

ENHANCED $\left(\frac{G'}{G}\right)$ -EXPANSION METHOD AND APPLICATIONS
TO THE $(2 + 1)$ D TYPICAL BREAKING SOLITON AND
BURGERS EQUATIONS

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*Dedicated to Professor E. M. E. Zayed
on the Occasion of his 60th Birthday*

ABSTRACT. In this paper, by introducing a new general framework, the enhanced $\left(\frac{G'}{G}\right)$ -expansion method and its algorithm are proposed by studying Wang's $\left(\frac{G'}{G}\right)$ -expansion method and constructing a first order nonlinear ordinary differential equation with a third-degree nonlinear term to the nonlinear $(2 + 1)$ -dimensional typical breaking soliton and Burgers equations. As results, some new exact traveling wave solutions are obtained which include solitary wave solutions.

1. INTRODUCTION

In recent years, nonlinear wave equations have been played essential roles in many scientific and engineering areas such as fluid mechanics, plasma and elastic media and optical fibers, etc. Thus, it has had a considerable attention to find explicit traveling wave solutions of those problems. Several methods have been presented to obtain new exact solutions for many nonlinear evolution equations such that the homogeneous balance method [15], the tanh-function method [3], the algebraic method [4], the Hirota bilinear method [8], [9], the F -function expansion method [16], [17], the inverse scattering transform [1], the exp-function expansion method [7], the Jacobi elliptic function expansion [12], [19], the Bäcklund transform [13], [14], the sub-ODE method [11], the original $\left(\frac{G'}{G}\right)$ -expansion method [18], the improved $\left(\frac{G'}{G}\right)$ -expansion method [5], the generalized $\left(\frac{G'}{G}\right)$ -expansion method [10], the extended $\left(\frac{G'}{G}\right)$ -expansion method [6], the modified $\left(\frac{G'}{G}\right)$ -expansion method [2] and so on. The objective of this article is to use the improved $\left(\frac{G'}{G}\right)$ -expansion method which proposed by Guo et al [5] together with the generalized $\left(\frac{G'}{G}\right)$ -expansion method which proposed by Lü et al [10], to find the exact solutions of nonlinear evolution equations

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via the (2+1)-dimensional typical breaking soliton and Burgers equations which play an important role in mathematical physics. The main idea of this method is that the traveling wave solutions of the nonlinear evolution equations can be expressed by polynomials in G where $G = G(\xi)$ is based on a first order nonlinear ordinary

differential equation $G' = \sum_{i=0}^3 h_i G^i$ with a third-degree nonlinear term, and $' = \frac{d}{d\xi}$.

The degree of these polynomials can be determined by considering the homogeneous balance between the highest order derivatives and the nonlinear terms appearing in the given nonlinear equations. The coefficients of these polynomials can be obtained by solving a set of algebraic equations resulted from the process of using the proposed method. This method will play an important role in expressing the traveling wave solutions in terms of hyperbolic, trigonometric and rational functions for the nonlinear evolution equations in mathematical physics. The enhanced $(\frac{G'}{G})$ -expansion method used in this article can be applied to further nonlinear equations as the difference-differential equations which can be done in forthcoming articles.

2. DESCRIPTION OF THE ENHANCED $(\frac{G'}{G})$ -EXPANSION METHOD

Suppose that a nonlinear equation is given by

$$F(u, u_t, u_x, u_{tt}, u_{xt}, u_{xx}, u_y, u_{yt}, u_{yy}, u_{xy}, \dots) = 0, \tag{2.1}$$

where $u = u(x, y, t)$ is an unknown function, F is a polynomial in $u = u(x, y, t)$ and its various partial derivatives, in which the highest order derivatives and nonlinear terms are involved. The main steps of the enhanced $(\frac{G'}{G})$ -expansion method are the following:

Step 1. The traveling wave variable

$$u(x, y, t) = u(\xi), \quad \xi = x + y - Vt, \tag{2.2}$$

where V is a constant, permits us reducing Equation (2.1) into an ODE in the form

$$P(u, u', u'', \dots) = 0, \tag{2.3}$$

where P is a polynomial in u and its total derivatives.

Step 2. Suppose that the solution of Equation (2.3) can be expressed by a polynomial in G as follows:

$$u(\xi) = \sum_{i=-m}^m \frac{A_i (\frac{G'}{G})^i}{[1 + \sigma(\frac{G'}{G})]^i}, \tag{2.4}$$

where $G = G(\xi)$ is the solution of the first order nonlinear ODE in the form

$$G' = h_0 + h_1 G + h_2 G^2 + h_3 G^3, \tag{2.5}$$

where $\sigma, A_j (-m, -m + 1, \dots, m - 1, m), h_0, h_1, h_2$ and h_3 are constants to be determined and $A_m \neq 0$, while m is called the balance number.

Step 3. The positive integer m can be determined by considering the homogeneous balance between the highest order derivatives and the nonlinear terms appearing in Equation (2.3) as follows: If we define the degree of $u(\xi)$ as $D[u(\xi)] = m$, then the degree of other expressions is defined by

$$D \left[u^r \left(\frac{d^q u}{d\xi^q} \right)^s \right] = mr + s(q + m). \tag{2.6}$$

Therefore, with (2.6), we can get the value of m in (2.4).

Step 4. Substituting (2.4) into (2.3) and using the ODE (2.5), collecting all terms with the same order of G together, we get a polynomial in G . Equating each coefficients of this polynomial to zero, yields a set of algebraic equations, which can be solved to get A_i and V . Since the general solution of Equation (2.5) is well known to us, then substituting A_i , V and the general solutions of (2.5) into (2.4), we have traveling wave solutions of Equation (2.1).

Remark 2.1. It is well known [10] that Equation (2.5) admits the following special solutions.

Theorem 2.1. Suppose that $h_0 = 0$, $h_2 = 0$ and $h_3 \neq 0$.

(i) If $h_1 \neq 0$, then Equation (2.5) has the solutions

$$G = \pm \frac{\sqrt{(-h_3 + C_1 h_1 \exp(-2h_1 \xi)) h_1}}{-h_3 + C_1 h_1 \exp(-2h_1 \xi)}.$$

(ii) If $h_1 = 0$, then Equation (2.5) has the solutions

$$G = \pm \frac{1}{\sqrt{-2h_3 \xi + C_2}}.$$

Theorem 2.2. Suppose $h_3 = 0$.

(i) If $h_0 \neq 0$, $h_1 \neq 0$ and $h_2 \neq 0$, then Eq. (2.5) has the solutions

$$G = -\frac{h_1}{2h_2} + \frac{\sqrt{4h_0 h_2 - h_1^2}}{2h_2} \tan \left(\frac{\xi + C_3}{h_2} \sqrt{4h_0 h_2 - h_1^2} \right).$$

(ii) If $h_0 \neq 0$, $h_1 \neq 0$ and $h_2 = 0$, then Eq. (2.5) has the solution

$$G = \frac{h_0}{-h_1} + C_4 \exp(h_1 \xi).$$

(iii) If $h_0 = 0$, $h_1 \neq 0$ and $h_2 \neq 0$, then Eq. (2.5) has the solution

$$G = \frac{h_1}{-h_2 + C_5 h_1 \exp(-h_1 \xi)}.$$

(iv) If $h_0 = 0$, $h_1 = 0$ and $h_2 \neq 0$, then Eq. (2.5) has the solution

$$G = \frac{1}{-h_2 \xi + C_6}.$$

(v) If $h_0 \neq 0$, $h_1 = 0$, and $h_2 = 0$, then Eq. (2.5) has the solution

$$G = h_0\xi + C_7,$$

where C_i ($0, 1, \dots, 7$) are arbitrary constants.

3. APPLICATIONS

In this section, we apply the enhanced $(\frac{G'}{G})$ -expansion method to find new traveling wave solutions for some nonlinear PDEs in mathematical physics.

