J. Adv. Math. Stud.
Vol. **4**(2011), No. 2, 01-12
http://journal.fairpartners.ro

FIXED POINT RESULTS FOR WEAKLY CONTRACTIVE MAPPINGS IN ORDERED PARTIAL METRIC SPACES

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ABSTRACT. We prove some fixed point theorems for weakly C-contractive mappings in ordered partial metric spaces. These results extend the main theorems of Harjani, Lopez and Sadarangani [Fixed point theorems for weakly contractive mappings in ordered metric spaces, Computers and Mathematics with Applications, 61 (2011), 790-796] to the class of partially ordered partial metric spaces.

1. INTRODUCTION

There are a lot of generalizations of the Banach contraction mapping principle in the literature. Fixed point theorems are very useful in the existence theory of differential equations, integral equations, functional equations, partial differential equations, random differential equations and other related areas. Existence of a fixed point for contraction type mappings in partially ordered metric spaces and applications have been considered recently by many authors (see, for details, [4, 5, 7, 12, 14, 17, 18]).

Chatterjea [8] introduced the following definition.

Definition 1.1. A mapping $T\colon X \to X$ where (X, d) is a metric space is said to be a *C-contraction* if there exists $\alpha \in \left(0, \frac{1}{2}\right)$ such that for all $x, y \in X$ the following inequality holds:

$$d(Tx, Ty) \leq \alpha(d(x, Ty) + d(y, Tx)).$$

Chatterjea in [8] proved that if X is complete, then every C-contraction has a unique fixed point. Choudhury in [9] introduced a generalization of C-contraction given by the following definition.

Received: January 30, 2011. *Revised*: May 26, 2011.
2010 *Mathematics Subject Classification*: 54H25, 47H10.
Key words and phrases: partially ordered, partial metrics, fixed point, weakly C-contraction.

Definition 1.2. A mapping $T\colon X \to X$, where (X,d) is a metric space is said to be *weakly C-contractive* (or a weak C-contraction) if for all $x, y \in X$,

$$d(Tx,Ty) \leq \frac{1}{2}(d(x,Ty)+d(y,Tx)) - \varphi(d(x,Ty),d(y,Tx)),$$

where $\varphi\colon [0,+\infty)^2 \to [0,+\infty)$ is a continuous function such that $\varphi(x,y)=0$ if and only if $x=y=0$.

In [9], the author proves that if X is complete then every weak C-contraction has a unique fixed point. Recently, Harjani et al, [11] presented this last result in the context of partially ordered metric spaces. In this paper, we extend the results of Harjani et al [11], to the class of partially ordered partial metric spaces. Also, we give an example illustrating our result.

2. PRELIMINARIES

The concept of partial metric space was introduced by Matthews [13] in 1994. In such spaces, the distance from a point to itself may not be zero. Matthews [13] extended the well known Banach contraction principle to complete partial metric spaces. After that, many interesting fixed point results were established in such spaces. For more details, we refer the reader to [1, 2, 3, 6, 10, 13, 15, 16, 19, 20].

First, we start by recalling some known definitions and properties of partial metric spaces.

Definition 2.1. Let X be a nonempty set. A partial metric on a X is a function $p\colon X \times X \to [0,+\infty)$ such that for all $x, y, z \in X$:
(p1) $x = y \Longleftrightarrow p(x,x) = p(x,y) = p(y,y)$,
(p2) $p(x,x) \leq p(x,y)$,
(p3) $p(x,y) = p(y,x)$,
(p4) $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$.
A partial metric space is a pair (X,p) such that X is a nonempty set and p is a partial metric on X.

Remark 2.1. It is clear that, if $p(x,y)=0$, then from (p1) and (p2), $x=y$. But if $x=y$, $p(x,y)$ may not be 0.

A basic example of a partial metric space is the pair $(\mathbb{R}_+, p)$, where $p(x,y) = \max\{x,y\}$ for all $x, y \in \mathbb{R}_+$.

