J. Adv. Math. Stud.
Vol. **4**(2011), No. 2, 97-108
http://journal.fairpartners.ro

ON MODIFIED ITERATIVE METHOD FOR GENERALIZED EQUILIBRIUM PROBLEMS AND FIXED POINT PROBLEMS

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ABSTRACT. In this paper, we present a new iterative method for finding a common element of the set of solutions of generalized equilibrium problems and the set of fixed points of a strict pseudocontractive mapping. Furthermore, we prove that the proposed iterative method has strong convergence under some mild conditions imposed on algorithm parameters.

1. INTRODUCTION

Let C be a nonempty closed convex subset of a real Hilbert space H. A mapping $A\colon C \to H$ is called monotone if

$$\langle Ax - Ay, x - y\rangle \geq 0, \quad \forall x, y \in C,$$

and $A\colon C \to H$ is called inverse strongly monotone with coefficient $\alpha > 0$ if

$$\langle Ax - Ay, x - y\rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C.$$

It is obvious that any inverse strongly monotone mapping A is monotone and Lipschitz continuous. Recall that $T\colon C \to H$ is said to be a strict pseudo-contraction if there exists a constant $0 \leq k < 1$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C.$$

For such case, we also say that $T\colon C \to H$ is a k-strict pseudo-contraction. From Acedo and Xu [1], we know that, if $T\colon C \to H$ is a k-strict pseudocontractive mapping, then T satisfies Lipschitz condition, that is,

$$\|Tx - Ty\| \leq \frac{1 + k}{1 - k}\|x - y\|, \quad \forall x, y \in C.$$

We use $Fix(T)$ to denote the set of fixed points of the mapping T. It is well-known that the class of strict pseudo-contractions strictly includes the class of nonexpansive mappings which are mappings $S\colon C \to H$ such that $\|Sx - Sy\| \leq \|x - y\|$, for all $x, y \in C$.

Received: January 16, 2011. *Revised*: July 6, 2011.
2000 *Mathematics Subject Classification*: 58E35, 47H09, 47H10, 65J25.
Key words and phrases: Equilibrium problem, fixed point, monotone mapping, pseudocontractive mapping.

Let $A\colon C \to H$ be a nonlinear mapping. Now, we concern the following variational inequality problem which is to find $x^* \in C$ such that

$$\langle Ax^*, x - x^*\rangle \geq 0, \quad \forall x \in C. \tag{1.1}$$

The set of solutions of the variational inequality (1.1) is denoted by $VI(A,C)$. The variational inequality problem has been extensively studied in the literature. For finding an element of $Fix(S) \cap VI(A,C)$ under the assumption that a set $C \subset H$ is closed and convex, a mapping S of C into itself is nonexpasive and a mapping A of C into H is α-inverse strongly monotone, Takahashi and Toyoda [21] introduced the following iterative scheme:

$$\begin{cases} x_0 = x \in C, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)SP_C(x_n - \lambda_n Ax_n), \quad \forall n \geq 0, \end{cases} \tag{1.2}$$

where P_C is the metric projection of H onto C, $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\lambda_n\}$ is a sequence in $(0,2\alpha)$. They showed that, if $Fix(S) \cap VI(A,C)$ is nonempty, then the sequence $\{x_n\}$ generated by (1.2) converges weakly to some $z \in Fix(S) \cap VI(A,C)$.

Nadezhkina and Takahashi [17] introduced a so-called extragradient method motivated by the idea of Korpelevich [8] for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of a variational inequality problem and obtained a weak convergence theorem. Zeng and Yao [28] introduced the following extragradient method

$$\begin{cases} y_n = P_C(x_n - \lambda_n Ax_n), \\ x_{n+1} = \alpha_n x_0 + (1-\alpha_n)SP_C(x_n - \lambda_n Ay_n), \quad \forall n \geq 0. \end{cases} \tag{1.3}$$

They obtained the strong convergence of the sequences $\{x_n\}$ and $\{y_n\}$ generated by (1.3) to $P_{Fix(S)\cap VI(A,C)}(x_0)$ under some assumptions.