3.1. The (2 + 1)-dimensional typical breaking soliton equation. Consider the following (2 + 1)-dimensional typical breaking soliton equation

$$u_{xt} - 4u_xu_{xy} - 2u_{xx}u_y + u_{xxx} = 0. \tag{3.1}$$

The traveling wave variable (2.2) permits us converting Eq. (3.1) into the following ODE:

$$C - Vu' - 3(u')^2 + u''' = 0, \tag{3.2}$$

where C is an integration constant. Consider the homogeneous balance between u''' and $(u')^2$ in (3.2) we get $m = 1$. From (2.4) we get

$$u(\xi) = \sum_{i=-1}^1 \frac{A_i(\frac{G'}{G})^i}{[1 + \sigma(\frac{G'}{G})]^i}, \tag{3.3}$$

where A_i are constants to be determined and $G = G(\xi)$ satisfies Eq. (2.5). It is easy to deduce that:

$$\begin{aligned} u' = & \frac{h_0 - h_2G^2 - 2h_3G^3}{(h_0 + h_1G + h_2G^2 + h_3G^3)(\sigma h_3G^3 + \sigma h_2G^2 + (1 + \sigma h_1)G + \sigma h_0)^2} \\ & \times [(A_2\sigma^2h_3^2 - A_1h_3^2)G^6 + (2A_2\sigma^2h_2h_3 - 2A_1h_2h_3)G^5 \\ & + (A_2\sigma^2h_2^2 + 2A_2\sigma h_3 - A_1h_2^2 + 2A_2\sigma^2h_1h_3 - 2A_1h_1h_3)G^4 \\ & + (-2A_1h_1h_2 - 2A_1h_0h_3 + 2A_2\sigma^2h_3h_0 + 2A_2\sigma h_2 + 2A_2\sigma^2h_1h_2)G^3 \\ & + (A_2 + 2A_2\sigma h_1 - A_1h_1^2 + A_2\sigma^2h_1^2 - 2A_1h_0h_2 + 2A_2\sigma^2h_2h_0)G^2 \\ & + (-2A_1h_0h_1 + 2A_2\sigma h_0 + 2A_2\sigma^2h_1h_0)G - A_1h_0^2 + A_2\sigma^2h_0^2]; \end{aligned} \tag{3.4}$$

$$\begin{aligned} u'' = & \frac{-1}{(h_0 + h_1G + h_2G^2 + h_3G^3)(\sigma h_3G^3 + \sigma h_2G^2 + (1 + \sigma h_1)G + \sigma h_0)^3} \\ & \times [(-A_1\sigma h_3^4h_2 + A_2\sigma^3h_3^4h_2)G^{13} + (-8A_1h_3^4 - 4A_1\sigma h_3^4h_1 - 3A_1\sigma h_3^3h_2^2 \\ & + 4A_2\sigma^3h_3^4h_1 + 3A_2\sigma^3h_3^3h_2^2)G^{12} + (-25A_1h_3^3h_2 - 3A_1\sigma h_3^2h_2^3 + 3A_2\sigma^2h_3^3h_2 \\ & - 16A_1\sigma h_3^3h_1h_2 - 9A_1\sigma h_3^4h_0 + 3A_2\sigma^3h_3^2h_2^3 + 9A_2\sigma^3h_3^4h_0 + 16A_2\sigma^3h_3^3h_1h_2)G^{11} \\ & + (12A_2\sigma^3h_3^3h_1^2 - 21A_1\sigma h_3^2h_1h_2^2 - 12A_1\sigma h_3^3h_1^2 - 28A_1h_3^2h_2^2 - 34A_1\sigma h_3^3h_0h_2 \end{aligned}$$

$$\begin{aligned}
 &+21A_2\sigma^3h_3^2h_1h_2^2 - A_1\sigma h_2^4h_3 + 12A_2\sigma^2h_3^3h_1 + 6A_2\sigma^2h_3^2h_2^2 + A_2\sigma^3h_3h_2^4 \\
 &-20A_1h_3^3h_1 + 34A_2\sigma^3h_3^3h_0h_2)G^{10} + (45A_2\sigma^3h_3^2h_2^2h_0 + 3A_2\sigma h_3^2h_2 - 17A_1h_0h_3^3 \\
 &-13A_1h_3h_3^3 - 40A_1\sigma h_3^3h_0h_1 + 33A_2\sigma^2h_3^2h_1h_2 + 40A_2\sigma^3h_3^3h_0h_1 + 30A_2\sigma^3h_3^2h_1^2h_2 \\
 &+10A_2\sigma^3h_3h_1h_2^3 - 30A_1\sigma h_3^2h_1^2h_2 - 10A_1\sigma h_1h_3h_2^3 - 45A_1\sigma h_3^2h_0h_2^2 - 43A_1h_3^2h_1h_2 \\
 &+27h_3^3A_2\sigma^2h_0 + 3A_2\sigma^2h_3h_3^2)G^9 + (-28A_1h_3h_1h_2^2 - A_1\sigma h_2^4h_1 - 27A_1\sigma h_3^3h_0^2 \\
 &+24h_2^3A_2\sigma^3h_3h_0 + 102A_2\sigma^3h_3^2h_1h_0h_2 - 21A_1\sigma h_1^2h_3h_2^2 + 12A_2\sigma h_3^2h_1 - 2h_2^4A_1 \\
 &+3A_2\sigma h_3h_2^2 + 24A_2\sigma^2h_3h_1h_2^2 - 102A_1\sigma h_3^2h_0h_1h_2 - 16A_1h_3^2h_1^2 + 21A_2\sigma^3h_3h_1^2h_2^2 \\
 &+27A_2\sigma^3h_3^3h_0^2 + 24A_2\sigma^2h_3^2h_1^2 + 72A_2\sigma^2h_3^2h_0h_2 - 24h_2^3A_1h_3\sigma h_0 + 12h_3^2A_2\sigma^3h_1^3 \\
 &-36A_1h_0h_2h_3^2 + A_2\sigma^3h_2^4h_1 - 12h_3^2A_1h_1^3\sigma)G^8 + (78A_2\sigma^3h_3h_1h_2^2h_0 + 3A_2\sigma^3h_3^3h_1^2 \\
 &-16h_2A_1h_1^3\sigma h_3 + 4h_2^4A_2\sigma^3h_0 - 4h_2^4A_1\sigma h_0 + 27A_2\sigma h_0h_3^2 - 23A_1h_0h_2^2h_3 \\
 &+81A_2\sigma^2h_3^2h_0h_1 + A_2h_2h_3 + 3A_2\sigma^2h_3^3h_1 - 54A_1\sigma h_3^2h_0h_1^2 - 19A_1h_3h_1^2h_2 \\
 &-3A_1\sigma h_2^3h_1^2 + 69A_2\sigma^3h_0^2h_2h_3^2 - 78A_1\sigma h_1h_3h_0h_2^2 - 27A_1h_0h_1h_3^2 + 18A_2\sigma h_3h_1h_2 \\
 &+54A_2\sigma^3h_3^2h_1^2h_0 - 69A_1\sigma h_3^2h_0^2h_2 + 16h_2A_2\sigma^3h_1^3h_3 + 33A_2\sigma^2h_3h_1^2h_2 - 5h_2^3A_1h_1 \\
 &+57A_2\sigma^2h_3h_2^2h_0)G^7 + (-12A_1h_0^2h_3^2 - 4A_1h_0h_2^3 - 4A_1\sigma h_1^4h_3 - 3A_1\sigma h_2^2h_1^3 \\
 &+54A_2\sigma^2h_3^2h_0^2 + 12A_2\sigma h_3h_1^2 + 12A_2\sigma^2h_3h_1^3 + 4A_2\sigma^3h_3h_1^4 - 4A_1h_3h_1^3 \\
 &-32A_1h_0h_1h_2h_3 + 4A_2h_3h_1 - 4h_2^2A_1h_1^2 - 54A_1\sigma h_2^2h_3h_0^2 - 69A_1\sigma h_3^2h_0^2h_1 \\
 &-78A_1\sigma h_1^2h_3h_0h_2 + 69A_2\sigma^3h_0^2h_1h_3^2 + 54A_2\sigma^3h_0^2h_2^2h_3 - 16A_1\sigma h_1h_0h_3^2 \\
 &+42A_2\sigma h_0h_2h_3 + 12A_2\sigma^2h_0h_2^3 + 3A_2\sigma h_2^2h_1 + 6A_2\sigma^2h_2^2h_1^2 + 3A_2\sigma^3h_2^2h_1^3 \\
 &+120A_2\sigma^2h_3h_1h_0h_2 + 78A_2\sigma^3h_3h_1^2h_0h_2 + 16A_2\sigma^3h_0h_2^3h_1)G^6 \\
 &+(-15A_1h_0^2h_2h_3 - 11A_1h_0h_1^2h_3 - 7A_1h_0h_1h_2^2 - 27A_1\sigma h_3^2h_0^3 - 12A_1\sigma h_2^3h_0^2 \\
 &+27A_2\sigma^3h_0^3h_3^2 + 12A_2\sigma^3h_0^2h_2^3 + 12A_2\sigma h_0h_2^2 + 9A_2h_0h_3 - h_2A_1h_1^3 + h_2A_2h_1 \\
 &-24A_1\sigma h_1^3h_3h_0 - 102A_1\sigma h_1h_3h_0^2h_2 + 102A_2\sigma^3h_0^2h_1h_2h_3 - 21A_1\sigma h_1^2h_0h_2^2 \\
 &+42A_2\sigma h_0h_1h_3 - h_2A_1h_1^4\sigma + 3h_2A_2h_1^2\sigma + 3h_2A_2\sigma^2h_1^3 + h_2A_2\sigma^3h_1^4 \\
 &+57A_2\sigma^2h_3h_1^2h_0 + 81A_2\sigma^2h_3h_2h_0^2 + 24A_2\sigma^3h_3h_1^3h_0 + 33A_2\sigma^2h_0h_1h_2^2 \\
 &+21A_2\sigma^3h_0h_1^2h_2^2)G^5 + (24A_2\sigma^2h_0h_1^2h_2 + 4A_2h_0h_2 + 10A_2\sigma^3h_0h_1^3h_2 \\
 &-40A_1\sigma h_2h_3h_0^3 + 27A_2\sigma h_0^2h_3 + 18A_2\sigma h_0h_1h_2 + 24A_2\sigma^2h_0^2h_2^2 + 30A_2\sigma^3h_0^2h_1h_2^2 \\
 &-4A_1h_0h_1^2h_2 - 30A_1\sigma h_1h_0^2h_2^2 + 72A_2\sigma^2h_3h_1h_0^2 + 40A_2\sigma^3h_0^3h_2h_3 - 45A_1\sigma h_1^2h_3h_0^2
 \end{aligned}$$