Each partial metric p on X generates a T_0 topology τ_p on X which has as a base the family of open p-balls $\{B_p(x,\varepsilon),\ x \in X, \varepsilon > 0\}$, where $B_p(x,\varepsilon) = \{y \in X : p(x,y) < p(x,x)+\varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

If p is a partial metric on X, then the function $p^s\colon X\times X\to\mathbb{R}_+$ given by

$$p^s(x,y)=2p(x,y)-p(x,x)-p(y,y), \tag{2.1}$$

is a metric on X.

Let $\{x_n\}$ be a sequence in X. Then
(i) $\{x_n\}$ converges to a point $x\in X$ if and only if $p(x,x)=\lim_{n\to+\infty}p(x,x_n)$,
(ii) $\{x_n\}$ is called a Cauchy sequence if there exists (and is finite) $\lim_{n,m\to+\infty}p(x_n,x_m)$.
(iii) (X,p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x\in X$, such that $p(x,x)=\lim_{n,m\to+\infty}p(x_n,x_m)$.

Lemma 2.1. *Let (X,p) be a partial metric space. Then*
(a) *$\{x_n\}$ is a Cauchy sequence in (X,p) if and only if it is a Cauchy sequence in the metric space (X,p^s),*
(b) *(X,p) is complete if and only if the metric space (X,p^s) is complete. Furthermore, $\lim_{n\to+\infty}p^s(x_n,x)=0$ if and only if*

$$p(x,x)=\lim_{n\to+\infty}p(x_n,x)=\lim_{n,m\to+\infty}p(x_n,x_m).$$

Definition 2.2. [1] Suppose that (X,p) is a partial metric space. A mapping $F\colon X\to X$ is said to be *continuous* at $x\in X$, if for every $\varepsilon>0$, there exists $\delta>0$ such that $F(B_p(x,\delta))\subseteq B_p(Fx,\varepsilon)$.

The following result is easy to check.

Lemma 2.2. *Let (X,p) be a partial metric space. $F\colon X\to X$ is continuous if and only if given a sequence $\{x_n\}\in\mathbb{N}$ and $x\in X$ such that $p(x,x)=\lim_{n\to+\infty}p(x,x_n)$, then $p(Fx,Fx)=\lim_{n\to+\infty}p(Fx,Fx_n)$.*

3. MAIN RESULTS

Before giving our result, we need the following lemma.

Lemma 3.1. *Consider non-negative real sequences $(u_n),(v_n),(a_n)$ and (b_n) such that there exists $\delta\geq 0$ with $\lim_{n\to+\infty}u_n=\lim_{n\to+\infty}v_n=\delta$, and*

$$u_n\leq v_n-\varphi(a_n,b_n),\quad \forall\ n\in\mathbb{N}$$

where $\varphi\colon[0,+\infty)^2\to[0,+\infty)$ is a continuous function such that $\varphi(x,y)=0$ if and only if $x=y=0$. Then,

$$\liminf_{n\to+\infty}a_n=\liminf_{n\to+\infty}b_n=0.$$

Proof. Taking the lim sup and using the continuity of φ, we have

$$\begin{aligned}\delta = \limsup_{n\to+\infty} u_n \leq& \limsup_{n\to+\infty} v_n - \liminf_{n\to+\infty} \varphi(a_n, b_n)\\ \leq&\delta - \varphi(\liminf_{n\to+\infty} a_n, \liminf_{n\to+\infty} b_n),\end{aligned}$$

which implies that $\varphi(\liminf_{n\to+\infty} a_n, \liminf_{n\to+\infty} b_n) = 0$, and by the fact that $\varphi(x, y) = 0$ if and only is $x = y = 0$, we get $\liminf_{n\to+\infty} a_n = \liminf_{n\to+\infty} b_n = 0$. □

Now, we state our result.

Theorem 3.1. *Let $(X, \leq_X)$ be a partially ordered set and suppose there is a partial metric p on X such that (X, p) is a complete partial metric space. Let $T\colon X \to X$ be a continuous and non-decreasing mapping such that*

$$p(Tx, Ty) \leq \frac{1}{2}(p(x, Ty) + p(y, Tx)) - \varphi(p(x, Ty), p(y, Tx)) \quad \text{for each} \quad x \geq_X y, \tag{3.1}$$

where $\varphi\colon [0, +\infty)^2 \to [0, +\infty)$ is a continuous function such that $\varphi(x, y) = 0$ if and only if $x = y = 0$.

