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SOME MARKOV-BERNSTEIN TYPE INEQUALITIES AND CERTAIN CLASS OF SOBOLEV POLYNOMIALS

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Abstract. Let (μ_0, μ_1) be a vector of non-negative measures on the real line, with μ_0 not identically zero, finite moments of all orders, compact or non compact supports, and at least one of them having an infinite number of points on its support. We show that for any linear operator T on the space of polynomials with complex coefficients and any integer $n \geq 0$, there is a constant $\gamma_n(T) \geq 0$, such that

$$\|Tp\|_S \leq \gamma_n(T)\|p\|_S,$$

for any polynomial p of degree $\leq n$, where $\gamma_n(T)$ is independent of p, and

$$\|p\|_S = \left\{ \int |p(x)|^2 d\mu_0(x) + \int |p'(x)|^2 d\mu_1(x) \right\}^{\frac{1}{2}}.$$

We find a formula for the best possible value ${}_n(T)$ of $\gamma_n(T)$ and inequalities for ${}_n(T)$. Also, we give some examples when T is a differentiation operator and (μ_0, μ_1) is a vector of orthogonalizing measures for classical orthogonal polynomials.

1. INTRODUCTION

Markov-Bernstein type inequalities are estimates connecting the norm of derivatives of a polynomial with the norm of the polynomial itself. These inequalities are interesting by themselves and fundamental for the proof of many inverse theoremsin polynomial approximation theory, (see [3], [7], [8], [15], [16], [17], [21], [24], [25], [26], [27]). For instance, let $\|\cdot\|_{L^2((a,b),w)}$ be a weighted L^2-norm on the space $\mathbb{P}(\mathbb{C})$ of polynomials with complex coefficients, given by

$$\|p\|_{L^2((a,b),w)} = \left(\int_a^b |p(x)|^2 w(x) dx \right)^{1/2},$$

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where w is an integrable function on (a,b), $-\infty \le a < b \le \infty$, such that $w > 0$ on (a,b) and all moments

$$r_n := \int_a^b x^n w(x)dx$$

are finite. Then, using the orthonormal polynomial system $\{p_n\}_{n=0}^{\infty}$ with respect to the positive measure $w(x)dx$, Mirsky [23] showed that there exists a constant $\gamma_n = \gamma_n(a,b,w)$ such that

$$\|p'\|_{L^2((a,b),w)} \le \gamma_n \|p\|_{L^2((a,b),w)}, \quad \forall p \in \mathbb{P}_n(\mathbb{C}), \tag{1.1}$$

where $\mathbb{P}_n(\mathbb{C})$ is the space of polynomials with complex coefficients of degree at most n. Furthermore, the best constant γ_n^* in (1.1) satisfies

$$\gamma_n^* := \sup_{p\in\mathbb{P}_n(\mathbb{C})} \left\{ \|p'\|_{L^2((a,b),w)} : \|p\|_{L^2((a,b),w)} = 1 \right\} \le \left\{ \sum_{\nu=1}^{n} \nu \|p_\nu'\|^2_{L^2((a,b),w)} \right\}^{\frac{1}{2}}. \tag{1.2}$$

Note that the main interest of this result is however qualitative, since the bound specified by (1.2) can be very crude. In fact, when $w(x) = e^{-x^2}$ on $(-\infty,\infty)$, the estimate (1.2) becomes

$$\gamma_n^* \le \left(\sum_{\nu=1}^{n} 2\nu^2 \right)^{\frac{1}{2}} = \sqrt{\frac{1}{3} n(n+1)(2n+1)} = O\left(n^{\frac{3}{2}}\right).$$

The contrast between this estimate and the classic result of Schmidt [28], which establishes $\gamma_n^* = \sqrt{2n}$, is evident.

In 1987, P. Dörfler [5] extended Mirsky's inequality to higher order derivatives and he suggested a way to find the best constant involved. Seven years later, A. Guessab and G. V. Milovanović [11] found the best constant for higher order derivatives when w is a weight for classical orthogonal polynomials. Finally, K. H. Kwon and D. W. Lee [13] showed that Markov-Bernstein type inequalities in weighted L^2-spaces hold not only for derivatives but also for any linear operator defined on $\mathbb{P}(\mathbb{C})$ even when the measure $w(x)dx$ is replaced by any positive Borel measure $d\mu(x)$. They also gave another way to find the best constant involved, which is easier to apply than Dörfler's way. Another interesting results about the asymptotic behavior of sharp constants in the Markov-Bernstein type inequalities involving Sobolev norms may be found in[2].

The aim of this paper is to introduce and study Markov-Bernstein type inequalities when the involved norm is associated to an inner product in a suitable Sobolev space of functions, containing the linear subspace $\mathbb{P}(\mathbb{C})$ of polynomials with complex coefficients. In order to do this, we will follow the ideas of [13], mainly. Our results give a partial answer to an open problem posted in 2008 by Professor F. Marcellán at a conference on Constructive Theory of Functions held in Campos do Jordão, Brazil. However, another approach related to the solutions of such problem may be found in [18].

The outline of the paper is as follows. Section 2 is devoted to some basic facts and notation. In section 3 we present the main results of the paper, Theorems 3.1, 3.2, 3.3 and we deduce some of their consequences. Section 4 contains some illustrative examples for the case $T = \dfrac{d^r}{dx^r}$, $1 \le r \le n$.

2. PREVIOUS DEFINITIONS AND NOTATIONS

We denote the degree of a polynomial p by $\deg(p)$, with the convention that $\deg(0) = -1$. If A is any matrix, its transpose is denoted by A^t.