$$\begin{aligned}
& -4A_1h_0^2h_2^2 + 45A_2\sigma^3h_0^2h_1^2h_3 - 12A_1h_0^2h_1h_3 - 10A_1\sigma h_1^3h_0h_2)G^4 \\
& + (-12A_1\sigma h_2^2h_0^3 + A_2\sigma^3h_0h_1^4 - 5A_1h_0^3h_3 + 33A_2\sigma^2h_0^2h_1h_2 + 21A_2\sigma^3h_0^2h_1^2h_2 \\
& - A_1\sigma h_1^4h_0 - 21A_1\sigma h_1^2h_0^2h_2 + 12A_2\sigma h_0^2h_2 + 3A_2\sigma h_0h_1^2 - 34A_1\sigma h_1h_3h_0^3 \\
& + 34A_2\sigma^3h_0^3h_1h_3 + 27A_2\sigma^2h_0^3h_3 + 3A_2\sigma^2h_0h_1^3 - 7A_1h_0^2h_1h_2 + A_2h_0h_1 \\
& + 12A_2\sigma^3h_0^3h_2^2 - A_1h_0h_1^3)G^3 + (3A_2\sigma h_0^2h_1 - 4A_1h_0^3h_2 - 9A_1\sigma h_3h_0^4 + 9A_2\sigma^3h_0^4h_3 \\
& + 12A_2\sigma^2h_0^3h_2 - 16A_1\sigma h_1h_0^3h_2 + 6A_2\sigma^2h_0^2h_1^2 - 3A_1\sigma h_1^3h_0^2 - 4A_1h_0^2h_1^2 + 3A_2\sigma^3h_0^2h_1^3 \\
& + 16A_2\sigma^3h_0^3h_1h_2)G^2 + (3A_2\sigma^2h_0^3h_1 - 5A_1h_0^3h_1 + 3A_2\sigma^3h_0^3h_1^2 + 4A_2\sigma^3h_0^4h_2 \\
& - 4A_1\sigma h_2h_0^4 - 3A_1\sigma h_1^2h_0^3)G - 2A_1h_0^4 + A_2\sigma^3h_0^4h_1 - A_1\sigma h_1h_0^4]; \quad (3.5)
\end{aligned}$$

$$\begin{aligned}
u''' = & \frac{-1}{(h_0 + h_1G + h_2G^2 + h_3G^3)(\sigma h_3G^3 + \sigma h_2G^2 + (1 + \sigma h_1)G + \sigma h_0)^4} \\
& \times [(A_2\sigma^4h_3^6h_2 - A_1\sigma^2h_3^6h_2)G^{18} + (6A_2\sigma^4h_3^5h_2^2 - 6A_1\sigma^2h_3^5h_2^2)G^{17} \\
& + (14A_2\sigma^4h_3^3h_3^4 - 14A_1\sigma h_2h_3^5 + 10A_2\sigma^4h_3^5h_1h_2 - 14A_1\sigma^2h_3^4h_3^3 \\
& - 9A_2\sigma^4h_0h_3^6 + 9A_1h_0\sigma^2h_3^6 - 10A_1\sigma^2h_3^5h_1h_2 + 4A_2\sigma^3h_3^5h_2)G^{16} \\
& + (-8A_1\sigma^2h_3^5h_1^2 - 44A_1\sigma^2h_3^4h_2^2h_1 - 16A_1\sigma^2h_3^3h_2^4 - 48A_1\sigma h_3^5h_1 + 44A_2\sigma^4h_3^4h_2^2h_1 \\
& + 8A_2\sigma^4h_3^5h_1^2 + 20A_2\sigma^3h_3^4h_2^2 + 16A_2\sigma^4h_2^4h_3^3 + 36A_1h_0\sigma^2h_2h_3^5 - 36A_2\sigma^4h_0h_3^5h_2 \\
& - 48A_1h_3^5 - 52A_1\sigma h_2^2h_3^4)G^{15} + (-72A_1\sigma^2h_3^3h_3^3h_1 - 181A_1h_3^4h_2 + 63A_2\sigma^4h_3^4h_2h_1^2 \\
& - 90A_1h_0\sigma h_3^5 - 9A_1\sigma^2h_3^2h_2^5 + 9A_2\sigma^4h_2^5h_3^2 - 18A_2\sigma^4h_0h_3^5h_1 + 72A_2\sigma^4h_2^3h_3^3h_1 \\
& + 6A_2\sigma^2h_3^4h_2 - 72A_1\sigma h_2^3h_3^3 + 54A_1h_0\sigma^2h_2^2h_3^4 + 36A_2\sigma^3h_3^3h_2^3 - 54A_2\sigma^4h_0h_3^4h_2^2 \\
& + 18A_1h_0\sigma^2h_1h_3^5 - 63A_1\sigma^2h_3^4h_1^2h_2 - 36A_2\sigma^3h_0h_3^5 - 234A_1\sigma h_2h_3^4h_1 \\
& + 36A_2\sigma^3h_3^4h_1h_2)G^{14} + (-262A_1h_3^3h_2^2 - 144A_1h_3^4h_1 + 32h_1^2A_2\sigma^3h_3^4 \\
& - 18A_2\sigma^4h_0^2h_3^5 + 24A_2\sigma^2h_3^3h_2^2 + 28A_2\sigma^3h_3^2h_2^4 + 32A_2\sigma^4h_1^3h_3^4 + 2A_2\sigma^4h_2^6h_3 \\
& - 16A_2\sigma^4h_0h_3^4h_1h_2 + 40A_1h_0\sigma^2h_2^3h_3^3 + 16A_1h_0\sigma^2h_3^4h_1h_2 - 400A_1h_0\sigma h_2h_3^4 \\
& - 408A_1\sigma h_2^2h_3^3h_1 - 148A_1\sigma^2h_3^3h_1^2h_2^2 - 52A_1\sigma^2h_3^2h_2^4h_1 - 40A_2\sigma^4h_0h_2^3h_3^3 \\
& + 148A_2\sigma^4h_3^3h_2^2h_1^2 + 120A_2\sigma^3h_3^3h_2^2h_1 + 52A_2\sigma^4h_2^4h_3^2h_1 - 112A_2\sigma^3h_0h_3^4h_2 \\
& + 18A_1h_0^2\sigma^2h_3^5 - 160A_1\sigma h_3^4h_1^2 - 44A_1\sigma h_2^4h_3^2 - 32A_1\sigma^2h_3^4h_1^3 - 2A_1\sigma^2h_3h_3^6)G^{13} \\
& + (-404A_1h_3^3h_1h_2 + 184A_2\sigma^3h_3^3h_1^2h_2 + 49A_2\sigma^4h_0h_3^4h_1^2 + 4A_2\sigma h_2h_3^3 + 30A_2\sigma^2h_2^3h_3^2
\end{aligned}$$

