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STRONGLY NONLINEAR NONCONVEX VARIATIONAL INEQUALITIES

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ABSTRACT. In this paper, we introduce and consider a new class of nonconvex variational inequalities involving two nonlinear operators, which is called the strongly nonlinear variational inequality. We establish the equivalence between the strongly nonlinear nonconvex variational inequalities and the fixed point problem. This alternative equivalence is used to study the existence of a solution of the strongly nonlinear nonconvex variational inequality and to suggest an iterative method. We consider the convergence criteria of the iterative method under suitable conditions. Some special cases are also discussed.

1. INTRODUCTION

Variational inequalities theory, which was introduced by Stampacchia [32], provides us with a simple, natural, general and unified framework to study a wide class of problems arising in pure and applied sciences. During the last three decades, there has been considerable activity in the development of numerical techniques for solving variational inequalities, see [1]-[32] and the references therein. It is worth mentioning that all the research work regarding the existence and iterative schemes for variational inequalities has been investigated and considered, if the underlying set is a convex set. This is because all the techniques are based on the properties of the projection operator over convex sets. We have noticed that these results may not hold if the choice set is nonconvex. Motivated and inspired by the research going on in this field, Noor [17]-[28] has introduced and studied a new class of variational inequalities, which is called the nonconvex variational inequality in conjunction with the uniformly prox-regular sets, which are nonconvex and include the convex sets as a special case, see [1], [2], [31]. Noor [17]-[28] has shown that the projection technique can be extended for the nonconvex variational inequalities and has established the equivalence between the nonconvex variational inequalities and fixed point problems using essentially the projection technique. This equivalent alternative formulation is used to discuss the existence of a solution of the nonconvex variational inequalities and to suggest some iterative method. In this paper, we introduce and

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consider a new class of nonconvex variational inequalities involving two nonlinear operators, which is called the *strongly nonlinear nonconvex variational inequality.* Using the technique and ideas of Noor [21]-[28], we prove that the strongly nonlinear nonconvex variational inequalities are equivalent to the fixed point problem. This equivalent is used to study the existence of a solution of the strongly nonlinear nonconvex variational inequality, which Theorem 3.1. We use this alternative equivalent formulation to suggest and analyze an iterative method for solving the strongly nonlinear nonconvex variational inequalities. We also consider the convergence (Theorem 3.2) of the new iterative method under some suitable conditions. Some special cases are also discussed. Results obtained in this paper can be viewed as refinement and improvement of the previously known results for the variational inequalities and related optimization problems.

2. PRELIMINARIES

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|.\|$ respectively. Let K be a nonempty and convex set in H. We, first of all, recall the following well-known concepts from nonlinear convex analysis and nonsmooth analysis [2], [31].

Definition 2.1. The proximal normal cone of K at $u \in H$ is given by

$$N_K^P(u) := \{\xi \in H : u \in P_K[u + \alpha\xi]\},$$

where $\alpha > 0$ is a constant and

$$P_K[u] = \{u^* \in K : d_K(u) = \|u - u^*\|\}.$$

Here $d_K(.)$ is the usual distance function to the subset K, that is

$$d_K(u) = \inf_{v\in K} \|v - u\|.$$

The proximal normal cone $N_K^P(u)$ has the following characterization.

Lemma 2.1. *Let K be a nonempty, closed and convex subset in H. Then $\zeta \in N_K^P(u)$, if and only if, there exists a constant $\alpha > 0$ such that*

$$\langle\zeta, v - u\rangle \le \alpha\|v - u\|^2, \quad \forall v \in K.$$

Definition 2.2. The Clarke normal cone, denoted by $N_K^C(u)$, is defined as

$$N_K^C(u) = \overline{co}[N_K^P(u)],$$

where $\overline{co}$ means the closure of the convex hull. Clearly $N_K^P(u) \subset N_K^C(u)$, but the converse is not true. Note that $N_K^P(u)$ is always closed and convex, whereas $N_K^C(u)$ is convex, but may not be closed.