Very recently, Yao and Yao [26] presented the following new extragradient method

$$\begin{cases} y_n = P_C(x_n - \lambda_n Ax_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n SP_C(y_n - \lambda_n Ay_n), \quad \forall n \geq 0. \end{cases} \tag{1.4}$$

They also obtained the strong convergence of the sequences $\{x_n\}$ and $\{y_n\}$ generated by (1.4) to $P_{Fix(S)\cap VI(A,C)}(u)$ under some mild assumptions.

Let $A\colon C \to H$ be a nonlinear mapping and F be a bifunction of $C \times C$ into $\mathbb{R}$. Now, we concern the following generalized equilibrium problem which denoted by $EP(F,A)$ is to find $z \in C$ such that

$$F(z,y) + \langle Az, y - z\rangle \geq 0, \quad \forall y \in C. \tag{1.5}$$

In the case of $A = 0$, (1.5) is denoted by $EP(F)$. In the case of $F = 0$, (1.5) reduces to the variational inequality (1.2). The problem (1.5) is very general in the sense that it includes, as special cases, variational inequality problem, optimization problems, minimax problems, Nash equilibrium problem in noncooperative games and others. Please see [2], [3], [5]-[7], [9], [11]-[27].

For solving the generalized equilibrium problem (1.5), Moudafi [10] introduced an iterative algorithm and proved a weak convergence theorem and Ceng et al. [4] introduced an iterative algorithm for finding an element of $EP(F) \cap Fix(S)$. Takahashi and Takahashi [23] introduced a new iterative algorithm for finding an element of $EP(F, A) \cap Fix(S)$ and proved a strong convergence theorem.

Inspired by the above results, in this paper, we present a new iterative method for finding a common element of the set of solutions of generalized equilibrium problems and the set of fixed points of a strict pseudocontractive mapping. Furthermore, we prove that the proposed iterative method has strong convergence under some mild conditions imposed on algorithm parameters.

2. PRELIMINARIES

Let C be a nonempty closed convex subset of H. Throughout this paper, let us assume that a bifunction $F : C \times C \to \mathbb{R}$ satisfies the following conditions:

(H1) $F(x, x) = 0$ for all $x \in C$;
(H2) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;
(H3) for each $x, y, z \in C$, $\lim_{t \downarrow 0} F(tz + (1 - t)x, y) \leq F(x, y)$;
(H4) for each $x \in C$, $y \mapsto F(x, y)$ is convex and lower semicontinuous.

Lemma 2.1. ([6], [22]) *Let C be a nonempty closed convex subset of H and F be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying conditions* (H1)-(H4). *Let $r > 0$ and $x \in C$. Then there exists $z \in C$ such that $F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C$. Further, if $T_r(x) = \left\{ z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C \right\}$, then the following hold:*

(1) *T_r is single-valued and T_r is firmly nonexpansive, i.e., for any $x, y \in H$,*

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle;$$

(2) *$Fix(T_r) = EP(F, A)$ and $EP(F, A)$ is closed and convex.*

We also need the following lemmas for proving our main result:

Lemma 2.2. ([20]) *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose that $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n, \quad \forall n \geq 0$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.*

Lemma 2.3. ([1]) *Assume that C is a closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a k-strict pseudocontractive mapping. Then the mapping $I - T$ is demiclosed at zero. That is, if $\{x_n\}$ is a sequence in C such that $x_n \to x^*$ weakly and $(I - T)x_n \to 0$ strongly, then $(I - T)x^* = 0$.*

Lemma 2.4. ([1]) *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that $a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n, \quad \forall n \geq 1$ where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence such that*

(1) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(2) $\limsup_{n\to\infty} \delta_n/\gamma_n \le 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

3. MAIN RESULTS

Let C be a nonempty closed convex subset of a real Hilbert space H. Let $F, G\colon C \times C \to \mathbb{R}$ be two bifunctions which satisfy the assumptions (H1)-(H4). Let $A, B\colon C \to H$ be two inverse strongly monotone mappings with coefficients $\alpha > 0$ and $\beta > 0$, respectively. Let $r > 0$ and $\lambda > 0$. From Lemma 2.1, we know that, for fix $x \in C$, there exist $u \in C$ and $v \in C$ such that

$$F(u, y) + \frac{1}{r}\langle y - u, u - x\rangle \ge 0, \quad \forall y \in C,$$

and

$$G(v, z) + \frac{1}{\lambda}\langle z - v, v - x\rangle \ge 0, \quad \forall z \in C.$$