Let (μ_0, μ_1) be a vector of non-negative measures on the real line, with compact or non compact supports Δ_0 and Δ_1, respectively, and at least one of them having an infinite number of points. From now on, we always assume that

$$\left| \int_{\Delta_j} x^n d\mu_j(x) \right| < \infty, \quad n = 0, 1, 2, \ldots, \text{ and } j = 0, 1.$$

Then,

$$\langle f, g \rangle_S := \int_{\Delta_0} f(x)\overline{g(x)} d\mu_0(x) + \int_{\Delta_1} f'(x)\overline{g'(x)} d\mu_1(x), \tag{2.1}$$

defines an inner product in a suitable Sobolev space of functions, containing the linear subspace $\mathbb{P}(\mathbb{C})$ of polynomials with complex coefficients. Standard arguments allow us to assure the existence of an orthonormal sequence $\{q_\nu\}_{\nu \ge 0}$ with respect to (2.1). That is, $\deg(q_\nu) = \nu$, and $\langle q_\nu, q_m \rangle_S = \delta_{\nu m}$, for $\nu, m \ge 0$. We use the following notations: for any a with $1 \le a \le \infty$,

$$\begin{aligned} \|c\|_a &:= \left(\sum_{\nu=0}^{n} |c_\nu|^a \right)^{\frac{1}{a}}, \text{ if } 1 \le a < \infty, \\ \|c\|_\infty &:= \max\{|c_\nu| : 0 \le \nu \le n\}, \end{aligned}$$

for any vector $c = (c_0, c_1, \ldots, c_n) \in \mathbb{C}^{n+1}$ and

$$\|p\|_S := \left\{ \int_{\Delta_0} |p(x)|^2 d\mu_0(x) + \int_{\Delta_1} |p'(x)|^2 d\mu_1(x) \right\}^{\frac{1}{2}}, \quad p \in \mathbb{P}(\mathbb{C}).$$

Let T be any linear operator from $P_n(\mathbb{C})$ into $\mathbb{P}(\mathbb{C})$, where n is any fixed non-negative integer. Then T is bounded so that there is a constant $\gamma_n(T)$, depending only on n and T, such that

$$\|Tp\|_S \le \gamma_n(T) \|p\|_S, \quad p \in \mathbb{P}_n(\mathbb{C}). \tag{2.2}$$

We denote by ${}_n(T)$ the smallest possible value of $\gamma_n(T)$ in (2.2). That is, ${}_n(T)$ is the operator normof T: ${}_n(T) := \sup\limits_{\|p\|_S = 1} \|Tp\|_S$, $p \in \mathbb{P}_n(\mathbb{C})$.

3. MAIN RESULTS

Let $\{\phi_\nu\}_{\nu\geq 0}$ be any sequence of polynomials with $\deg(\phi_\nu) = \nu$, $\nu \geq 0$; then $\{\phi_\nu\}_{\nu\geq 0}$ is a basis of $\mathbb{P}(\mathbb{C})$.

Theorem 3.1. *Let n be any fixed non-negative integer, T be any linear operator from $P_n(\mathbb{C})$ into $\mathbb{P}(\mathbb{C})$ and ${}_n(T)$ its operator norm with respect to the Sobolev norm induced by* (2.1). *Then*

$$ {}_n(T) = \sup_{c\in\mathbb{C}^{n+1}\setminus\{0\}} \sqrt{\frac{cD^tR(m+1)\overline{D}\overline{c}^t}{cR(n+1)\overline{c}^t}}, $$

where $D = \left(d_\nu^j\right)_{0\leq j\leq m, 0\leq \nu\leq n}$ and $R(n+1) = (r_{ij})_{0\leq i,j\leq n}$ are matrices whose entries are given by

$$ (T\phi_\nu)(x) := \sum_{j=0}^{m} d_\nu^j \phi_j(x), \quad m := \max_{0\leq\nu\leq n} \deg(T\phi_\nu) $$

and

$$ r_{ij} := \int_{\Delta_0} \phi_i(x)\overline{\phi_j(x)}d\mu_0(x) + \int_{\Delta_1} \phi_i'(x)\overline{\phi_j'(x)}d\mu_1(x). $$

Proof. For any polynomial $p \in \mathbb{P}_n(\mathbb{C})$, we can write it as $p(x) = \sum_{\nu=0}^{n} c_\nu\phi_\nu(x)$. Then, we have

$$
\begin{aligned}
\|p\|_S^2 &= \int_{\Delta_0} |p(x)|^2 d\mu_0(x) + \int_{\Delta_1} |p'(x)|^2 d\mu_1(x) \\
&= \sum_{\nu=0}^{n}\sum_{j=0}^{n} c_\nu\overline{c}_j \left[\int_{\Delta_0} \phi_\nu(x)\overline{\phi_j(x)}d\mu_0(x) + \int_{\Delta_1} \phi_\nu'(x)\overline{\phi_j'(x)}d\mu_1(x)\right] \\
&= cR(n+1)\overline{c}^t; \\
\|Tp\|_S^2 &= \int_{\Delta_0} |(Tp)(x)|^2 d\mu_0(x) + \int_{\Delta_1} \left|\frac{d}{dx}(Tp)(x)\right|^2 d\mu_1(x) \\
&= \int_{\Delta_0} \left[\sum_{\nu=0}^{n} c_\nu \left(\sum_{i=0}^{m} d_\nu^i \phi_i(x)\right)\right] \left[\sum_{l=0}^{n} \overline{c}_l \left(\sum_{j=0}^{m} \overline{d_l^j}\, \overline{\phi_j(x)}\right)\right] d\mu_0(x) \\
&\quad + \int_{\Delta_1} \left[\sum_{\nu=0}^{n} c_\nu \left(\sum_{i=0}^{m} d_\nu^i \phi_i'(x)\right)\right] \left[\sum_{l=0}^{n} \overline{c}_l \left(\sum_{j=0}^{m} \overline{d_l^j}\, \overline{\phi_j'(x)}\right)\right] d\mu_1(x)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i=0}^{m}\left(\sum_{\nu=0}^{n} c_\nu d_\nu^i\right)\sum_{j=0}^{m}\left(\sum_{l=0}^{n}\overline{c}_l\overline{d_l^j}\right)\int_{\Delta_0}\phi_i(x)\overline{\phi_j(x)}d\mu_0(x) \\
&\quad +\sum_{i=0}^{m}\left(\sum_{\nu=0}^{n} c_\nu d_\nu^i\right)\sum_{j=0}^{m}\left(\sum_{l=0}^{n}\overline{c}_l\overline{d_l^j}\right)\int_{\Delta_1}\phi_i'(x)\overline{\phi_j'(x)}d\mu_1(x) \\
&= \sum_{i=0}^{m}\left(\sum_{\nu=0}^{n} c_\nu d_\nu^i\right)\sum_{j=0}^{m}\left(\sum_{l=0}^{n}\overline{c}_l\overline{d_l^j}\right) r_{ij} \\
&= (Dc^t)^t R(m+1)(\overline{D}\overline{c}^t).
\end{aligned}
$$

