-connected, then Y is β-connected;

(2) If $f\colon (X,\tau) \to (Y,\sigma)$ is a surjective θ-βg-irresolute function and the set X is $T\beta GO$-connected, then Y is $T\beta GO$-connected.

Proof. (1) Suppose that Y is not β-connected. There exist non-empty β-open sets V and W of Y such that $V \cup W = Y$ and $V \cap W = \emptyset$. Since f is θ-βg-continuous and surjective $X = f^{-1}(A) \cup f^{-1}(B)$ where $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint non-empty and θ-βg-open in X. This contradicts the fact that X is $T\beta GO$-connected.

(2) The argument is a minor modification of the proof (i). □

5. CONCLUSION

Maps have always been of tremendous importance in all branches of mathematics and the whole science. On the other hand, topology plays a significant role in quantum physics, high energy physics and superstring theory [9, 10]. Thus we have obtained a new class of mappings called θ-βg-continuous which may have possible application in quantum physics, high energy physics and superstring theory.

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Universidade Federal Fluminense
Departamento de Matematica Aplicada
Rua Mário Santos Braga, s/n - CEP: 24020-140*, Niteroi-RJ, Brasil*
E-mail address: `gmamccs@vm.uff.br`