Set

$$P_A(x) = \left\{ u \in C : F(u, y) + \frac{1}{r}\langle y - u, u - x\rangle \ge 0,\ \forall y \in C \right\},$$

and

$$P_B(x) = \left\{ v \in C : G(v, z) + \frac{1}{\lambda}\langle z - v, v - x\rangle \ge 0,\ \forall z \in C \right\}.$$

Again, from Lemma 2.1, we know that $P_A(x)$ and $P_B(x)$ are single-valued and firmly nonexpansive mappings. Let $T\colon C \to C$ be a k-strict pseudocontractive mapping. We use Ω to denote $EP(F, A) \cap EP(G, B) \cap Fix(T)$, that is,

$$\Omega = EP(F, A) \cap EP(G, B) \cap Fix(T).$$

Now we state and prove the following strong convergence theorem:

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $F, G\colon C \times C \to \mathbb{R}$ be two bifunctions which satisfy the assumptions* (H1)-(H4). *Let $A, B\colon C \to H$ be two inverse strongly monotone mappings with coefficients $\alpha > 0$ and $\beta > 0$, respectively. Let $r \in (0, 2\alpha)$ and $\lambda \in (0, 2\beta)$. Let $T\colon C \to C$ be a k-strict pseudocontractive mapping such that $\Omega \neq \emptyset$. For fixed $u \in C$ and $x_0 \in C$ arbitrarily, let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be the sequences generated iteratively by*

$$\begin{cases} z_n = P_A(x_n - rAx_n), \\ y_n = P_B(z_n - \lambda B z_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n y_n + \delta_n T y_n, \quad \forall n \ge 0, \end{cases} \tag{3.1}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ are four sequences in $[0, 1]$such that:

(i) $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ *and* $(\gamma_n + \delta_n)k \le \gamma_n$ *for all* $n \ge 0$;

(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$*;*

(iii) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$ *and* $\liminf_{n\to\infty} \delta_n > 0$*;*

(iv) $\lim_{n\to\infty} \left(\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n}\right) = 0.$

Then the sequences $\{x_n\}$*,* $\{y_n\}$ *and* $\{z_n\}$ *generated by* (3.1) *converge strongly to* $P_\Omega(u)$*.*

Proof. We divide the proof into several steps:

Step 1. $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are bounded. In fact, take a point $x^* \in \Omega$ to get

$$\begin{aligned} \|x_{n+1} - x^*\| &= \|\alpha_n(u - x^*) + \beta_n(x_n - x^*) + \gamma_n(y_n - x^*) + \delta_n(Ty_n - x^*)\| \\ &\le \alpha_n\|u - x^*\| + \beta_n\|x_n - x^*\| + \|\gamma_n(y_n - x^*) + \delta_n(Ty_n - x^*)\|. \end{aligned} \tag{3.2}$$

From (3.1) and (3.2), we obtain

$$\begin{aligned} &\|\gamma_n(y_n - x^*) + \delta_n(Ty_n - x^*)\|^2 \\ &\quad = \gamma_n^2\|y_n - x^*\|^2 + \delta_n^2\|Ty_n - x^*\|^2 + 2\gamma_n\delta_n\langle Ty_n - x^*, y_n - x^*\rangle \\ &\quad \le (\gamma_n + \delta_n)^2\|y_n - x^*\|^2 + [\delta_n^2 k - (1-k)\gamma_n\delta_n]\|y_n - Ty_n\|^2 \\ &\quad = (\gamma_n + \delta_n)^2\|y_n - x^*\|^2 + \delta_n[(\gamma_n + \delta_n)k - \gamma_n]\|y_n - Ty_n\|^2 \\ &\quad \le (\gamma_n + \delta_n)^2\|y_n - x^*\|^2, \end{aligned}$$

which implies that

$$\|\gamma_n(y_n - x^*) + \delta_n(Ty_n - x^*)\| \le (\gamma_n + \delta_n)\|y_n - x^*\|. \tag{3.3}$$

Since A is α-inverse strongly monotone and B is β-inverse strongly monotone, we have

$$\|(I - rA)x - (I - rA)y\|^2 \le \|x - y\|^2 + r(r - 2\alpha)\|Ax - Ay\|^2 \tag{3.4}$$

and

$$\|(I - \lambda B)x - (I - \lambda B)y\|^2 \le \|x - y\|^2 + \lambda(\lambda - 2\beta)\|Bx - By\|^2. \tag{3.5}$$

It is clear that, if $0 < r \le 2\alpha$ and $0 < \lambda \le 2\beta$, then $(I - rA)$ and $(I - \lambda B)$ are all nonexpansive.