Therefore,

$$
{}_n(T) = \sup_{p\in\mathbb{P}_n(\mathbb{C})\setminus\{0\}} \frac{\|Tp\|_S}{\|p\|_S} = \sup_{c\in\mathbb{C}^{n+1}\setminus\{0\}} \sqrt{\frac{cD^tR(m+1)\overline{D}\overline{c}^t}{cR(n+1)\overline{c}^t}},
$$

which completes the proof. □

Theorem 3.2. *Let T, ${}_n(T)$ be the same as in Theorem 3.1, with $\{q_\nu\}_{\nu\geq 0}$ an orthonormal sequence with respect to (2.1). Then*

$$
{}_n(T) = \sup_{\substack{\|c\|_2=1\\ c\in\mathbb{C}^{n+1}}} \left\{\sum_{j=0}^{m}\left|\sum_{\nu=0}^{n} c_\nu d_\nu^j\right|^2\right\}^{\frac{1}{2}}, \tag{3.1}
$$

where $D = \left(d_\nu^j\right)_{0\leq j\leq m, 0\leq\nu\leq n}$ is the matrix whose entries are given by

$$
(Tq_\nu)(x) := \sum_{j=0}^{m} d_\nu^j q_j(x). \tag{3.2}
$$

Moreover, ${}_n(T)$ satisfies the following estimate: for any a with $1\leq a\leq\infty$,

$$
\max_{0\leq\nu\leq n}\|Tq_\nu\|_S \leq {}_n(T) \leq C(a,n)\|(\|Tq_0\|_S,\ldots,\|Tq_n\|_S)\|_b, \tag{3.3}
$$

where $C(a,n) = \sup_{c\in\mathbb{C}^{n+1}\setminus\{0\}} \frac{\|c\|_a}{\|c\|_2}$ and $\frac{1}{a}+\frac{1}{b}=1$.

Proof. If we take an orthonormal basis $\{q_0,\ldots,q_n\}$ of $\mathbb{P}_n(\mathbb{C})$ in Theorem 3.1, then the matrix $R(n+1)$ becomes an identity matrix. Hence, $cR(n+1)\overline{c}^t = \|c\|^2$ and so

$$
{}_n(T) = \sup_{\substack{\|c\|_2=1\\ c\in\mathbb{C}^{n+1}}} \sqrt{cD^t\overline{D}\overline{c}^t},
$$

which is the matrix form of the equation (3.1). On the other hand, for any a with $1 \leq a \leq \infty$ and for any $p(x) = \sum_{\nu=0}^{n} c_\nu q_\nu(x)$ in $\mathbb{P}_n(\mathbb{C})$, Hölder's inequality implies

$$\|Tp\|_S \leq \sum_{\nu=0}^{n} |c_\nu| \|Tq_\nu\|_S \leq \|c\|_a \|(\|Tq_0\|_S, \dots, \|Tq_n\|_S)\|_b, \text{ with } \frac{1}{a} + \frac{1}{b} = 1.$$

Let us consider the linear operator $H : (\mathbb{P}_n(\mathbb{C}), \| \cdot \|_S) \to (\mathbb{C}^{n+1}, \| \cdot \|_a)$, defined by

$$H(p) = H\left(\sum_{\nu=0}^{n} c_\nu q_\nu\right) = (c_0, \dots, c_n).$$

Then $\|c\|_a = \|H(p)\|_a \leq \|H\| \|p\|_S = C(a, n)\|p\|_S$, hence

$$\|Tp\|_S \leq C(a, n) \|(\|Tq_0\|_S, \dots, \|Tq_n\|_S)\|_b \|p\|_S, \quad p \in \mathbb{P}_n(\mathbb{C}),$$

which gives the upper bound for $_n(T)$ in (3.3). The lower bound for $_n(T)$ in (3.3) is immediate since $\|q_\nu\|_S = 1$, $0 \leq \nu \leq n$. □

Note that we can easily deduce that the constant $C(a, n)$ is given by

$$C(a, n) = \begin{cases} (n+1)^{\frac{1}{a} - \frac{1}{2}}, \text{ if } 1 \leq a \leq 2, \\ \\ 1, \text{ if } 2 \leq a \leq \infty. \end{cases}$$

Then for the cases $a = 1$ and $a = 2$, we have the following result.

Corollary 3.1. *Let T and $_n(T)$ be the same as in Theorem* 3.2. *Then*

$$\max_{0 \leq \nu \leq n} \|Tq_\nu\|_S \leq {}_n(T) \leq \sqrt{n+1} \max_{0 \leq \nu \leq n} \|Tq_\nu\|_S. \tag{3.4}$$

$$\max_{0 \leq \nu \leq n} \|Tq_\nu\|_S \leq {}_n(T) \leq \left\{ \sum_{\nu=0}^{n} \|Tq_\nu\|_S^2 \right\}^{\frac{1}{2}}. \tag{3.5}$$

In the case $\mu_1 = 0$, the best constant $_n(T)$ and the inequality (3.5) (when μ_0 is absolutely continuous) were obtained by Dörfler [5] and Kwon and Lee [13], using different approaches. Since $_n(T)$ is the smallest value λ satisfying

$$\frac{cD^t R(m+1)\overline{D}\overline{c}^t}{cR(n+1)\overline{c}^t} \leq \lambda^2, \quad c \in \mathbb{C}^{n+1} \setminus \{0\},$$

then, $_n(T)$ is the smallest constant λ such that $\lambda^2 R(n+1) - D^t R(m+1)\overline{D}$ is positive-semidefinite. Since μ_0 and μ_1 are positive measures on $\mathbb{C}$, $R(n+1)$ is Hermitian for $n \geq 0$ and $D^t R(m+1)\overline{D}$ is Hermitian and positive-semidefinite for $m \geq 0$.

If $R(n+1)$ and $D^t R(m+1)\overline{D}$ commute, then they have $n+1$ common linearly independent eigenvectors $\{u_i\}_{i=0}^n$ such that

$$R(n+1)u_i = \zeta_i u_i, \quad D^t R(m+1)\overline{D}u_i = \eta_i u_i, \quad i = 0, \dots, n, \tag{3.6}$$

since both $R(n+1)$ and $D^t R(m+1)\overline{D}$ are Hermitian (see for example, [14]).

Consequently, we have the following result.

Theorem 3.3. *Let* T *and* ${}_n(T)$ *be the same as in Theorem* 3.1. *If* $R(n+1)$ *and* $D^t R(m+1)\overline{D}$ *commute, then*

$$ {}_n(T) = \max_{0\le i\le n} \sqrt{\frac{\eta_i}{\zeta_i}}, $$

where $\zeta_i > 0$ *and* $\eta_i \ge 0$, $0 \le i \le n$, *are the eigenvalues of* $R(n+1)$ *and* $D^t R(m+1)\overline{D}$, *respectively, as in* (3.6).