If there exists $x_0 \in X$ such that $x_0 \leq_X Tx_0$, then T has a fixed point, say z. Moreover, $p(z, z) = 0$.

Proof. If $Tx_0 = x_0$, then the proof is finished. Suppose that $x_0 < Tx_0$. Since $x_0 < Tx_0$ and T is a non-decreasing mapping, we obtain by induction that

$$x_0 < Tx_0 \leq_X T^2x_0 \leq_X T^3x_0 \leq_X \cdots \leq_X T^nx_0 \leq_X T^{n+1}x_0 \leq_X \cdots .$$

Put $x_{n+1} = Tx_n$. Then, for each integer $n \geq 1$, from (3.1) and, as the elements x_{n-1} and x_n are comparable, we get using (p4) and a the fact that φ is non-negative

$$\begin{aligned}p(x_{n+1}, x_n) =&p(Tx_n, Tx_{n-1})\\ \leq_X&\frac{1}{2}(p(x_n, Tx_{n-1}) + p(x_{n-1}, Tx_n)) - \varphi(p(x_n, Tx_{n-1}), p(x_{n-1}, Tx_n))\\ =&\frac{1}{2}(p(x_n, x_n) + p(x_{n-1}, x_{n+1})) - \varphi(p(x_n, x_n), p(x_{n-1}, x_{n+1}))\\ \leq_X&\frac{1}{2}(p(x_n, x_n) + p(x_{n-1}, x_{n+1}))\\ \leq_X&\frac{1}{2}(p(x_{n-1}, x_n) + p(x_n, x_{n+1})).\end{aligned} \tag{3.2}$$

Therefore,

$$p(x_{n+1}, x_n) \leq_X p(x_n, x_{n-1}).$$

Thus $\{p(x_{n+1}, x_n)\}$ is a non-increasing sequence of non-negative real numbers and hence it is convergent. Let

$$\lim_{n\to+\infty} p(x_{n+1}, x_n) = \delta. \tag{3.3}$$

Letting $n \to +\infty$ in (3.2) we have

$$\delta \leq \frac{1}{2} \lim_{n\to+\infty} (p(x_n, x_n) + p(x_{n-1}, x_{n+1})) \leq \frac{1}{2}(\delta + \delta),$$

that is

$$\lim_{n\to+\infty} (p(x_n, x_n) + p(x_{n-1}, x_{n+1})) = 2\delta. \tag{3.4}$$

By (3.2) and the condition (p2)

$$2p(x_{n+1}, x_n) \leq p(x_n, x_n) + p(x_{n-1}, x_{n+1}) \leq p(x_{n+1}, x_n) + p(x_{n-1}, x_{n+1}),$$

so for any $n \in \mathbb{N}^*$, we get

$$p(x_{n+1}, x_n) \leq p(x_{n-1}, x_{n+1}). \tag{3.5}$$

Again, by (3.2)

$$p(x_{n+1}, x_n) \leq \frac{1}{2}(p(x_n, x_n) + p(x_{n-1}, x_{n+1})) - \varphi(p(x_n, x_n), p(x_{n-1}, x_{n+1})).$$

Thanks to (3.3), (3.4) and Lemma 3.1 , we obtain

$$\liminf_{n\to+\infty} p(x_n, x_n) = \liminf_{n\to+\infty} p(x_{n-1}, x_{n+1}) = 0.$$