Note that $x^* = P_A(x^* - rAx^*) = P_B(x^* - \lambda Bx^*)$. From (3.1), we obtain

$$\begin{aligned} \|y_n - x^*\| &= \|P_B(z_n - \lambda Bz_n) - P_B(x^* - \lambda Bx^*)\| \\ &\le \|z_n - x^*\| \\ &= \|P_A(x_n - rAx_n) - P_A(x^* - rAx^*)\| \\ &\le \|x_n - x^*\|. \end{aligned} \tag{3.6}$$

It follows from (3.2), (3.3) and (3.6) that

$$\begin{aligned} \|x_{n+1} - x^*\| &\le \alpha_n\|u - x^*\| + \beta_n\|x_n - x^*\| + (\gamma_n + \delta_n)\|y_n - x^*\| \\ &\le \alpha_n\|u - x^*\| + \beta_n\|x_n - x^*\| + (\gamma_n + \delta_n)\|x_n - x^*\| \\ &= \alpha_n\|u - x^*\| + (1 - \alpha_n)\|x_n - x^*\|. \end{aligned}$$

By induction, we obtain

$$\|x_n - p\| \le \max\Big\{\|x_0 - x^*\|,\ \|u - x^*\|\Big\}, \quad \forall n \ge 0.$$

Hence $\{x_n\}$ is bounded. Consequently, we deduce immediately that $\{y_n\}$ and $\{z_n\}$ are also bounded.

Step 2. $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$

Define $x_{n+1} = \beta_n x_n + (1-\beta_n) w_n$ for all $n \ge 1$. It follows that

$$\begin{aligned} w_{n+1} - w_n &= \frac{x_{n+2} - \beta_{n+1}x_{n+1}}{1-\beta_{n+1}} - \frac{x_{n+1} - \beta_n x_n}{1-\beta_n} \\ &= \frac{\alpha_{n+1}u}{1-\beta_{n+1}} - \frac{\alpha_n u}{1-\beta_n} + \frac{\gamma_{n+1}(y_{n+1}-y_n) + \delta_{n+1}(Ty_{n+1} - Ty_n)}{1-\beta_{n+1}} \\ &\quad + \Big(\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n}\Big) y_n + \Big(\frac{\delta_{n+1}}{1-\beta_{n+1}} - \frac{\delta_n}{1-\beta_n}\Big) Ty_n. \end{aligned} \tag{3.7}$$

Observe that

$$\begin{aligned} &\|\gamma_{n+1}(y_{n+1}-y_n) + \delta_{n+1}(Ty_{n+1} - Ty_n)\|^2 \\ &= \gamma_{n+1}^2\|y_{n+1}-y_n\|^2 + \delta_{n+1}^2\|Ty_{n+1} - Ty_n\|^2 \\ &\quad + 2\gamma_{n+1}\delta_{n+1}\langle Ty_{n+1} - Ty_n, y_{n+1} - y_n\rangle \\ &\le \gamma_{n+1}^2\|y_{n+1}-y_n\|^2 + \delta_{n+1}^2[\|y_{n+1}-y_n\|^2 + k\|(y_{n+1} - Ty_{n+1}) \\ &\quad - (y_n - Ty_n)\|^2] + 2\gamma_{n+1}\delta_{n+1}[\|y_{n+1}-y_n\|^2 \\ &\quad - \frac{1-k}{2}\|(y_{n+1} - Ty_{n+1}) - (y_n - Ty_n)\|^2] \\ &= (\gamma_{n+1} + \delta_{n+1})^2\|y_{n+1}-y_n\|^2 \\ &\quad + \delta_{n+1}[(\gamma_{n+1}+\delta_{n+1})k - \gamma_{n+1}]\|(y_{n+1} - Ty_{n+1}) - (y_n - Ty_n)\|^2 \\ &\le (\gamma_{n+1} + \delta_{n+1})^2\|y_{n+1}-y_n\|^2, \end{aligned}$$

which implies that

$$\|\gamma_{n+1}(y_{n+1}-y_n) + \delta_{n+1}(Ty_{n+1} - Ty_n)\| \le (\gamma_{n+1} + \delta_{n+1})\|y_{n+1}-y_n\|. \tag{3.8}$$