Proof. It is sufficient to follow the proof offered in [13, Theorem 2.5], taking into consideration Theorem 3.1. ☐

Combining the above result with the developed approach by Milovanović (see [20]), which consists of finding the largest eigenvalue of a matrix of Gram's type, we can obtain an interesting result when $T = \dfrac{d^k}{dx^k}$, $1 \le k \le n$. Namely,

Corollary 3.2. *Let* (μ_0, μ_1) *be a vector of non-negative measures on the real line, with* μ_0 *not identically zero, compact or non compact supports* Δ_0 *and* Δ_1, *respectively, at least one of them having an infinite number of points, and all moments exist and are finite. Let us consider the Sobolev norm,* $\{q_\nu\}_{\nu\ge 0}$ *an orthonormal sequence induced by* (2.1) *and the following extremal problem*

$$ {}_n\left(\frac{d^k}{dx^k}\right) = \sup_{p\in\mathbb{P}_n(C)} \frac{\|p^{(k)}\|_S}{\|p\|_S}, \quad 1 \le k \le n. $$

Then ${}_n\left(\dfrac{d^k}{dx^k}\right) = (r_{\max}(B_{n,k}))^{1/2}$, *where* $r_{\max}(B_{n,k})$ *is the maximal eigenvalue of the matrix* $B_{n,k} = \left[b_{i,j}^{(k)}\right]_{k\le i,j\le n}$, *whose elements are given by*

$$ b_{i,j}^{(k)} = \left\langle q_i^{(k)}, q_j^{(k)} \right\rangle_S, \quad k \le i, j \le n. $$

An extremal polynomial is

$$ P^*(x) = \sum_{\nu=k}^{n} c_{\nu-k} q_{\nu-k}(x), $$

where $[c_0, c_1, \ldots, c_{n-k}]^t$ *is an eigenvector of the matrix* $B_{n,k}$ *corresponding to the eigenvalue* $r_{\max}(B_{n,k})$.

Proof. If we take an orthonormal basis $\{q_0, \ldots, q_n\}$ of $\mathbb{P}_n(\mathbb{C})$ in Theorem 3.3, then the matrix $R(n+1)$ becomes an identity matrix. Therefore, $R(n+1)$ and $D^t R(m+1)\overline{D} = D^t\overline{D}$ commute and for $T = \dfrac{d^k}{dx^k}$, $1 \le k \le n$, we have

$$ {}_n\left(\frac{d^k}{dx^k}\right) = \max_{0\le i\le n} \sqrt{\eta_i}, $$

where η_i, $0 \le i \le n$ are the eigenvalues of $D^t\overline{D}$.

On the other hand, $\dfrac{d^k}{dx^k}(q_\nu) = \displaystyle\sum_{j=0}^{m} d_\nu^j q_j, \quad m = \max_{0\le\nu\le n} \deg\left(\frac{d^k}{dx^k}(q_\nu)\right) = n-k.$ Consequently,

$$b_{i,j}^{(k)} = \left\langle \sum_{s=0}^{n-k} d_i^s q_s, \sum_{l=0}^{n-k} d_j^l q_l \right\rangle_S = \sum_{s=0}^{n-k} d_i^s \overline{d}_j^s = \sum_{\nu=k}^{n} d_i^{\nu-k}\overline{d}_j^{\nu-k},$$

i.e., $B_{n,k} = D^t\overline{D}$.

Finally, if $[c_0, c_1, \ldots, c_{n-k}]^t$ is an eigenvector of the matrix $B_{n,k}$ corresponding to the eigenvalue $r_{\max}(B_{\lambda,n,k})$, that $P^*(x) = \displaystyle\sum_{\nu=k}^{n} c_{\nu-k} q_{\lambda,\nu-k}(x)$ is an extremal polynomial is straightforward. □

Another similar extremal problem arises on the space of polynomials with non-negative coefficients. Assume that T is a linear operator on the real polynomials $\mathbb{P}$, and consider the following extremal problem

$${}^{\star}_{n}(T) = \sup_{p\in S_n} \frac{\|T(p)\|_S}{\|p\|_S},$$

where $S_n := \left\{ p \in \mathbb{P}_n : p(x) = \displaystyle\sum_{\nu=0}^{n} c_\nu \phi_\nu(x), \quad c_\nu \ge 0, \quad 0 \le \nu \le n \right\}$, and $\{\phi_\nu\}_{\nu\ge 0}$ is asequence of real polynomials with $\deg(\phi_\nu) = \nu$, $\nu \ge 0$. By the same arguments as before, we can see that, ${}^{\star}_{n}(T)$ is the smallest value of λ such that

$$c[\lambda^2 R(n+1) - D^t R(m+1) D]c^t \ge 0, \tag{3.7}$$

for all $c = (c_0, c_1, \ldots, c_n) \in \mathbb{R}^{n+1}$ with $c_\nu \ge 0$, $0 \le \nu \le n$, where the matrices $R(n+1)$ and$D$ are the same as in Theorem 3.1.

We set $D^t R(m+1) D = (\tilde{r}_{ij})_{0\le i,j\le n}$ and assume that $r_{ij} \ge 0$, $\tilde{r}_{ij} \ge 0$ for $0 \le i, j \le n$, and that $r_{ij} = 0$, then $\tilde{r}_{ij} = 0$. Let

$$\sigma := \max_{0\le i,j\le n} \left\{ \sqrt{\frac{\tilde{r}_{ij}}{r_{ij}}} : r_{ij} > 0 \right\}.$$

Since the components of c are non-negative, the inequality (3.7) holds for $r = \sigma$. Hence,

$${}^{\star}_{n}(T) \le \sigma.$$

Theorem 3.4. *If σ is attained for $i = j = s$, then ${}^{\star}_{n}(T) = \sigma$, that is,*

$$\|Tp\|_S \le \sqrt{\frac{\tilde{r}_{ss}}{r_{ss}}}\, \|p\|_S, \quad p \in S_n,$$

and the equality holds for $p(x) = b\phi_s(x)$, where b is a non-negative constant.