$$\begin{aligned}
 &+8A_2\sigma^3h_2^5h_3 - 123A_1h_0h_3^4 - 42A_2\sigma^4h_0^2h_3^4h_2 + 56A_2\sigma^4h_0h_1h_2^2h_3^3 - 179A_1h_2^3h_3^3 \\
 &-49A_1h_0\sigma^2h_1^2h_3^4 + 21A_1h_0\sigma^2h_2^4h_3^3 - 56A_1h_0\sigma^2h_2^2h_3^3h_1 + 42A_1h_0^2\sigma^2h_3^4h_2 \\
 &-438A_1h_0\sigma h_1h_3^4 - 664A_1h_0\sigma h_2^2h_3^3 - 14A_1\sigma^2h_3h_2^5h_1 - 54A_2\sigma^2h_0h_3^4 \\
 &-312A_1\sigma h_2^3h_3^2h_1 - 524A_1\sigma h_2h_3^3h_1^2 - 140A_1\sigma^2h_3^3h_1^3h_2 - 140A_1\sigma^2h_3^2h_2^3h_1^2 \\
 &-21A_2\sigma^4h_0h_2^4h_3^2 + 48A_2\sigma^2h_3^3h_1h_2 + 140A_2\sigma^4h_2^3h_3^2h_1^2 + 140A_2\sigma^4h_1^2h_2h_3^3 \\
 &+132A_2\sigma^3h_3^3h_2^3h_1 + 14A_2\sigma^4h_2^5h_3h_1 - 36A_2\sigma^3h_0h_3^4h_1 - 124A_2\sigma^3h_0h_3^3h_2^2 \\
 &-10A_1\sigma h_2^5h_3)G^{12} + (-56A_1h_2^4h_3 - 396A_1h_2^3h_1h_2^2 - 36A_2\sigma^3h_0^2h_3^4 + 48A_2\sigma^2h_3^3h_1^2 \\
 &+12A_2\sigma h_2^2h_3^2 + 12A_2\sigma^2h_2^4h_3 + 96A_2\sigma^3h_1^3h_3^3 + 48A_2\sigma^4h_1^4h_3^3 + 48A_2\sigma^3h_0h_3^3h_1h_2 \\
 &-152A_1h_3^3h_1^2 + 36A_2\sigma^4h_0^2h_3^4h_1 + 264A_2\sigma^4h_0h_3^3h_1^2h_2 - 24A_2\sigma^4h_0^2h_2^3h_3^3 \\
 &+72A_2\sigma^4h_0h_1h_2^3h_3^2 - 120A_2\sigma^2h_0h_2h_3^3 + 12A_1h_0\sigma^2h_2^5h_3 - 264A_1h_0\sigma^2h_1^2h_3^3h_2 \\
 &-72A_1h_0\sigma^2h_2^3h_3^2h_1 + 24A_1h_0^2\sigma^2h_2^2h_3^3 - 36A_1h_0^2\sigma^2h_3^4h_1 - 504A_1h_0\sigma h_2^3h_3^2 \\
 &-192A_1\sigma^2h_2^2h_3^3h_1^2 - 48A_1\sigma^2h_3h_2^4h_1^2 - 1392A_1h_0\sigma h_3^3h_1h_2 - 576A_1\sigma h_2^2h_3^2h_1^2 \\
 &-96A_1\sigma h_2^4h_3h_1 - 12A_2\sigma^4h_0h_2^5h_3 + 288A_2\sigma^3h_2^2h_1^2 + 48A_2\sigma^4h_2^4h_3h_1^2 \\
 &+192A_2\sigma^4h_1^3h_2^2h_3^2 + 108A_2\sigma^2h_1h_2^3h_3^2 + 48A_2\sigma^3h_2^4h_3h_1 - 72A_2\sigma^3h_0h_2^3h_3^2 \\
 &-340A_1h_0h_2h_3^3 - 252A_1h_0^2\sigma h_3^4 - 192A_1\sigma h_3^3h_1^3 - 48A_1\sigma^2h_3^3h_1^4)G^{11} \\
 &+(A_2h_2h_3^2 - 154A_1h_1h_3h_2^3 - 282A_1h_2^2h_1^2h_2 - 6h_1A_1\sigma h_2^5 + 28A_2h_2^2h_2\sigma h_1 \\
 &+8A_2\sigma h_2^3h_3 + A_2\sigma^4h_2^5h_1^2 + 200A_2\sigma^3h_0h_3^3h_1^2 + 372A_2\sigma^4h_0h_1^2h_2^2h_3^2 \\
 &-20A_2\sigma^3h_0^2h_3^3h_2 - 6h_2^5A_1 + 2A_2\sigma^4h_0h_2^4h_1h_3 + 232A_2\sigma^4h_0^2h_3^3h_1h_2 \\
 &+180A_2\sigma^3h_0h_2^2h_3^2h_1 - 6A_2\sigma^4h_0^2h_2^3h_3^2 - 90A_2\sigma^2h_0h_2^2h_3^2 - 164A_1h_0\sigma^2h_1^2h_3^3 \\
 &-232A_1h_0^2\sigma^2h_3^3h_1h_2 - 372A_1h_0\sigma^2h_1^2h_2^2h_3^2 + 6A_1h_0^2\sigma^2h_2^3h_3^2 - 2A_1h_0\sigma^2h_1h_2^4h_3 \\
 &-676A_1h_0\sigma h_1^2h_3^3 - 166A_1h_0\sigma h_2^4h_3 - 88A_1\sigma^2h_3h_2^3h_1^3 - 36A_2\sigma h_0h_3^3 + 18A_2\sigma^4h_0^3h_3^4 \\
 &-1512A_1h_0\sigma h_2^2h_1h_3^2 - 776A_1h_0^2\sigma h_3^3h_2 - 404A_1\sigma h_2h_3^2h_1^3 - 232A_1\sigma h_2^3h_3h_1^2 \\
 &-127A_1\sigma^2h_3^2h_1^4h_2 - 40A_2\sigma^3h_0h_2^4h_3 + 164A_2\sigma^4h_0h_1^3h_3^3 + 180A_2\sigma^2h_2h_3^2h_1^2 \\
 &+140A_2\sigma^3h_2^3h_3h_1^2 + 88A_2\sigma^4h_1^3h_2^3h_3 + 60A_2\sigma^2h_2^3h_3h_1 + 280A_2\sigma^3h_1^3h_3^2h_2 \\
 &+127A_2\sigma^4h_1^4h_2h_2 + 4A_1h_0\sigma^2h_2^6 - 18A_1h_0^3\sigma^2h_3^4 - 252A_1h_0h_1h_3^3 \\
 &-327A_1h_0h_2^2h_3^2 - A_1\sigma^2h_2^5h_1^2 - 4A_2\sigma^4h_0h_2^6)G^{10} + (-64h_3^2A_1h_1^3 - 18h_2^4A_1h_1 \\
 &-102A_1h_0^2h_3^3 + 2A_2h_2^2h_3 + 200A_2\sigma^3h_1^3h_2^2h_3 + 86A_2\sigma^4h_1^4h_2^2h_3 - 150A_1h_1^2h_3h_2^2
 \end{aligned}$$