Next, we estimate $\|y_{n+1} - y_n\|$ and $\|z_{n+1} - z_n\|$. From (3.1), we have

$$\begin{aligned} \|z_{n+1} - z_n\| &= \|P_A(x_{n+1} - rAx_{n+1}) - P_A(x_n - rAx_n)\| \\ &\le \|x_{n+1} - x_n\| \end{aligned}$$

and

$$\begin{aligned} \|y_{n+1} - y_n\| &= \|P_B(z_{n+1} - \lambda Bz_{n+1}) - P_B(z_n - \lambda Bz_n)\| \\ &\le \|z_{n+1} - z_n\| \\ &\le \|x_{n+1} - x_n\|. \end{aligned} \tag{3.9}$$

From (3.7), (3.8) and (3.9), we have

$$\|w_{n+1}-w_n\| \le \frac{\alpha_{n+1}}{1-\beta_{n+1}}(\|u\|+Ty_n\|)+\frac{\alpha_n}{1-\beta_n}(\|u\|+\|Ty_n\|) \\ +\|x_{n+1}-x_n\|+\Big|\frac{\gamma_{n+1}}{1-\beta_{n+1}}-\frac{\gamma_n}{1-\beta_n}\Big|(\|y_n\|+\|Ty_n\|).$$

Therefore, it follows that

$$\limsup_{n\to\infty}\Big(\|w_{n+1}-w_n\|-\|x_{n+1}-x_n\|\Big) \\ \le \limsup_{n\to\infty}\Big\{\frac{\alpha_{n+1}}{1-\beta_{n+1}}(\|u\|+Ty_n\|)+\frac{\alpha_n}{1-\beta_n}(\|u\|+\|Ty_n\|) \\ +\Big|\frac{\gamma_{n+1}}{1-\beta_{n+1}}-\frac{\gamma_n}{1-\beta_n}\Big|(\|y_n\|+\|Ty_n\|)\Big\} \le 0.$$

This together with Lemma 2.2 imply that $\lim_{n\to\infty}\|w_n-x_n\|=0$ and so

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{3.10}$$

Step 3. $\lim_{n\to\infty}\|x_n-Tx_n\|=0.$

It follows from (3.4) and (3.5) that

$$\begin{aligned}\|y_n-x^*\|^2 &= \|P_B(z_n-\lambda Bz_n)-P_B(x^*-\lambda Bx^*)\|^2\\ &\le \|z_n-x^*\|^2+\lambda(\lambda-2\beta)\|Bz_n-Bx^*\|^2\\ &= \|P_A(x_n-rAx_n)-P_A(x^*-rAx^*)\|^2+\lambda(\lambda-2\beta)\|Bz_n-Bx^*\|^2\\ &\le \|x_n-x^*\|^2+r(r-2\alpha)\|Ax_n-Ax^*\|^2\\ &\quad +\lambda(\lambda-2\beta)\|Bz_n-Bx^*\|^2.\end{aligned} \tag{3.11}$$

From (3.1) and (3.3), we also have

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &= \alpha_n\langle u-x^*,x_{n+1}-x^*\rangle+\beta_n\langle x_n-x^*,x_{n+1}-x^*\rangle\\ &\quad +\langle\gamma_n(y_n-x^*)+\delta_n(Ty_n-x^*),x_{n+1}-x^*\rangle\\ &\le \alpha_n\langle u-x^*,x_{n+1}-x^*\rangle+\beta_n\|x_n-x^*\|\|x_{n+1}-x^*\|\\ &\quad +\|\gamma_n(y_n-x^*)+\delta_n(Ty_n-x^*)\|\|x_{n+1}-x^*\|\\ &\le \alpha_n\langle u-x^*,x_{n+1}-x^*\rangle+\frac{\beta_n}{2}(\|x_n-x^*\|^2+\|x_{n+1}-x^*\|^2)\\ &\quad +\frac{(\gamma_n+\delta_n)}{2}(\|y_n-x^*\|^2+\|x_{n+1}-x^*\|^2),\end{aligned}$$

that is,

$$\|x_{n+1}-x^*\|^2 \le \frac{2\alpha_n}{1+\alpha_n}\langle u-x^*,x_{n+1}-x^*\rangle+\frac{\beta_n}{1+\alpha_n}\|x_n-x^*\|^2 \\ +\frac{\gamma_n+\delta_n}{1+\alpha_n}\|y_n-x^*\|^2. \tag{3.12}$$