Letting $n \to +\infty$ in (3.5), we have

$$\delta = \lim_{n\to+\infty} p(x_{n+1}, x_n) = \liminf_{n\to+\infty} p(x_{n+1}, x_n) \leq \liminf_{n\to+\infty} p(x_{n-1}, x_{n+1}) = 0,$$

so $\delta = 0$, that is,

$$\lim_{n\to+\infty} p(x_{n+1}, x_n) = 0. \tag{3.6}$$

In what follows we prove that $\{x_n\}$ is a Cauchy sequence in (X, p). From Lemma 2.1, it is sufficient to prove that $\{x_n\}$ is a Cauchy sequence in the metric space (X, p^s). Suppose to the contrary. then there exists $\varepsilon > 0$ for which we can find subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ with $n(k) > m(k) > k$ such that for every k

$$p^s(x_{n(k)}, x_{m(k)}) \geq \varepsilon. \tag{3.7}$$

Further, corresponding to $m(k)$ we can choose $n(k)$ in such a way that it is the smallest integer with $n(k) > m(k)$ and satisfying (3.7). Then

$$p^s(x_{n(k)-1}, x_{m(k)}) < \varepsilon. \tag{3.8}$$

Using (3.7), (3.8) and the triangular inequality, we have

$$\begin{aligned}\varepsilon \leq& p^s(x_{n(k)}, x_{m(k)})\\ \leq& p^s(x_{n(k)}, x_{n(k)-1}) + p^s(x_{n(k)-1}, x_{m(k)})\\ <& p^s(x_{n(k)}, x_{n(k)-1}) + \varepsilon.\end{aligned}$$

Making $k \to +\infty$ in the above inequality and using (3.6)

$$\lim_{k\to+\infty} p^s(x_{n(k)}, x_{m(k)}) = \lim_{k\to+\infty} p^s(x_{n(k)-1}, x_{m(k)}) = \varepsilon. \tag{3.9}$$

Again, the triangular inequality gives us

$$p^s(x_{m(k)}, x_{n(k)-1}) \leq p^s(x_{m(k)}, x_{m(k)-1}) + p^s(x_{m(k)-1}, x_{n(k)}) + p^s(x_{n(k)}, x_{n(k)-1}).$$

$$p^s(x_{m(k)-1}, x_{n(k)}) \leq p^s(x_{m(k)-1}, x_{m(k)}) + p^s(x_{m(k)}, x_{n(k)}).$$

Letting $k \to +\infty$ in the above two inequalities and using (3.6) and (3.9) we get

$$\lim_{k\to+\infty} p^s(x_{m(k)-1}, x_{n(k)}) = \varepsilon. \tag{3.10}$$

On the other hand, by definition of p^s given by (2.1), we have

$$p^s(x_{m(k)-1}, x_{n(k)}) = 2p(x_{m(k)-1}, x_{n(k)}) - p(x_{m(k)-1}, x_{m(k)-1}) - p(x_{n(k)}, x_{n(k)}),$$

then referring to (3.6)-(3.10), we obtain

$$\lim_{k\to+\infty} p(x_{m(k)-1}, x_{n(k)}) = \frac{\varepsilon}{2}. \tag{3.11}$$

Similarly, using (3.6)-(3.9) one can shows

$$\lim_{k\to+\infty} p(x_{n(k)}, x_{m(k)}) = \lim_{k\to+\infty} p(x_{n(k)-1}, x_{m(k)}) = \frac{\varepsilon}{2}. \tag{3.12}$$

As $n(k) > m(k)$ and $x_{n(k)-1}$ and $x_{m(k)-1}$ are comparable, using (3.1) we have

$$\begin{aligned}\varepsilon \leq& p^s(x_{n(k)}, x_{m(k)})\\ \leq& 2p(x_{n(k)}, x_{m(k)})\\ =& 2p(Tx_{n(k)-1}, Tx_{m(k)-1})\\ \leq& p(x_{n(k)-1}, Tx_{m(k)-1}) + p(x_{m(k)-1}, Tx_{n(k)-1})\\ &-2\varphi\Big(p(x_{n(k)-1}, Tx_{m(k)-1}), p(x_{m(k)-1}, Tx_{n(k)-1})\Big)\\ =& p(x_{n(k)-1}, x_{m(k)}) + p(x_{m(k)-1}, x_{n(k)}) - 2\varphi\Big(p(x_{n(k)-1}, x_{m(k)}), p(x_{m(k)-1}, x_{n(k)})\Big).\end{aligned}$$