Proof. It is sufficient to follow the proof offered in [13, Theorem2.6], taking into consideration the previous arguments. □

4. SOME EXAMPLES

In general, it is very hard to compute ${}_n\left(\dfrac{d^r}{dx^r}\right)$ explicitly. The reader is referred to [13] for some examples corresponding to the case $\mu_1 = 0$. In order to illustrate Theorem 3.2 when $T = \dfrac{d^r}{dx^r}$, $1 \le r \le n$, we consider the following examples.

Example 4.1. Let us consider (μ_0, μ_1) a vector of non-negative measures on the real line, given by

$$d\mu_0(x) = \chi_{[-1,1]}(x)dx, \quad d\mu_1(x) = (1-x^2)\chi_{[-1,1]}(x)dx,$$

where $\chi_{[-1,1]}$ is the characteristic function of the interval $[-1,1]$. The Sobolev inner product on $\mathbb{P}(\mathbb{C})$, is defined by

$$\langle p, q\rangle_S := \int_{\mathbb{R}} p(x)\overline{q(x)}d\mu_0(x) + \int_{\mathbb{R}} p'(x)\overline{q'(x)}d\mu_1(x). \tag{4.1}$$

It is known (see for instance, [1], [9]) that the classical orthonormal Legendre polynomials $\{P_\nu\}_{\nu\ge 0}$, defined by

$$\begin{aligned} P_\nu(x) &= \sqrt{\frac{2\nu+1}{2}}\, P_\nu^{(0,0)}(x) \\ &= \sqrt{\frac{2\nu+1}{2}} \sum_{j=0}^{[\nu/2]} \frac{(-1)^j(2\nu-2j)!}{2^\nu j!(\nu-j)!(\nu-2j)!} x^{\nu-2j}, \quad \nu \ge 0, \end{aligned}$$

satisfy the Sobolev orthogonality relationship:

$$\langle P_\nu, P_s\rangle_S = (\nu(\nu+1)+1)\delta_{\nu s}, \quad \nu, s \ge 0.$$

Then, we can easily deduce that the polynomials $\{q_\nu\}_{\nu\ge 0}$, defined by

$$q_\nu(x) = \sqrt{\frac{2\nu+1}{2\nu(\nu+1)+2}} \sum_{j=0}^{[\nu/2]} \frac{(-1)^j(2\nu-2j)!}{2^\nu j!(\nu-j)!(\nu-2j)!} x^{\nu-2j}, \quad \nu \ge 0,$$

are orthonormal with respect to Sobolev inner product (4.1).

On the other hand, if $C_m^{(\xi)}$ is the m-th Gegenbauer polynomial given by

$$C_m^{(\xi)}(x) = \frac{\Gamma(\xi+1/2)\Gamma(m+2\xi)}{\Gamma(2\xi)\Gamma(m+\xi+1/2)}\, P_m^{(\xi-1/2,\xi-1/2)}(x), \quad \xi > -1/2,$$

where $P_m^{(\xi-1/2,\xi-1/2)}$ is the m-th Jacobi polynomial normalized by

$$P_m^{(\xi-1/2,\xi-1/2)}(1) = \binom{m+\xi-1/2}{m}, \quad \xi > -1/2.$$

Then using the identity

$$\frac{d}{dx}C_m^{(\xi)}(x) = 2\xi C_{m-1}^{(\xi+1)}(x),$$

we can deduce that

$$\frac{d^r}{dx^r}C_m^{(\xi)}(x) = 2^r(\xi)_r C_{m-r}^{(\xi+r)}(x), \quad 1 \leq r \leq m.$$

Hence, for $1 \leq r \leq \nu$, we obtain

$$\begin{aligned} \frac{d^r}{dx^r}q_\nu(x) &= \left(\frac{2\nu+1}{2\nu(\nu+1)+2}\right)^{1/2} \frac{d^r}{dx^r}C_\nu^{(\frac{1}{2})}(x) \\ &= \left(\frac{2\nu+1}{2\nu(\nu+1)+2}\right)^{1/2} \frac{\Gamma(2r+1)}{2^r\Gamma(r+1)}C_{\nu-r}^{(r+\frac{1}{2})}(x). \end{aligned} \tag{4.2}$$

Furthermore, the relation (see [29, p. 99])

$$C_m^{(\xi)}(x) = \sum_{k=0}^{[m/2]} a_{k,m}C_{m-2k}^{(\zeta)}(x), \quad \xi > \zeta > 0,$$

where

$$a_{k,m} = \frac{\Gamma(\zeta)(m-2k+\zeta)\Gamma(k+\xi-\zeta)\Gamma(m-k+\xi)}{\Gamma(\xi)k!\Gamma(\xi-\zeta)\Gamma(m-k+\zeta+1)},$$

allows us to deduce for $\xi = r+1/2$, $m = \nu - r$ and $\zeta = 1/2$ the following identity

$$C_{\nu-r}^{(r+\frac{1}{2})}(x) = \sum_{k=0}^{[\frac{\nu-r}{2}]} a_{k,\nu-r}\, C_{\nu-r-2k}^{(\frac{1}{2})}(x).$$

Thus, the equality (4.2) is equivalent to

$$\begin{aligned} \frac{d^r}{dx^r}q_\nu(x) &= \left(\frac{2\nu+1}{2\nu(\nu+1)+2}\right)^{1/2} \frac{\Gamma(2r+1)}{2^r\Gamma(r+1)} \sum_{k=0}^{[\frac{\nu-r}{2}]} a_{k,\nu-r}\, C_{\nu-r-2k}^{(\frac{1}{2})}(x) \\ &= \left(\frac{2\nu+1}{2\nu(\nu+1)+2}\right)^{1/2} \frac{\Gamma(2r+1)}{2^r\Gamma(r+1)} \sum_{k=0}^{[\frac{\nu-r}{2}]} b_{k,\nu-r}\, q_{\nu-r-2k}(x), \end{aligned} \tag{4.3}$$

where

$$\begin{aligned} b_{k,\nu-r} &= \left(\frac{2(\nu-r-2k)(\nu-r-2k+1)+2}{2(\nu-r-2k)+1}\right)^{1/2} \\ &\quad \times \frac{\Gamma(1/2)(\nu-r-2k+1/2)\Gamma(k+r)\Gamma(\nu-k+1/2)}{\Gamma(r+1/2)k!\Gamma(r)\Gamma(\nu-r-k+3/2)}, \end{aligned}$$

or equivalently,

$$b_{k,\nu-r} = \frac{2^{2r-1}(\nu-r-2k+1/2)\Gamma(k+r)\Gamma(\nu-k+1/2)}{\Gamma(2r)k!\Gamma(\nu-r-k+3/2)} \times \left(\frac{2(\nu-r-2k)(\nu-r-2k+1)+2}{2(\nu-r-2k)+1}\right)^{1/2}.$$