Thus it follow from (3.11) and (3.12) that

$$\begin{aligned}\|x_{n+1}-x^*\|^2 \le& \frac{2\alpha_n}{1+\alpha_n}\|u-x^*\|\|x_{n+1}-x^*\| + \frac{\beta_n}{1+\alpha_n}\|x_n-x^*\|^2\\ &+\frac{\gamma_n+\delta_n}{1+\alpha_n}\|x_n-x^*\|^2 + \frac{\gamma_n+\delta_n}{1+\alpha_n}r(r-2\alpha)\|Ax_n-Ax^*\|^2\\ &+\frac{\gamma_n+\delta_n}{1+\alpha_n}\lambda(\lambda-2\beta)\|Bz_n-Bx^*\|^2\end{aligned}$$

and hence

$$\begin{aligned}&\frac{\gamma_n+\delta_n}{1+\alpha_n}r(2\alpha-r)\|Ax_n-Ax^*\|^2 + \frac{\gamma_n+\delta_n}{1+\alpha_n}\lambda(2\beta-\lambda)\|Bz_n-Bx^*\|^2\\ &\quad\le \frac{2\alpha_n}{1+\alpha_n}\|u-x^*\|\|x_{n+1}-x^*\| + \|x_n-x^*\|^2 - \|x_{n+1}-x^*\|^2\\ &\quad\le \frac{2\alpha_n}{1+\alpha_n}\|u-x^*\|\|x_{n+1}-x^*\| + (\|x_n-x^*\| + \|x_{n+1}-x^*\|)\|x_n-x_{n+1}\|.\end{aligned}$$

From $\liminf_{n\to\infty}\frac{\gamma_n+\delta_n}{1+\alpha_n}r(2\alpha-r)>0$, $\liminf_{n\to\infty}\frac{\gamma_n+\delta_n}{1+\alpha_n}\lambda(2\beta-\lambda)>0$, $\|x_n-x_{n+1}\|\to 0$ and $\alpha_n\to 0$, we have

$$\lim_{n\to\infty}\|Ax_n-Ax^*\|=0, \quad \lim_{n\to\infty}\|Bz_n-Bx^*\|=0. \tag{3.13}$$

Noting that P_A and P_B are all firmly nonexpansive, then we have

$$\begin{aligned}\|z_n-x^*\|^2 \le& \langle (x_n-rAx_n)-(x^*-rAx^*), z_n-x^*\rangle\\ =& \frac{1}{2}\Big(\|(x_n-rAx_n)-(x^*-rAx^*)\|^2 + \|z_n-x^*\|^2\\ &-\|(x_n-rAx_n)-(x^*-rAx^*)-(z_n-x^*)\|^2\Big)\\ \le& \frac{1}{2}\Big(\|x_n-x^*\|^2 + \|z_n-x^*\|^2 - \|(x_n-z_n)-r(Ax_n-Ax^*)\|^2\Big)\\ =& \frac{1}{2}\Big(\|x_n-x^*\|^2 + \|z_n-x^*\|^2 - \|x_n-z_n\|^2\\ &+2r\langle x_n-z_n, Ax_n-Ax^*\rangle - r^2\|Ax_n-Ax^*\|^2\Big)\end{aligned}$$

and

$$\begin{aligned}\|y_n-x^*\|^2 \le& \langle (z_n-\lambda Bz_n)-(x^*-\lambda Bx^*), y_n-x^*\rangle\\ =& \frac{1}{2}\Big(\|(z_n-\lambda Bz_n)-(x^*-\lambda Bx^*)\|^2 + \|y_n-x^*\|^2\\ &-\|(z_n-\lambda Bz_n)-(x^*-\lambda Bx^*)-(y_n-x^*)\|^2\Big)\\ \le& \frac{1}{2}\Big(\|z_n-x^*\|^2 + \|y_n-x^*\|^2 - \|(z_n-y_n)-\lambda(Bz_n-Bx^*)\|^2\Big)\end{aligned}$$

$$= \frac{1}{2}\Big(\|z_n - x^*\|^2 + \|y_n - x^*\|^2 - \|z_n - y_n\|^2 \\ +2\lambda\langle z_n - y_n, Bz_n - Bx^*\rangle - \lambda^2\|Bz_n - Bx^*\|^2\Big).$$