$$\begin{aligned}
& -20A_1\sigma h_2^4 h_1^2 + 144A_2\sigma^2 h_3 h_1^2 h_2^2 + 32A_2\sigma h_3^2 h_1^2 + 96A_2\sigma^2 h_1^3 h_3^2 + 96A_2\sigma^3 h_1^4 h_3^2 \\
& + 32A_2\sigma^4 h_1^5 h_3^2 + 4A_2\sigma^4 h_2^4 h_1^3 + 4A_2\sigma^3 h_2^4 h_1^2 - 48A_2\sigma h_0 h_2 h_3^2 + 336A_2\sigma^4 h_0^2 h_2^2 h_3^2 h_1 \\
& + 432A_2\sigma^4 h_0 h_1^3 h_2 h_3^2 + 216A_2\sigma^3 h_0^2 h_3^3 h_1 + 624A_2\sigma^3 h_0 h_3^2 h_1^2 h_2 - 18A_2\sigma^4 h_0^2 h_2^4 h_3 \\
& + 136A_2\sigma^4 h_0 h_2^3 h_1^2 h_3 + 104A_2\sigma^4 h_0^3 h_3^3 h_2 + 48A_2\sigma^3 h_0^2 h_2^2 h_3^2 + 260A_2\sigma^4 h_0^2 h_3^3 h_1^2 \\
& - 48A_2\sigma^2 h_0 h_3^2 h_3 - 456A_1 h_0 h_1 h_2 h_3^2 + 16A_1 h_0 \sigma^2 h_1 h_2^5 - 336A_1 h_0^2 \sigma^2 h_2^2 h_1 h_3^2 \\
& - 104A_1 h_0^3 \sigma^2 h_3^3 h_2 - 136A_1 h_0 \sigma^2 h_1^2 h_2^3 h_3 - 432A_1 h_0 \sigma^2 h_1^3 h_3^2 h_2 - 96A_1 \sigma h_3^2 h_1^4 \\
& - 260A_1 h_0^2 \sigma^2 h_1^2 h_3^3 + 18A_1 h_0^2 \sigma^2 h_2^4 h_3 - 624A_1 h_0 \sigma h_2^3 h_1 h_3 - 1392A_1 h_0 \sigma h_1^2 h_2 h_3^2 \\
& - 86A_1 \sigma^2 h_3 h_2^2 h_1^4 - 16A_2\sigma^3 h_0 h_2^5 - 816A_1 h_0^2 \sigma h_2^2 h_3^2 - 696A_1 h_0^2 \sigma h_3^3 h_1 \\
& - 232A_1 \sigma h_2^2 h_3 h_1^3 - 4A_1 \sigma^2 h_2^4 h_1^3 - 16A_2\sigma^4 h_0 h_2^5 h_1 + 32A_2 \sigma h_2^2 h_3 h_1 - 16A_1 h_0 \sigma h_2^5 \\
& + 144A_2\sigma^2 h_0 h_3^2 h_1 h_2 + 48A_2\sigma^3 h_0 h_3^2 h_1 h_3 - 124A_1 h_0 h_2^3 h_3 - 32A_1 \sigma^2 h_3^2 h_1^5)G^9 \\
& + (-19h_2^3 A_1 h_1^2 - 14A_1 h_0 h_2^4 - 9A_2 h_0 h_3^2 + 132A_2\sigma^3 h_1^4 h_2 h_3 + 42A_2\sigma^4 h_1^5 h_2 h_3 \\
& + 6A_2 h_2 h_3 h_1 - 60h_2 A_1 h_1^3 h_3 - 24A_1 \sigma h_2^3 h_1^3 + 252A_2\sigma^2 h_0 h_3^2 h_1^2 - 21A_2\sigma^4 h_0 h_2^4 h_1^2 \\
& - 12A_2\sigma^4 h_0^2 h_2^5 + 12A_2\sigma^3 h_2^3 h_1^3 + 6A_2\sigma^2 h_1^2 h_2^3 + 6A_2\sigma^4 h_1^4 h_2^3 - 24A_2\sigma h_0 h_2^2 h_3 \\
& + 36A_2\sigma h_0 h_1 h_3^2 + 684A_2\sigma^4 h_0^2 h_1^2 h_3^2 h_2 + 150A_2\sigma^4 h_0^3 h_2^2 h_3^2 + 264A_2\sigma^4 h_0 h_1^3 h_2^2 h_3 \\
& + 684A_2\sigma^3 h_0^2 h_3^2 h_1 h_2 + 108A_2\sigma^4 h_0^2 h_2^3 h_1 h_3 + 216A_2\sigma^4 h_0^3 h_3^3 h_1 - 48A_2\sigma^3 h_0 h_2^4 h_1 \\
& - 234A_1 h_0 h_1 h_2^2 h_3 + 21A_1 h_0 \sigma^2 h_1^2 h_2^4 - 153A_1 h_0 \sigma^2 h_1^4 h_3^2 - 108A_1 h_0^2 \sigma^2 h_2^3 h_1 h_3 \\
& - 150A_1 h_0^3 \sigma^2 h_2^2 h_3^2 - 216A_1 h_0^3 \sigma^2 h_3^3 h_1 - 684A_1 h_0^2 \sigma^2 h_1^2 h_2 h_3^2 - 264A_1 h_0 \sigma^2 h_1^3 h_2^2 h_3 \\
& - 1368A_1 h_0^2 \sigma h_1 h_2 h_3^2 - 396A_1 h_0 \sigma h_1^3 h_3^2 - 66A_1 h_0 \sigma h_1 h_2^4 - 792A_1 h_0 \sigma h_1^2 h_2^2 h_3 \\
& - 42A_1 \sigma^2 h_3 h_1^5 h_2 + 108A_2\sigma^3 h_0^3 h_3^3 - 324A_1 h_0^2 \sigma h_2^3 h_3 - 102A_1 \sigma h_2 h_3 h_1^4 \\
& + 60A_2 \sigma h_2 h_3 h_1^2 + 144A_2\sigma^2 h_1^3 h_2 h_3 + 360A_2\sigma^3 h_0 h_1^3 h_3^2 + 153A_2\sigma^4 h_0 h_1^4 h_3^2 \\
& + 108A_2\sigma^2 h_0 h_2^2 h_1 h_3 + 90A_2\sigma^2 h_0^2 h_3^2 h_2 + 396A_2\sigma^3 h_0 h_2^2 h_1^2 h_3 - 177A_1 h_0^2 h_2 h_3^2 \\
& - 150A_1 h_0 h_1^2 h_3^2 + 12A_1 h_0^2 \sigma^2 h_2^5 - 216A_1 h_0^3 \sigma h_3^3 - 6A_1 \sigma^2 h_3^2 h_1^4 - 24A_2\sigma^2 h_0 h_2^4)G^8 \\
& + (-8h_2^2 A_1 h_1^3 + 8A_2 h_3 h_1^2 - 8A_1 h_1^4 h_3 - 16A_1 \sigma h_1^5 h_3 - 12A_1 \sigma h_2^2 h_1^4 + 32A_2 h_3 h_1^3 \sigma \\
& + 4A_2 h_2^2 h_1^2 \sigma - 8A_1 \sigma^2 h_1^6 h_3 - 4A_1 \sigma^2 h_2^2 h_1^5 + 48A_2\sigma^2 h_3 h_1^4 + 32A_2\sigma^3 h_3 h_1^5 \\
& + 12A_2\sigma^2 h_2^2 h_1^3 + 12A_2\sigma^3 h_2^2 h_1^4 + 8A_2\sigma^4 h_3 h_1^6 + 4A_2\sigma^4 h_2^2 h_1^5 - 40A_2\sigma^3 h_0 h_2^3 h_1^2 \\
& - 32A_2\sigma^3 h_0^2 h_2^4 - 4A_2 h_0 h_2 h_3 + 80A_2 \sigma h_0 h_1 h_2 h_3 + 408A_2\sigma^4 h_0^2 h_1^2 h_2^2 h_3 \\
& + 568A_2\sigma^4 h_0^3 h_1 h_3^2 h_2 + 336A_2\sigma^4 h_0^2 h_1^3 h_3^2 + 328A_2\sigma^3 h_0^3 h_3^2 h_2 + 360A_2\sigma^2 h_0 h_1^2 h_2 h_3
\end{aligned}$$