This last identity is a consequence of the duplication formula for the Gamma function, $\Gamma(r)\Gamma(r+1/2) = 2^{1-2r}\sqrt{\pi}\Gamma(2r)$ and the well known equality $\Gamma(1/2) = \sqrt{\pi}$. Consequently, substituting the last expression for $b_{k,\nu-r}$ in (4.3), we obtain

$$\frac{d^r}{dx^r}q_\nu(x) = \left(\frac{2\nu+1}{\nu(\nu+1)+1}\right)^{1/2} \frac{2^r}{\Gamma(r)} \sum_{k=0}^{[\frac{\nu-r}{2}]} e_{k,\nu-r}\, q_{\nu-r-2k}(x), \tag{4.4}$$

where

$$e_{k,\nu-r} = \frac{(\nu-r-2k+1/2)\Gamma(k+r)\Gamma(\nu-k+1/2)}{k!\Gamma(\nu-r-k+3/2)} \times \left(\frac{(\nu-r-2k)(\nu-r-2k+1)+1}{2(\nu-r-2k)+1}\right)^{1/2}.$$

If we take $T = \dfrac{d^r}{dx^r}$, then the relation (4.4) implies that the coefficients in (3.2) are given by

$$d_\nu^{\nu-r-2j} = \begin{cases} \left(\dfrac{2\nu+1}{\nu(\nu+1)+1}\right)^{1/2} \dfrac{2^r}{(r-1)!}\, e_{j,\nu-r}, & 0 \le j \le \left[\frac{\nu-r}{2}\right], \\ 0, & \text{otherwise}, \end{cases}$$

so that

$$_n(T) = \sup_{\substack{\|c\|_2=1 \\ c\in\mathbb{C}^{n+1}}} \left\{ \sum_{j=0}^{n-r} \left| \sum_{\nu=0}^{n} c_\nu d_\nu^j \right|^2 \right\}^{\frac{1}{2}}.$$

Furthermore, we have

$$\begin{aligned} \max_{r\le\nu\le n} \left\| \frac{d^r}{dx^r}q_\nu \right\|_S &= \max_{r\le\nu\le n} \left(\frac{2\nu+1}{\nu(\nu+1)+1}\right)^{1/2} \frac{2^r}{(r-1)!} \left\| \sum_{j=0}^{[\frac{\nu-r}{2}]} e_{j,\nu-r}\, q_{\nu-r-2j} \right\|_S \\ &= \max_{r\le\nu\le n} \left(\frac{2\nu+1}{\nu(\nu+1)+1}\right)^{1/2} \frac{2^r}{(r-1)!} \left(\sum_{j=0}^{[\frac{\nu-r}{2}]} |e_{j,\nu-r}|^2 \right)^{1/2}. \end{aligned}$$

Since the function $g(x) = \left(\dfrac{2x+1}{x(x+1)+1}\right)^{1/2}$ is decreasing if $x \geq 1$, we have $\left(\dfrac{2n+1}{n(n+1)+1}\right)^{1/2} \leq \left(\dfrac{2\nu+1}{\nu(\nu+1)+1}\right)^{1/2} \leq \left(\dfrac{2r+1}{r(r+1)+1}\right)^{1/2}$, $r \leq \nu \leq n$. Therefore,

$$\frac{2^r}{(r-1)!}\left(\frac{2n+1}{n(n+1)+1}\right)^{1/2} \max_{r\leq\nu\leq n} \Delta_\nu \leq \max_{r\leq\nu\leq n} \left\| \frac{d^r}{dx^r} q_\nu \right\|_S,$$

and

$$\max_{r\leq\nu\leq n} \left\| \frac{d^r}{dx^r} q_\nu \right\|_S \leq \frac{2^r}{(r-1)!}\left(\frac{2r+1}{r(r+1)+1}\right)^{1/2} \max_{r\leq\nu\leq n} \Delta_\nu,$$

where

$$\Delta_\nu = \left(\sum_{j=0}^{[\frac{\nu-r}{2}]} |e_{j,\nu-r}|^2 \right)^{1/2}, \; r \leq \nu \leq n.$$

Hence, inequality (3.4) gives

$$\frac{2^r}{(r-1)!}\left(\frac{2n+1}{n(n+1)+1}\right)^{1/2} \max_{r\leq\nu\leq n} \Delta_\nu \leq {}_n(T),$$

and

$${}_n(T) \leq \frac{2^r}{(r-1)!}\left(\frac{(2r+1)(n+1)}{r(r+1)+1}\right)^{1/2} \max_{r\leq\nu\leq n} \Delta_\nu.$$

Example 4.2. Let us consider (μ_0, μ_1) a vector of non-negative measures on the real line, given by

$$d\mu_0(x) = x^\alpha e^{-x} H(x)dx, \quad d\mu_1(x) = x^{\alpha+1} e^{-x} H(x)dx,$$

where $\alpha > -1$ and H is the Heaviside step function. The Sobolev inner product on $\mathbb{P}(\mathbb{C})$, is defined by

$$\langle p, q \rangle_S := \int_{\mathbb{R}} p(x)\overline{q(x)} d\mu_0(x) + \int_{\mathbb{R}} p'(x)\overline{q'(x)} d\mu_1(x). \tag{4.5}$$

The classical orthonormal Laguerre polynomials $\{P_\nu^{(\alpha)}\}_{\nu\geq 0}$, defined by

$$\begin{aligned} P_\nu^{(\alpha)}(x) &= \left(\frac{\nu!}{\Gamma(\nu+\alpha+1)}\right)^{1/2} L_\nu^{(\alpha)}(x) \\ &= \left(\frac{\nu!}{\Gamma(\nu+\alpha+1)}\right)^{1/2} \sum_{j=0}^{\nu} (-1)^j \binom{\nu+\alpha}{\nu-j} \frac{x^j}{j!}, \quad \nu \geq 0, \end{aligned}$$

satisfy the Sobolev orthogonality relationship:

$$\langle P_\nu^{(\alpha)}, P_s^{(\alpha)} \rangle_S = (\nu+1)\delta_{\nu s}, \quad \nu, s \geq 0,$$

since

$$\frac{d}{dx}L_\nu^{(\alpha)}(x) = -L_{\nu-1}^{(\alpha+1)}(x). \tag{4.6}$$

Then, we can easily deduce that the polynomials $\{q_\nu^{(\alpha)}\}_{\nu\geq 0}$, defined by

$$q_\nu^{(\alpha)}(x) = \left(\frac{\nu!}{(\nu+1)\Gamma(\nu+\alpha+1)}\right)^{1/2}\sum_{j=0}^{\nu}(-1)^j\binom{\nu+\alpha}{\nu-j}\frac{x^j}{j!}, \quad \nu\geq 0,$$

are orthonormal with respect to the Sobolev inner product (4.5).