Hence it follows that

$$\|z_n - x^*\|^2 \leq \|x_n - x^*\|^2 - \|x_n - z_n\|^2 + 2r\|x_n - z_n\|\,\|Ax_n - Ax^*\| \tag{3.14}$$

and

$$\|y_n - x^*\|^2 \leq \|z_n - x^*\|^2 - \|z_n - y_n\|^2 + 2\lambda\|z_n - y_n\|\,\|Bz_n - Bx^*\|. \tag{3.15}$$

Therefore, combining (3.14) and (3.15), we get

$$\|y_n - x^*\|^2 \leq \|x_n - x^*\|^2 - \|x_n - z_n\|^2 - \|z_n - y_n\|^2 \\ +2r\|x_n - z_n\|\|Ax_n - Ax^*\| + 2\lambda\|z_n - y_n\|\|Bz_n - Bx^*\|. \tag{3.16}$$

From (3.12) and (3.16), we get

$$\|x_{n+1} - x^*\|^2 \leq \frac{2\alpha_n}{1+\alpha_n}\|u - x^*\|\|x_{n+1} - x^*\| + \frac{\beta_n}{1+\alpha_n}\|x_n - x^*\|^2 \\ +\frac{\gamma_n + \delta_n}{1+\alpha_n}\Big[\|x_n - x^*\|^2 - \|x_n - z_n\|^2 - \|z_n - y_n\|^2 \\ +2r\|x_n - z_n\|\|Ax_n - Ax^*\| + 2\lambda\|z_n - y_n\|\|Bz_n - Bx^*\|\Big].$$

It follows that

$$\frac{\gamma_n + \delta_n}{1+\alpha_n}\Big[\|x_n - z_n\|^2 + \|z_n - y_n\|^2\Big] \\ \leq \frac{2\alpha_n}{1+\alpha_n}\|u - x^*\|\|x_{n+1} - x^*\| + (\|x_n - x^*\| + \|x_{n+1} - x^*\|)\|x_n - x_{n+1}\| \\ +\frac{\gamma_n + \delta_n}{1+\alpha_n}\Big[2r\|x_n - z_n\|\|Ax_n - Ax^*\| + 2\lambda\|z_n - y_n\|\|Bz_n - Bx^*\|\Big].$$

From (3.13), $\liminf_{n\to\infty} \frac{\gamma_n + \delta_n}{1+\alpha_n} > 0$, $\lim_{n\to\infty} \alpha_n = 0$ and $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$, we deduce that

$$\lim_{n\to\infty} \|x_n - z_n\| = 0, \quad \lim_{n\to\infty} \|z_n - y_n\| = 0. \tag{3.17}$$

Thus, from (3.1), (3.10) and (3.17), we have

$$\lim_{n\to\infty} \|Ty_n - x_n\| = 0.$$

At the same time, we note that

$$\|x_n - Tx_n\| \leq \|x_n - Ty_n\| + \|Ty_n - Tx_n\| \leq \|x_n - Ty_n\| + \frac{1+k}{1-k}\|y_n - x_n\|$$

and so

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0 \text{ and } \lim_{n\to\infty} \|y_n - Ty_n\| = 0.$$

Step 4. $\limsup_{n\to\infty}\langle u - x^*, x_n - x^*\rangle \le 0$ where $x^* = P_\Omega(u)$.

To show this inequality, take a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle u - x^*, x_n - x^*\rangle = \lim_{i\to\infty}\langle u - x^*, x_{n_i} - x^*\rangle.$$

Without loss of generality, we may assume that $x_{n_i} \rightharpoonup \omega$. Since C is closed and convex, C is weakly closed. So, we have $\omega \in C$. Let us show $\omega \in \Omega$. By the similar argument as that in [23], we have $\omega \in EP(F, A)$ and $\omega \in EP(G, B)$.