$$\begin{aligned}
 & -32A_2\sigma^4 h_0^2 h_2^4 h_1 + 56A_2\sigma^4 h_0^3 h_2^3 h_3 + 624A_2\sigma^3 h_0^2 h_3^2 h_1^2 - 48A_2\sigma^2 h_0 h_1 h_2^3 \\
 & -132A_1 h_0 h_1^2 h_2 h_3 + 8A_1 h_0 \sigma^2 h_1^3 h_2^3 - 568A_1 h_0^3 \sigma^2 h_1 h_2 h_3^3 - 56A_1 h_0^3 \sigma^2 h_2^3 h_3 \\
 & + 32A_1 h_0^2 \sigma^2 h_1 h_2^4 - 408A_1 h_0^2 \sigma^2 h_1^2 h_2^2 h_3 - 336A_1 h_0^2 \sigma^2 h_1^3 h_3^2 - 720A_1 h_0^2 \sigma h_1 h_2^2 h_3 \\
 & -188A_1 h_0 \sigma^2 h_1^4 h_2 h_3 - 88A_1 h_0 \sigma h_1^2 h_2^3 - 400A_1 h_0 \sigma h_1^3 h_2 h_3 - 528A_1 h_0^2 \sigma h_1^2 h_3^2 \\
 & + 36A_2 \sigma h_0^2 h_3^2 - 16A_2 \sigma h_0 h_3^3 + 72A_2 \sigma^4 h_0^4 h_3^3 - 392A_1 h_0^3 \sigma h_3^2 h_2 - 8A_2 \sigma^4 h_0 h_1^3 h_3^2 \\
 & + 72A_2 \sigma^2 h_0^2 h_2^2 h_3 + 324A_2 \sigma^2 h_0^2 h_3^2 h_1 + 464A_2 \sigma^3 h_0 h_1^3 h_2 h_3 + 432A_2 \sigma^3 h_0^2 h_2^2 h_1 h_3 \\
 & + 188A_2 \sigma^4 h_0 h_1^4 h_2 h_3 - 108A_1 h_0^2 h_1 h_3^2 - 84A_1 h_0^2 h_2^2 h_3 - 72A_1 h_0^4 \sigma^2 h_3^3 - 32A_1 h_0^2 \sigma h_2^4 \\
 & - 28A_1 h_0 h_1 h_2^3)G^7 + (-h_2 A_1 h_1^4 - 8A_1 h_0^2 h_2^3 - 4A_2 h_0 h_2^2 + 156A_2 \sigma^3 h_0 h_1^4 h_3 \\
 & + 46A_2 \sigma^4 h_0 h_1^5 h_3 - 24A_2 \sigma^2 h_0^2 h_2^3 + 18A_2 h_0 h_3 h_1 - 16A_2 \sigma h_0 h_1 h_2^2 \\
 & + 72A_2 \sigma h_0^2 h_2 h_3 + 356A_2 \sigma^4 h_0^3 h_1 h_2^2 h_3 + 408A_2 \sigma^4 h_0^2 h_1^3 h_2 h_3 + 404A_2 \sigma^4 h_0^3 h_1^2 h_3^2 \\
 & + 189A_2 \sigma^4 h_0^4 h_2 h_3^2 + 540A_2 \sigma^3 h_0^3 h_2^2 h_1 + 468A_2 \sigma^2 h_0^2 h_1 h_2 h_3 + 192A_2 \sigma^2 h_0 h_1^3 h_3 \\
 & - 18A_2 \sigma^2 h_0 h_2^2 h_1^2 - 22A_2 \sigma^4 h_0^2 h_2^3 h_1^2 + h_2 A_2 h_1^2 - 46A_1 h_0 \sigma^2 h_1^5 h_3 - 2A_1 h_0 \sigma^2 h_1^4 h_2^2 \\
 & - 356A_1 h_0^3 \sigma^2 h_1 h_2^2 h_3 - 189A_1 h_0^4 \sigma^2 h_2^2 h_2 - 78A_1 h_0^2 h_1 h_2 h_3 - 408A_1 h_0^2 \sigma^2 h_1^3 h_2 h_3 \\
 & - 404A_1 h_0^3 \sigma^2 h_1^2 h_3^2 + 22A_1 h_0^2 \sigma^2 h_1^2 h_3^2 - 66A_1 h_0 \sigma h_1^4 h_3 - 40A_1 h_0 \sigma h_1^3 h_2^2 \\
 & - 172A_1 h_0^3 \sigma h_2^2 h_3 + 4h_2 A_2 \sigma^3 h_1^5 + 100A_2 \sigma h_0 h_1^2 h_3 + 162A_2 \sigma^2 h_0^3 h_2^2 - 8A_2 \sigma^4 h_0^3 h_2^4 \\
 & - 456A_1 h_0^2 \sigma h_2 h_1^2 h_3 - 60A_1 h_0^2 \sigma h_2^3 h_1 - 264A_1 h_0^3 \sigma h_2^2 h_1 - 4A_2 \sigma^3 h_0 h_1^3 h_2^2 \\
 & + 2A_2 \sigma^4 h_0 h_1^4 h_2^2 + 804A_2 \sigma^3 h_0^2 h_1^2 h_2 h_3 - h_2 A_1 h_1^6 \sigma^2 - 48A_2 \sigma^3 h_0^2 h_2^3 h_1 \\
 & + 224A_2 \sigma^3 h_0^3 h_2^2 h_3 - 21A_1 h_0^3 h_2^2 + 6h_2 A_2 \sigma^2 h_1^4 - 20A_1 h_0 h_1^3 h_3 + 4h_2 A_2 h_1^3 \sigma \\
 & - 2h_2 A_1 h_1^5 \sigma + h_2 A_2 \sigma^4 h_1^6 - 15A_1 h_0 h_1^2 h_2^2 + 8A_1 h_0^3 \sigma^2 h_2^4)G^6 + (18A_2 h_0^2 h_3 \\
 & - 36A_1 h_0^4 \sigma h_3^2 + 252A_2 \sigma^4 h_0^4 h_1 h_3^2 + 180A_2 \sigma^3 h_0^4 h_3^2 + 504A_2 \sigma^4 h_0^3 h_1^2 h_2 h_3 \\
 & + 360A_2 \sigma^3 h_0^2 h_1^3 h_3 + 360A_2 \sigma^2 h_0^2 h_1^2 h_3 - 4A_1 h_0^3 h_2 h_3 - 72A_1 h_0^2 \sigma h_1^3 h_3 \\
 & + 240A_2 \sigma^2 h_0^3 h_2 h_3 - 126A_1 h_0^2 \sigma^2 h_1^4 h_3 + 126A_2 \sigma^4 h_0^4 h_2^2 h_3 + 144A_2 h_0^2 h_1 h_3 \sigma \\
 & - 504A_1 h_0^3 \sigma^2 h_2 h_1^2 h_3 - 126A_1 h_0^4 \sigma^2 h_2^2 h_3 - 6A_1 h_0^2 h_1^2 h_3 + 126A_2 \sigma^4 h_0^2 h_1^4 h_3 \\
 & - 144A_1 h_0^3 \sigma h_1 h_2 h_3 + 720A_2 \sigma^3 h_0^3 h_1 h_2 h_3 - 252A_1 h_0^4 \sigma^2 h_3^2 h_1)G^5 \\
 & + (A_1 h_0 h_1^4 + 8A_1 h_0^3 h_2^2 + 63A_2 \sigma^4 h_0^5 h_3^2 + 16A_2 \sigma h_0^2 h_1 h_2 + 326A_2 \sigma^4 h_0^4 h_1 h_2 h_3 \\
 & + 22A_2 \sigma^4 h_0^3 h_1^2 h_2^2 + 18A_2 \sigma^2 h_0^2 h_1^2 h_2 - A_2 h_0 h_1^2 + 324A_2 \sigma^2 h_0^3 h_1 h_3 + 4A_2 h_0^2 h_2 \\
 & + 200A_2 \sigma^4 h_0^3 h_1^3 h_3 + 248A_2 \sigma^3 h_0^4 h_2 h_3 + 2A_1 h_0^2 \sigma^2 h_1^4 h_2 - 326A_1 h_0^4 \sigma^2 h_1 h_2 h_3
 \end{aligned}$$