On the other hand, using (4.6) we can deduce that

$$\frac{d^r}{dx^r}L_\nu^{(\alpha)}(x) = (-1)^r L_{\nu-r}^{(\alpha+r)}(x), \quad 1\leq r\leq \nu.$$

Hence, for $1\leq r\leq\nu$, we obtain

$$\frac{d^r}{dx^r}q_\nu^{(\alpha)}(x) \;=\; (-1)^r\left(\frac{\nu!}{(\nu+1)\Gamma(\nu+\alpha+1)}\right)^{1/2} L_{\nu-r}^{(r+\alpha)}(x). \tag{4.7}$$

Now, using the addition formula (see [29, p. 391, Problem 90])

$$L_\nu^{(\alpha+\beta+1)}(x+y) = \sum_{k=0}^{\nu} L_{\nu-k}^{(\alpha)}(x)L_k^{(\beta)}(y),$$

we can deduce that equality (4.7) is equivalent to:

$$\begin{aligned}\frac{d^r}{dx^r}q_\nu^{(\alpha)}(x) &= (-1)^r\left(\frac{\nu!}{(\nu+1)\Gamma(\nu+\alpha+1)}\right)^{1/2}\sum_{k=0}^{\nu-r}\binom{r+k-1}{k}L_{\nu-r-k}^{(\alpha)}(x)\\ &= (-1)^r\left(\frac{\nu!}{(\nu+1)\Gamma(\nu+\alpha+1)}\right)^{1/2}\sum_{k=0}^{\nu-r}\tilde{e}_{k,\nu-r}\, q_{\nu-r-k}^{(\alpha)}(x) \qquad (4.8)\end{aligned}$$

where

$$\tilde{e}_{k,\nu-r} = \left(\frac{(\nu-r-k+1)\Gamma(\nu-r-k+\alpha+1)}{(\nu-r-k)!}\right)^{\frac{1}{2}}\binom{r+k-1}{k}.$$

Substituting $j=\nu-r-k$ in (4.8), we have

$$\frac{d^r}{dx^r}q_\nu^{(\alpha)}(x) = (-1)^r\left(\frac{\nu!}{(\nu+1)\Gamma(\nu+\alpha+1)}\right)^{1/2}\sum_{j=0}^{\nu-r}\tilde{e}_{\nu-r-j,\nu-r}\, q_j^{(\alpha)}(x).$$

Again, if we take $T=\dfrac{d^r}{dx^r}$, then the relation (4.8) implies that the coefficients in (3.2) are given by

$$d_\nu^j = \begin{cases} (-1)^r\left(\dfrac{\nu!(j+1)\Gamma(j+\alpha+1)}{j!(\nu+1)\Gamma(\nu+\alpha+1)}\right)^{1/2}\dbinom{\nu-j-1}{\nu-j-r}, & 0\leq j\leq \nu-r,\\ 0, & \text{otherwise,}\end{cases}$$

so that

$$ {}_n(T) = \sup_{\substack{\|c\|_2=1 \\ c\in\mathbb{C}^{n+1}}} \left\{ \sum_{j=0}^{n-r} \left| \sum_{\nu=0}^{n} c_\nu d_\nu^j \right|^2 \right\}^{\frac{1}{2}} . $$

Furthermore,

$$ \begin{aligned} \max_{r\le\nu\le n} \left\| \frac{d^r}{dx^r} q_\nu^{(\alpha)} \right\|_S &= \max_{r\le\nu\le n} \left(\frac{\nu!}{(\nu+1)\Gamma(\nu+\alpha+1)} \right)^{1/2} \left\| \sum_{j=0}^{\nu-r} \tilde{e}_{\nu-r-j,\nu-r}\, q_j^{(\alpha)} \right\|_S \\ &= \max_{r\le\nu\le n} \left(\frac{\nu!}{(\nu+1)\Gamma(\nu+\alpha+1)} \right)^{1/2} \Delta_\nu^{(\alpha)}, \end{aligned} $$

where

$$ \Delta_\nu^{(\alpha)} = \left(\sum_{j=0}^{\nu-r} \frac{(j+1)\Gamma(j+\alpha+1)}{j!} \binom{\nu-j-1}{\nu-j-r}^2 \right)^{1/2}, \; r\le\nu\le n. $$

Since the binomial coefficient is an increasing function of ν, we have

$$ \max_{r\le\nu\le n} \Delta_\nu^{(\alpha)} = \Delta_n^{(\alpha)}. $$

Also, $\dfrac{\nu!}{\Gamma(\nu+\alpha+1)}$ is decreasing (increasing) if $\alpha\ge 0$ $(-1<\alpha<0)$, (see [4] or [13]), then

$$ \begin{aligned} a_{n,n}^{(\alpha)}\Delta_n^{(\alpha)} \le \max_{r\le\nu\le n} \left\| \frac{d^r}{dx^r} q_\nu^{(\alpha)} \right\|_S \le a_{r,r}^{(\alpha)}\Delta_n^{(\alpha)}, &\qquad \text{if } \alpha\ge 0, \\ a_{r,n}^{(\alpha)}\Delta_n^{(\alpha)} \le \max_{r\le\nu\le n} \left\| \frac{d^r}{dx^r} q_\nu^{(\alpha)} \right\|_S \le a_{n,r}^{(\alpha)}\Delta_n^{(\alpha)}, &\qquad \text{if } -1<\alpha<0, \end{aligned} $$

where

$$ a_{n,r}^{(\alpha)} = \left(\frac{n!}{(r+1)\Gamma(n+\alpha+1)} \right)^{1/2}. $$

Hence, inequality (3.4) gives

$$ \begin{cases} a_{n,n}^{(\alpha)}\Delta_n^{(\alpha)} \le {}_n(T) \le (n+1)^{1/2} a_{r,r}^{(\alpha)}\Delta_n^{(\alpha)}, & \text{if } \alpha\ge 0, \\ a_{r,n}^{(\alpha)}\Delta_n^{(\alpha)} \le {}_n(T) \le (n+1)^{1/2} a_{n,r}^{(\alpha)}\Delta_n^{(\alpha)}, & \text{if } -1<\alpha<0. \end{cases} $$