Next, we show that $\omega \in Fix(T)$. In fact, this follows from $\|x_n - Tx_n\| \to 0$ and Lemma 2.3. Hence $\omega \in \Omega$. Further, we have

$$\limsup_{n\to\infty}\langle u - x^*, x_n - x^*\rangle = \lim_{i\to\infty}\langle u - x^*, x_{n_i} - x^*\rangle = \langle u - x^*, \omega - x^*\rangle \le 0.$$

Step 5. $x_n \to x^*$ as $n \to \infty$.

From (3.3), we have

$$\|x_{n+1} - x^*\|^2 \le \left(1 - \frac{2\alpha_n}{1+\alpha_n}\right)\|x_n - x^*\|^2 + \frac{2\alpha_n}{1+\alpha_n}\langle u - x^*, x_{n+1} - x^*\rangle.$$

It is clear that $\sum_{n=0}^{\infty}\frac{2\alpha}{1+\alpha_n} = \infty$. Hence all the conditions of Lemma 2.4 are satisfied. Therefore, we immediately deduce that $x_n \to x^*$ as $n \to \infty$. Consequently, we also have $y_n \to x^*$ and $z_n \to x^*$ as $n \to \infty$. This completes the proof. □

Corollary 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $F, G\colon C \times C \to \mathbb{R}$ be two bifunctions which satisfy the assumptions* (H1)-(H4). *Let $A, B\colon C \to H$ be two inverse strongly monotone mappings with coefficients $\alpha > 0$ and $\beta > 0$, respectively. Let $r \in (0, 2\alpha)$ and $\lambda \in (0, 2\beta)$. Let $T\colon C \to C$ be a nonexpansive mapping such that $\Omega \neq \emptyset$. For fixed $u \in C$ and $x_0 \in C$ arbitrarily, let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be the sequences generated iteratively by*

$$\begin{cases} z_n = P_A(x_n - rAx_n), \\ y_n = P_B(z_n - \lambda B z_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n y_n + \delta_n T y_n, \quad \forall n \ge 0 \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ are four sequences in $[0, 1]$ *such that:*

(i) $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ *for all* $n \ge 0$;

(ii) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=0}^{\infty}\alpha_n = \infty$;

(iii) $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$ *and* $\liminf_{n\to\infty}\delta_n > 0$;

(iv) $\lim_{n\to\infty}\left(\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n}\right) = 0.$

Then the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ converge strongly to $P_\Omega(u)$.

We note that, if $T\colon C \to C$ is a k-strict pseudocontractive mapping, then the mapping $kI + (1-k)T$ is nonexpansive. Hence, from Corollary 3.1, we have the following conclusion:

Corollary 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $F, G\colon C \times C \to \mathbb{R}$ be two bifunctions which satisfy the assumptions* (H1)-(H4). *Let $A, B\colon C \to H$ be two inverse strongly monotone mappings with coefficients $\alpha > 0$ and $\beta > 0$, respectively. Let $r \in (0, 2\alpha)$ and $\lambda \in (0, 2\beta)$. Let $T\colon C \to C$ be a nonexpansive mapping such that $\Omega \neq \emptyset$. For fixed $u \in C$ and $x_0 \in C$ arbitrarily, let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be the sequences generated iteratively by*

$$\begin{cases} z_n = P_A(x_n - rAx_n), \\ y_n = P_B(z_n - \lambda B z_n), \\ x_{n+1} = \alpha'_n u + \beta'_n x_n + \gamma'_n T y_n, \quad \forall n \geq 0, \end{cases}$$

where $\{\alpha'_n\}$, $\{\beta'_n\}$, $\{\gamma'_n\}$ are three sequences in $[0,1]$ such that:

(i) $\alpha'_n + \beta'_n + \gamma'_n = 1$ *for all* $n \geq 0$;

(ii) $\lim_{n\to\infty} \alpha'_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha'_n = \infty$;

(iii) $0 < \liminf_{n\to\infty} \beta'_n \leq \limsup_{n\to\infty} \beta'_n < 1$.

Then the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ converge strongly to $P_\Omega(u)$.

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