Making $k \to +\infty$ and taking into account (3.11), (3.12) and the continuity of φ, we have

$$\varepsilon \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} - 2\varphi\left(\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right) \leq \varepsilon,$$

and from the last inequality, $\varphi\left(\frac{\varepsilon}{2},\frac{\varepsilon}{2}\right)=0$. From the fact that $\varphi(x,y)=0 \Longleftrightarrow x=y=0$, we have $\varepsilon=0$, which is a contradiction. This proves that $\{x_n\}$ is a Cauchy sequence in (X,p^s). From lemma 2.1, (X,p^s) is complete, so $\{x_n\}$ converges to some $z\in X$, that is

$$\lim_{n\to+\infty} p^s(x_n,z)=0.$$

Therefore, from lemma 2.1, using (3.6) and the property (p2), we have

$$p(z,z)=\lim_{n\to+\infty} p(x_n,z)=\lim_{n\to+\infty} p(x_n,x_n)=0. \tag{3.13}$$

Since T is continuous, we have

$$p(Tz,Tz)=\lim_{n\to+\infty} p(Tx_n,Tz)=\lim_{n\to+\infty} p(x_{n+1},Tz). \tag{3.14}$$

On the other hand, thanks to (3.13), it is obvious that $\lim_{n\to+\infty} p(x_{n+1},Tz)=p(Tz,z)$, so (3.14) reads

$$p(Tz,Tz)=p(Tz,z).$$

By (3.1), one can write

$$p(z,Tz)=p(Tz,Tz)\le p(z,Tz)-\varphi(p(z,Tz),p(z,Tz)),$$

which implies that $\varphi(p(z,Tz),p(z,Tz))=0$, and from the fact that $\varphi(x,y)=0 \Longleftrightarrow x=y=0$, we get $p(z,Tz)=0$, so $Tz=z$, that is, T has a fixed point. This completes the proof of Theorem 3.1. □

In the following theorem we remove the continuity of T. But, we add a condition on X:

$$\text{if a non-decreasing sequence } \{x_n\} \text{ converges to } x, \text{ then } x_n\le_X x \text{ for all } n. \tag{3.15}$$

Theorem 3.2. *Let $(X,\le_X)$ be a partially ordered set and suppose there is a partial metric p on X such that (X,p) is a complete partial metric space. Assume that X satisfies* (3.15). *Let $T\colon X\to X$ be a non-decreasing mapping such that for each $x\ge_X y$*

$$p(Tx,Ty)\le\frac{1}{2}(p(x,Ty)+p(y,Tx))-\varphi(p(x,Ty),p(y,Tx)),$$

where $\varphi\colon[0,+\infty)^2\to[0,+\infty)$ is a continuous function such that $\varphi(x,y)=0$ if and only if $x=y=0$.
If there exists $x_0\in X$ such that $x_0\le_X Tx_0$, then T has a fixed point, say z. Moreover, $p(z,z)=0$.

Proof. Following the proof of Theorem 3.1 we only have to check that $Tz = z$. As $\{x_n\}$ is a non-decreasing sequence in X and $x_n \to z$ in (X, p), then, the condition (3.15) yields that $x_n \leq_X z$ for every $n \in \mathbb{N}$ and, consequently, the contractive condition (3.1) leads to

$$\begin{aligned} p(x_{n+1}, Tz) =& p(Tx_n, Tz) \\ \leq& \frac{1}{2}(p(x_n, Tz) + p(z, Tx_n)) - \varphi(p(x_n, Tz), p(z, Tx_n)) \\ =& \frac{1}{2}(p(x_n, Tz) + p(z, x_{n+1})) - \varphi(p(x_n, Tz), p(z, x_{n+1})). \end{aligned}$$

Since $\lim_{n\to+\infty} p(x_n, Tz) = p(Tz, z)$, then letting $n \to +\infty$ and using (3.13) together with the continuity of φ we have

$$\begin{aligned} p(z, Tz) \leq& \frac{1}{2}(p(z, Tz) + p(z, z)) - \varphi(p(z, Tz), p(z, z)) \\ =& \frac{1}{2}(p(z, Tz) + 0) - \varphi(p(z, Tz), 0) \leq \frac{1}{2}p(z, Tz), \end{aligned}$$

which implies that $p(z, Tz) = 0$, so $z = Tz$. This completes the proof. □

Now we give a sufficient condition for the uniqueness of the fixed point.