$$\begin{aligned}
& -200A_1h_0^3\sigma^2h_1^3h_3 - 22A_1h_0^3\sigma^2h_1^2h_2^2 + 60A_1h_0^3\sigma h_2^2h_1 + 8A_1h_0^3\sigma h_1^2h_3 \\
& + 40A_1h_0^2\sigma h_1^3h_2 + 72A_2\sigma h_0^3h_3 - A_2\sigma^4h_0h_1^6 - 6A_2\sigma^2h_0h_1^4 - 4A_2\sigma h_0h_1^3 \\
& + 24A_2\sigma^2h_0^3h_2^2 + 8A_2\sigma^4h_0^4h_2^3 + 14A_1h_0^4\sigma h_2h_3 + 4A_2\sigma^3h_0^2h_1^3h_2 \\
& + 452A_2\sigma^3h_0^3h_1^2h_3 + 48A_2\sigma^3h_0^3h_2^2h_1 - 2A_2\sigma^4h_0^2h_1^4h_2 + 18A_1h_0^3h_1h_3 \\
& + 15A_1h_0^2h_1^2h_2 - 63A_1h_0^5\sigma^2h_2^3 + 2A_1h_0\sigma h_1^5 + A_1h_0\sigma^2h_1^6 - 8A_1h_0^4\sigma^2h_2^3 \\
& - 4A_2\sigma^3h_0h_1^5)G^4 + (8A_1h_0^2h_1^3 + 12A_1h_0^4h_3 - 84A_1\sigma^2h_2h_0^5h_3 - 4A_2\sigma^4h_0^2h_1^5 \\
& + 84A_2\sigma^4h_0^5h_2h_3 - 12A_2\sigma^2h_0^2h_1^3 - 12A_2\sigma^3h_0^2h_1^4 + 184A_2\sigma^4h_0^4h_1^2h_3 \\
& + 32A_2\sigma^4h_0^4h_1h_2^2 + 48A_2\sigma^2h_0^3h_1h_2 + 288A_2\sigma^3h_0^4h_1h_3 - 32A_1h_0^4\sigma^2h_2^2h_1 \\
& - 184A_1h_0^4\sigma^2h_1^2h_3 - 8A_1h_0^3\sigma^2h_1^3h_2 + 88A_1h_0^3\sigma h_1^2h_2 + 48A_1h_0^4\sigma h_1h_3 \\
& + 16A_2\sigma h_0^3h_2 + 108A_2\sigma^2h_0^4h_3 + 32A_2\sigma^3h_0^4h_2^2 + 40A_2\sigma^3h_0^3h_1^2h_2 + 8A_2\sigma^4h_0^3h_1^3h_2 \\
& + 28A_1h_0^3h_1h_2 + 4A_1h_0^2\sigma^2h_1^5 + 32A_1h_0^4\sigma h_2^2 - 4A_2\sigma h_0^2h_1^2 + 12A_1h_0^2\sigma h_1^4)G^3 \\
& + (14A_1h_0^4h_2 + 18A_1h_0^5\sigma h_3 + 19A_1h_0^3h_1^2 - 21A_1h_0^4\sigma^2h_1^2h_2 - 12A_2\sigma^3h_0^3h_1^3 \\
& - 90A_1h_0^5\sigma^2h_1h_3 - 6A_2\sigma^2h_0^3h_1^2 - 12A_1h_0^5\sigma^2h_2^2 + 24A_1h_0^3\sigma h_1^3 + 66A_1h_0^4\sigma h_1h_2 \\
& + 48A_2\sigma^3h_0^4h_1h_2 + 6A_1h_0^3\sigma^2h_1^4 + 90A_2\sigma^4h_0^5h_1h_3 + 72A_2\sigma^3h_0^5h_3 + 24A_2\sigma^2h_0^4h_2 \\
& + 12A_2\sigma^4h_0^5h_2^2 - 6A_2\sigma^4h_0^3h_1^4 + 21A_2\sigma^4h_0^4h_1^2h_2)G^2 + (-16h_1A_1\sigma^2h_2h_0^5 \\
& + 16h_1A_2\sigma^4h_0^5h_2 + 16A_2\sigma^3h_0^5h_2 + 18A_1h_1h_0^4 + 18A_2\sigma^4h_0^6h_3 + 20A_1\sigma h_1^2h_0^4 \\
& - 18A_1h_0^6\sigma^2h_3 - 4A_2\sigma^4h_0^4h_1^3 + 4A_1h_0^4\sigma^2h_1^3 + 16A_1h_0^5\sigma h_2 - 4A_2\sigma^3h_0^4h_1^2)G \\
& + A_1h_0^5\sigma^2h_1^2 + 4A_2\sigma^4h_0^6h_2 + 6A_1h_0^5\sigma h_1 + 6A_1h_0^5 - A_2\sigma^4h_0^5h_1^2 - 4A_1h_0^6\sigma^2h_2]. \quad (3.6)
\end{aligned}$$

Substituting (3.4) and (3.6) into (3.2) we obtain a polynomial in the power of G . Equating the coefficients of this polynomial to zero, we get a system of algebraic equations which can be solved by *Maple* or *Mathematica*. We obtain the results:

$$\begin{aligned}
u_1 &= A_0 + 2h_1 + \frac{2h_1h_2}{-h_2 + C_5h_1 \exp[-h_1(x + y - h_1^2t)]}, \\
u_2 &= A_0 + \frac{2C_5h_1^2(1 + \sigma h_1) \exp[-h_1(x + y - h_1^2t)]}{-h_2 + C_5h_1(1 + \sigma h_1) \exp[-h_1(x + y - h_1^2t)]},
\end{aligned}$$

where $h_1 \neq 0$, $h_2 \neq 0$ and C_5 are arbitrary constants.

$$u_3 = A_0 - 2h_1 - \frac{2h_0h_1}{-h_0 + C_4h_1 \exp[h_1(x + y - h_1^2t)]},$$

$$u_4 = A_0 - \frac{2C_4 h_1^2 (1 + \sigma h_1) \exp [h_1 (x + y - h_1^2 t)]}{-h_0 + C_4 h_1 (1 + \sigma h_1) \exp [h_1 (x + y - h_1^2 t)]},$$

where $h_0 \neq 0$, $h_1 \neq 0$ and C_4 are arbitrary constants.

$$u_5 = A_0 + \frac{4C_1 h_1^2 \exp [-2h_1 (x + y - 4h_1^2 t)]}{-h_3 + C_1 h_1 \exp [-2h_1 (x + y - 4h_1^2 t)]},$$

$$u_6 = A_0 + \frac{4C_1 h_1^2 (1 + \sigma h_1) \exp [-2h_1 (x + y - 4h_1^2 t)]}{-h_3 + C_1 h_1 (1 + \sigma h_1) \exp [-2h_1 (x + y - 4h_1^2 t)]},$$

where $h_1 \neq 0$, $h_3 \neq 0$ and C_1 are arbitrary constants.

$$u_7 = A_0 + \frac{4h_3}{C_2 + h_3 [\sigma - 2(x + y - 2\sqrt{-3Ct})]} + \frac{\sqrt{-3C} \{C_2 + h_3 [\sigma - 2(x + y - 2\sqrt{-3Ct})]\}}{6h_3},$$

where $h_3 \neq 0$, and C_2 are arbitrary constants.

$$u_8 = A_0 + \frac{2h_2}{C_6 + h_2 (\sigma - x - y + 2\sqrt{-3Ct})} + \frac{\sqrt{-3C} [C_6 + h_2 (\sigma - x - y + 2\sqrt{-3Ct})]}{3h_2},$$

where $h_2 \neq 0$ and C_6 are arbitrary constants.

$$u_9 = A_0 - \frac{2h_0}{C_7 + h_0 (\sigma + x + y + 2\sqrt{-3Ct})} + \frac{\sqrt{-3C} [C_7 + h_0 (\sigma + x + y + 2\sqrt{-3Ct})]}{3h_0},$$

where $h_0 \neq 0$, and C_7 are arbitrary constants.

3.2. The $(2+1)$ dimensional Burgers equations. In this section, we consider the following $(2+1)$ -dimensional Burgers equations:

$$\begin{aligned} -u_t + uu_y + \alpha v u_x + \beta u_{yy} + \alpha \beta u_{xx} &= 0, \\ u_x - v_x &= 0, \end{aligned} \tag{3.7}$$

where α and β are nonzero constants. The traveling wave variable (3.7) permits us converting Eq. (3.7) into the following ODE:

$$\begin{aligned} V u' + u u' + \alpha v u' + \beta (1 + \alpha) u'' &= 0, \\ u' - v' &= 0. \end{aligned} \tag{3.8}$$

Consider the homogeneous balance between u'' with $v u'$ and u'' with $u u'$ in (3.8) we get $n = 1$ and $m = 1$. Using the same idea in Sec. 3.1, we may choose the solution of Equation (3.7) in the form

$$u(\xi) = \sum_{i=-1}^1 \frac{A_i (\frac{G'}{G})^i}{[1 + \sigma (\frac{G'}{G})]^i}, \quad v(\xi) = \sum_{i=-1}^1 \frac{B_i (\frac{G'}{G})^i}{[1 + \sigma (\frac{G'}{G})]^i}, \tag{3.9}$$

where A_i , B_i and σ are constants to be determined and $G = G(\xi)$ satisfies Eq. (2.5). Substituting (3.4), (3.5) and (3.9) into Eqs. (3.8), we obtain a polynomial in the power of G . On equating the coefficients of this polynomial to zero, we get a system of algebraic equations which can be solved by *Maple* or *Mathematica* and we obtain the following results:

$$u_1 = \beta(1 + \alpha)h_1 - V - \alpha B_0 - \frac{2\beta C_5 h_1^2 \exp[-h_1(x + y - Vt)]}{-h_2 + C_5 h_1 \exp[-h_1(x + y - Vt)]};$$

$$v_1 = B_0 - \frac{2\beta C_5 h_1^2 \exp[-h_1(x + y - Vt)]}{-h_2 + C_5 h_1 \exp[-h_1(x + y - Vt)]};$$

$$u_2 = A_0 - \frac{2\beta C_5 h_1^2 (1 + \sigma h_1) \exp[-h_1(x + y - Vt)]}{-h_2 + C_5 h_1 (1 + \sigma h_1) \exp[-h_1(x + y - Vt)]};$$

$$v_2 = \frac{\beta(1 + \alpha)h_1 - V - A_0}{\alpha} - \frac{2\beta C_5 h_1^2 (1 + \sigma h_1) \exp[-h_1(x + y - Vt)]}{-h_2 + C_5 h_1 (1 + \sigma h_1) \exp[-h_1(x + y - Vt)]},$$

where $h_1 \neq 0$, $h_2 \neq 0$ and C_5 are arbitrary constants.