In particular, when $\alpha=0$, we obtain

$$ \left(\sum_{j=0}^{n-r} \frac{(j+1)}{(n+1)} \binom{n-j-1}{n-j-r}^2 \right)^{1/2} \le {}_n(T) \le \left(\sum_{j=0}^{n-r} \frac{(n+1)(j+1)}{r+1} \binom{n-j-1}{n-j-r}^2 \right)^{1/2}. $$

Finally, other examples arise when the vector of measures (μ_0,μ_1) is either a coherent or a symmetrically coherent pair (see[12], [19]), and the Sobolev inner product on $\mathbb{P}(\mathbb{C})$ is defined as in (2.1). Meijer has proved that if (μ_0, μ_1) is either a coherent or a symmetrically coherent pair, then at least μ_0 or μ_1 is a classical measure (that is, Jacobi, Laguerre or Hermite measure), and has completely classified all the coherent and the symmetrically coherent pairs (see [19]). In these cases, applying Theorem3.1 with an appropriate basis $\{\phi_\nu\}_{\nu \geq 0}$, it is possible to determine an expression for $_n(T)$ when $T = \dfrac{d^r}{dx^r}$, $1 \leq r \leq n$.

REFERENCES

[1] G. E. Andrews and L. L. Littlejohn: *A combinatorial interpretation of the Legendre-Stirling numbers*, Proc. Amer. Math. Soc., **137**(2009), No. 8, 2581–2590.

[2] A. I. Aptekarev, A. Draux and V. A. Kalyagin: *On the asymptotics of sharp constants in Markov-Bernstein inequalities in integral metrics with classical weight*, Communications of the Moscow Mathematical Society, **55**(2000), 163–165.

[3] P. Borwein and T. Erdélyi: *Polynomials and Polynomials Inequalities*, Springer-Verlag, New York, 1995.

[4] P. J. Davis: Gamma Function and Related Functions, in *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables*, (M. Abramowitz and I. A. Stegun (Eds.)), Dover, New York, 1965, 253–293.

[5] P. Dörfler: *New inequalities of Markov type*, SIAM J. Math. Anal., **18** (1987), 490–494.

[6] P. Dörfler: *A Markov Type Inequality for Higher Derivatives of Polynomials*, Mh. Math., **109**(1990), 113–122.

[7] T. Erdélyi: *Markov -and Bernstein- type inequalities for Müntz polynomials and exponential sums in L_p*, J. Approx. Theory, **104**(2000), 142–152.

[8] T. Erdélyi: *Markov-Nikolskii type inequalities for exponential sums on finite intervals*, Adv. Math., **208**(2007), No. 1, 135–146.

[9] W. N. Everitt, L. L. Littlejohn and R. Wellman: *Legendre polynomials, Legendre-Stirling numbers, and the left definite spectral analysis of the Legendre differential expression*, J. Comput. Appl. Math., **48**(2002), 213–238.

[10] N. K. Govil and G. Labelle: *On Bernstein's inequality*, J. Math. Anal. Appl., **126**(1987), 494–500.

[11] A. Guessab and G. V. Milovanović: *Weighted L^2 analogues of Bernstein's inequality and classical orthogonal polynomials*, J. Math. Anal. Appl., **182**(1994), 244–249.

[12] A. Iserles, P. E. Koch, S. P. Nørsett and J. M. Sanz-Serna: *On polynomials orthogonal with respect to certain Sobolev inner products*, J. Approx. Theory, **6**(1991), 151–175.

[13] K. H. Kwon and D. W. Lee: *Markov-Bernstein type inequalities for polynomials*, Bull. Korean Math. Soc., **36**(1999), 63–78.

[14] P. Lancaster and M. Tismenetsky: *The Theory of Matrices*, 2nd ed., Acad. Press, New York, 1985.

[15] E. Levin and D. S. Lubinsky: *Orthogonal Polynomials for Exponential Weights*, Springer-Verlag, New York, 2001.

[16] G. G Lorentz: *Approximation of Functions*, 2nd ed., Chelsea, New York, NY, 1986.

[17] D. S. Lubinsky: *A survey of weighted polynomial approximation with exponential weights*, Surveys in Approximation Theory, **3**(2007), 1–105.

[18] F. Marcellán and J. M. Rodríguez: *Markov type inequalities in weighted Sobolev spaces*, manuscript.

[19] H. G. Meijer: *Determination of all coherent pairs*, J. Approx. Theory, **89**(1997), 321-343.

[20] G. V. Milovanović: *Various extremal problems of Markov's type for algebraic polynomials*, Facta Univ. Ser. Math. Inform., **2**(1987), 7–28.

[21] G. V. Milovanović, D. S. Mitrinović and Th. M. Rassias: *Topics in Polynomials: Extremal problems, inequalities, zeros*, World Scientific Publishing Co. Pte. Ltd., Singapore, 1994.

[22] I. Ž. Milovanović and M. A. Kovačević: *An extremal problem for real algebraic polynomials*, Tamkang J. Math., **24**(1993), 203–208.

[23] L. Mirsky: *An inequality of the Markov-Bernstein type for Polynomials*, SIAM J. Math. Anal., **14**(1983), 1004–1008.

[24] P. Nevai and V. Totik: *Weighted polynomial inequalities*, Constr. Approx., **2**(1986), 113–127.

[25] A. Pinkus: *Negative Theorems in Approximation Theory*, The American Mathematical Monthly, **110**(2003), 900–911.

[26] A. Portilla, Y. Quintana, J. M. Rodríguez and E. Tourís: *Weighted Weierstrass' Theorem with first derivatives*, J. Math. Anal. Appl., **334**(2007), 1667–1698.

[27] G. V. Radzievskiĭ: *Direct and inverse theorems on approximation by root functions of a regular boundary-value problem*, Sbornik: Mathematics, **197**(2006), 1037–1083.

[28] E. Schmidt: *Über die nebst ihren Ableitungen orthogonalen Polynomensysteme und das zugehörige Extremum*, Math. Ann., **119**(1944), 165–204.

[29] G. Szegő: *Orthogonal Polynomials*, Coll. Publ. Amer. Math. Soc., Vol. 23, (4th ed.), Providence, R.I., 1975.

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