Theorem 3.3. *In addition to the hypotheses of Theorem* 3.1 *(or Theorem* 3.2*), suppose that for $x, y \in X$ there exists $z \in X$ which is comparable to x and y. Then T has a unique fixed point.*

Proof. Let z and y be two fixed points of T. We consider two cases.

• If y is comparable to z, then, for every $n = 1, 2, \ldots$, we have $T^n y = y$ is comparable to $T^n z = z$. Also

$$\begin{aligned} p(y, z) =& p(T^n y, T^n z) \\ \leq& \frac{1}{2}(p(T^{n-1}y, T^n z) + p(T^{n-1}z, T^n y)) - \varphi(p(T^{n-1}y, T^n z), p(T^{n-1}z, T^n y)) \\ =& \frac{1}{2}(p(y, z) + p(z, y)) - \varphi(p(y, z), p(z, y)) \\ =& p(y, z) - \varphi(p(y, z), p(y, z)) \leq p(y, z), \end{aligned}$$

and this inequality gives us $\varphi(p(y, z), p(y, z)) = 0$, and, by our assumption about φ, we get $p(y, z) = 0$, or, equivalently, $y = z$.

• If y is not comparable to z, then there exists $x \in X$ such that x is comparable to y and z. Then, for every $n = 1, 2, \ldots$, monotonicity of T implies that $T^n x$ is comparable to $T^n y = y$ and $T^n z = z$. Therefore, from (3.1), we have

$$\begin{aligned} p(z, T^n x) =& p(T^n z, T^n x) \\ \leq& \frac{1}{2}(p(T^{n-1} z, T^n x) + p(T^{n-1} x, T^n z)) - \varphi(p(T^{n-1} z, T^n x), p(T^{n-1} x, T^n z)) \\ =& \frac{1}{2}(p(z, T^n x) + p(T^{n-1} x, z)) - \varphi(p(z, T^n x), p(T^{n-1} x, z)) \\ \leq& \frac{1}{2}(p(z, T^n x) + p(T^{n-1} x, z)). \end{aligned} \tag{3.16}$$

This yields that

$$p(z, T^n x) \leq p(z, T^{n-1} x).$$

This proves that the nonnegative non-increasing sequence $\{p(z, T^n x)\}$ is convergent. Put $\lim_{n \to +\infty} p(z, T^n x) = r$. Letting $n \to +\infty$ in (3.16) and taking into account the continuity of φ we obtain

$$r \leq \frac{1}{2}(r + r) - \varphi(r, r) \leq r.$$

This gives us $\varphi(r, r) = 0$, and, by our assumption about φ, $r = 0$. Consequently, $\lim_{n \to +\infty} p(z, T^n x) = 0$. Analogously, it can be proved that

$$\lim_{n \to +\infty} p(y, T^n x) = 0.$$

By triangular inequality, $p(z, y) \leq p(z, T^n x) + p(T^n x, z)$, and letting $n \to +\infty$, we get

$$p(y, z) = 0,$$

so $y = z$. The proof is finished. □

Remark 3.1. Notice that if $(X, \leq_X)$ is a totally ordered set, the condition given in Theorem 3.3 is obviously satisfied and we obtain uniqueness of the fixed point.

Remark 3.2. The Theorems 3.1, 3.2 and 3.3 are the extension of the main results of Harjani et al. [11] on the class of partially ordered partial metric spaces.