$$u_3 = 2\beta(1 + \alpha)h_1 - V - \alpha B_0 - \frac{4\beta C_1 h_1^2 \exp[-2h_1(x + y - Vt)]}{-h_3 + C_1 h_1 \exp[-2h_1(x + y - Vt)]},$$

$$v_3 = B_0 - \frac{4\beta C_1 h_1^2 \exp[-2h_1(x + y - Vt)]}{-h_3 + C_1 h_1 \exp[-2h_1(x + y - Vt)]},$$

$$u_4 = A_0 - \frac{4\beta C_1 h_1^2 (1 + \sigma h_1) \exp[-2h_1(x + y - Vt)]}{-h_3 + C_1 h_1 (1 + \sigma h_1) \exp[-2h_1(x + y - Vt)]},$$

$$v_4 = \frac{2\beta(1 + \alpha)h_1 - V - A_0}{\alpha} - \frac{4\beta C_1 h_1^2 (1 + \sigma h_1) \exp[-2h_1(x + y - Vt)]}{-h_3 + C_1 h_1 (1 + \sigma h_1) \exp[-2h_1(x + y - Vt)]},$$

where $h_1 \neq 0$, $h_3 \neq 0$ and C_1 are arbitrary constants.

$$u_5 = -\beta(1 + \alpha)h_1 - V - \alpha B_0 + \frac{2\beta C_4 h_1^2 \exp[h_1(x + y - Vt)]}{-h_0 + C_4 h_1 \exp[h_1(x + y - Vt)]},$$

$$v_5 = B_0 + \frac{2\beta C_4 h_1^2 \exp[h_1(x + y - Vt)]}{-h_0 + C_4 h_1 \exp[h_1(x + y - Vt)]},$$

$$u_6 = A_0 + \frac{2\beta C_4 h_1^2 (1 + \sigma h_1) \exp[h_1(x + y - Vt)]}{-h_0 + C_4 h_1 (1 + \sigma h_1) \exp[h_1(x + y - Vt)]},$$

$$v_6 = -\frac{\beta(1 + \alpha)h_1 + V + A_0}{\alpha} + \frac{2\beta C_4 h_1^2 (1 + \sigma h_1) \exp[h_1(x + y - Vt)]}{-h_0 + C_4 h_1 (1 + \sigma h_1) \exp[h_1(x + y - Vt)]},$$

where $h_0 \neq 0$, $h_1 \neq 0$ and C_4 are arbitrary constants.

Remark 3.1. The ansatz (2.4) and (2.5) proposed in this paper is more general than the ansatz in the original $(\frac{G'}{G})$ -expansion method [18], the improved $(\frac{G'}{G})$ -expansion method [5], the generalized $(\frac{G'}{G})$ -expansion method [10], the extended $(\frac{G'}{G})$ -expansion method [6] and the modified $(\frac{G'}{G})$ -expansion method [2]. If we set the parameters in (2.4) and (2.5) to special values the above methods can be recovered by our proposed method. Therefore our new method is more powerful than the above methods and some new types of traveling wave solutions and solitary wave solutions would be expected for other nonlinear equations.

Remark 3.2. By writing the exact solutions of nonlinear equations as polynomials of $\frac{(\frac{G'}{G})}{1+\sigma(\frac{G'}{G})}$, the equations can be changed into the nonlinear system of algebraic equations which can be solved with the help of symbolic computation. Therefore our method is a pure algebraic algorithm which can be applied to integrable system and non-integrable system.

Remark 3.3. All solutions presented in this article have been checked with Mathematica by putting them back into the original equations (3.1) and (2.2).

4. CONCLUSIONS

In this work, the enhanced $(\frac{G'}{G})$ -expansion method has been successfully applied to find the exact solutions of the $(2+1)$ -dimensional typical breaking soliton and Burgers equations. In this paper, we investigated the case when $G' = \sum_{i=0}^3 h_i G^i$. In the future, the proposed method can be extended to the case $i \geq 4$. The present work shows that the enhanced $(\frac{G'}{G})$ -expansion method is direct, concise and effective, and can be applied to other nonlinear equations in mathematical physics.

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REFERENCES

- [1] M. J. Ablowitz and P. A. Clarkson: *Solitons, Nonlinear Evolution Equations and Inverse Scattering Transform*, Cambridge University Press, Cambridge, 1991.
- [2] Z. Y. Bin and L. Chao: *Application of modified $(\frac{G'}{G})$ -expansion method to traveling wave solutions for Whitham-Broer-Kaup-Like equations*, Comm. Theor. Phys., **51**(2009), 664-670.
- [3] E. G. Fan: *Extended tanh-function method and its applications to nonlinear equations*, Phys. Lett., A, **277**(2000), 212-218.
- [4] E. G. Fan: *Multiple travelling wave solutions of nonlinear evolution equations using a unified algebraic method*, J. Phys., A, **35**(2002), 6853-6872.
- [5] S. Guo and Y. Zhou: *The extended $(\frac{G'}{G})$ -expansion method and its applications to the Whitham-Broer-Kaup-Like equations and coupled Hirota-Satsuma KdV equations*, Appl. Math. Comput., **215**(2010), 3214-3221.

- [6] S. Guo, Y. Zhou and C. Zhao: *The improved $(\frac{G'}{G})$ -expansion method and its applications to the Broer-Kaup equations and approximate long water wave equations*, Appl. Math. Comput., **216**(2010), 1965-1971.
- [7] J. H. He and X. H. Wu: *Exp-function method for nonlinear wave equations*, Chaos, Solitons and Fractals, **30**(2006), 700-708.
- [8] R. Hirota: *Exact N-soliton solutions of the wave equation of long waves in shallow-water and in nonlinear lattices*, J. Math. Phys., **14**(1973), 810-816.
- [9] R. Hirota and J. Satsuma: *Soliton solutions of a coupled Korteweg-de Vries equation*, Phys. Lett., A, **85**(1981), 407-408.
- [10] H. L. Lü, X. Q. Liu and L. Niu: *A generalized $(\frac{G'}{G})$ -expansion method and its applications to nonlinear evolution equations*, Appl. Math. Comput., **215**(2010), 3811-3816.
- [11] X. Z. Li and M. L. Wang: *A sub-ODE method for finding exact solutions of a generalized KdV-mKdV equation with high-order nonlinear terms*, Phys. Lett., A, **361**(2007), 115-118.
- [12] S. K. Liu, Z. T. Fu, S. D. Liu and Q. Zhao: *Jacobi elliptic function expansion method and periodic wave solutions of nonlinear wave equations*, Phys. Lett., A, **289**(2001), 69-74.
- [13] M. R. Miura: *Bäcklund Transformation*, Springer-Verlag, Berlin, 1978.
- [14] C. Rogers and W. F. Shadwick: *Backlund Transformation*, Academic, New York, 1982.
- [15] M. L. Wang, Y. B. Zhou and Z. B. Li: *Application of homogeneous balance method to exact solutions of nonlinear equations in mathematical physics*, Phys. Lett., A, **216**(1996), 67-75.
- [16] M. L. Wang and X. Li: *Extended F-expansion and periodic wave solutions for the generalized Zakharov equations*, Phys. Lett., A, **343**(2005), 48-54.
- [17] M. L. Wang and X. Li: *Applications of F-expansion to periodic wave solutions for a new Hamiltonian amplitude equation*, Chaos, Solitons and Fractals, **24**(2005), 1257-1268.
- [18] M. L. Wang, X. Z. Li and J. L. Zhang: *The $(\frac{G'}{G})$ -expansion method and travelling wave solutions of nonlinear evolution equations in mathematical physics*, Phys. Lett., A, **372**(2008), 417-423.
- [19] Z. Y. Yan: *Abundant families of Jacobi elliptic function solutions of the $(2 + 1)$ -dimensional integrable Davey-Stewartson-type equation via a new method*, Chaos, Solitons and Fractals, **18**(2003), 299-309.

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