Example 3.1. Let $X = [0, +\infty)$ endowed with the natural ordering of real numbers $\leq$. Let $p(x, y) = \max(x, y)$. For any $x, y \in X$, we have $p^s(x, y) = |x - y|$. Then, (X, p^s) is a complete metric space, and so for (X, p). We define $T \colon X \to X$ by $Tx = \frac{1}{5}$. It is clear that the mapping T is continuous with respect to the partial metric p. Consider $\varphi(a, b) = \frac{1}{4}a + \frac{1}{2}b$ for any $a, b \geq 0$. For each $x \geq y$, we have

$$p(Tx, Ty) = \frac{x}{5} \leq \frac{x}{4} = \frac{1}{4}p(x, Ty) = \frac{1}{2}(p(x, Ty) + p(Tx, y)) - \varphi(p(x, Ty), p(Tx, y))$$

which is the contractive condition (3.1). Let $x_0 = 0$, we have $x_0 = 0 \leq 0 = Tx_0$. All the hypotheses of Theorem 3.1 are satisfied. Clearly T has a fixed point, which is $z = 0$. It is unique, since X is totally ordered.

Corollary 3.1. *Let $(X, \leq_X)$ be a partially ordered set and suppose there is a partial metric p on X such that (X, p) is a complete partial metric space. Let $\alpha \in \left[0, \frac{1}{2}\right)$ and $T\colon X \to X$ be a non-decreasing mapping such that*

$$p(Tx, Ty) \leq \alpha(p(x, Ty) + p(y, Tx)) \quad \text{for each} \quad x \geq_X y. \tag{3.17}$$

Also suppose either
a) *T is continuous or*
b) *X has the following property: if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \leq_X x$ for all n.*
If there exists $x_0 \in X$ such that $x_0 \leq_X Tx_0$, then T has a fixed point, say z. Moreover, $p(z, z) = 0$.

Proof. Let $\alpha \in \left[0, \frac{1}{2}\right)$. Here, it suffices to take the function $\varphi\colon [0, +\infty)^2 \to [0, +\infty)$ defined by $\varphi(a, b) = \left(\frac{1}{2} - \alpha\right)(a + b)$. Obviously, φ satisfies that $\varphi(a, b) = 0$ if and only if $a = b = 0$, and $\varphi(x, y) = \left(\frac{1}{2} - \alpha\right)(x + y) = \varphi(x + y, 0)$. Then, we can apply Theorems 3.1 and 3.2. □

Remark 3.3. Note that our result given by Corollary 3.1 is an extension of Chatterjea's fixed point theorem [8] to ordered partial metric spaces.

We give an example to illustrate our obtained result.

Example 3.2. Let $X = \{0, 1, 2\}$ endowed with the natural ordering $\leq$. Let $p(x, y) = \max(x, y)$. (X, p) is a complete partial metric space. We define $T\colon X \to X$ by $T0 = 0$, $T1 = 0$ and $T2 = 1$. It is easy that T is non-decreasing and the condition (b) given in Corollary 3.1 holds. Letting $\alpha = \frac{2}{5}$, we have

$$p(T0, T1) = 0 \leq \alpha = \alpha(p(0, T1) + p(T0, 1)),$$
$$p(T0, T2) = 1 \leq \frac{6}{5} = \alpha(p(0, T2) + p(T0, 2)),$$
$$p(T1, T2) = 1 \leq \frac{6}{5} = \alpha(p(1, T2) + p(T1, 2)).$$

Also, it is obvious that (3.17) is satisfied for $x = y$. Thus, the contractive condition (3.17) holds for any $x, y \in X$. Let $x_0 = 0$, we have $x_0 \leq Tx_0$. All the hypotheses of corollary 3.1 are satisfied, and T has a unique fixed point $z = 0$.

Now, we will show that Chatterjea's fixed point theorem [8] is not applicable in this case. Suppose that

$$d(Tx,Ty) \leq k\left(d(x,Ty)+d(y,Tx)\right)$$

for each $k \in \left[0, \frac{1}{2}\right)$ and all $x, y \in X$, where d is the standard metric given by $d(x,y) = |x-y|$ for all $x, y \in X$. Then for $x = 1$ and $y = 2$, we have

$$1 = d(Tx,Ty) \leq k(d(x,Ty)+d(y,Tx)) = k\left(0+2\right) = 2k.$$

This implies that $k \geq \frac{1}{2}$, it is a contradiction.

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