Poliquin et al. [31] and Clarke et al. [2] have introduced and studied a new class of nonconvex sets, which are called uniformly prox-regular sets. This class of uniformly prox-regular sets has played an important part in many nonconvex applications such as optimization, dynamic systems and differential inclusions.

Definition 2.3. For a given $r \in (0, \infty]$, a subset K_r is said to be normalized uniformly r-prox-regular if and only if every nonzero proximal normal to K_r can be realized by an r-ball, that is, $\forall u \in K_r$ and $0 \neq \xi \in N^P_{K_r}(u)$, one has

$$\langle (\xi)/\|\xi\|, v - u\rangle \leq (1/2r)\|v - u\|^2, \quad \forall v \in K.$$

It is clear that the class of normalized uniformly prox-regular sets is sufficiently large to include the class of convex sets, p-convex sets, $C^{1,1}$ submanifolds (possibly with boundary) of H, the images under a $C^{1,1}$ diffeomorphism of convex sets and many other nonconvex sets; see [7], [32]. It is clear that if $r = \infty$, then uniformly prox-regularity of K_r is equivalent to the convexity of K. It is known that if K_r is a uniformly prox-regular set, then the proximal normal cone $N^P_{K_r}(u)$ is closed as a set-valued mapping. Thus, we have $N^P_{K_r}(u) = N^C_{K_r}(u)$.

For given nonlinear operators T, A, we consider the problem of finding $u \in K_r$ such that

$$\langle Tu, v - u\rangle \geq \langle A(u), v - u\rangle, \quad \forall v \in K_r, \tag{2.1}$$

which is called the *strongly nonlinear nonconvex variational inequality.*

We note that, if $K_r \equiv K$, the convex set in H, then problem (2.1) is equivalent to finding $u \in K$ such that

$$\langle Tu, v - u\rangle \geq \langle A(u), v - u\rangle, \qquad \forall v \in K. \tag{2.2}$$

Inequality of type (2.2) is called the *strongly nonlinear variational inequality*, which was introduced and studied by Noor [5]-[8]. For the applications, numerical methods and other aspects of the strongly nonlinear variational inequalities and related optimization problems, see [5]-[30].

If $A(u) \equiv 0$, then problem (2.1) is equivalent to finding $u \in K_r$ such that

$$\langle Tu, v - u\rangle \geq 0, \quad \forall v \in K_r, \tag{2.3}$$

which is called the *nonconvex variational inequality* introduced and studied by Noor [21]-[28].

We note that, if $K_r \equiv K$, the convex set in H, then problem (2.3) is equivalent to finding $u \in K$ such that

$$\langle Tu, v - u\rangle \geq 0, \qquad \forall v \in K. \tag{2.4}$$

Problem (2.4) is known as the classical variational inequality, which was introduced and studied by Stampacchia [32] in 1964. It turned out that a number of unrelated obstacle, free, moving, unilateral and equilibrium problems arising in various branches of pure and applied sciences can be studied via variational inequalities, see [1]-[32] and the references therein.

If K_r is a nonconvex (uniformly prox-regular) set, then problem (2.1) is equivalent to finding $u \in K_r$ such that

$$0 \in Tu - A(u) + N^P_{K_r}(u), \tag{2.5}$$

where $N^P_{K_r}(u)$ denotes the normal cone of K_r at u in the sense of nonconvex analysis. Problem (2.4) is called the nonconvex variational inclusion problem associated with nonconvex variational inequality (2.1). This implies that the variational inequality (2.1) is equivalent to finding a zero of the sum of two monotone operators (2.4). This equivalent formulation plays a crucial and basic part in this paper. We would like to point out this equivalent formulation allows us to use the projection operator technique for solving the strongly nonlinear nonconvex variational inequality (2.1).

We now recall the well known proposition which summarizes some important properties of the uniform prox-regular sets.

Lemma 2.2. *Let K be a nonempty closed subset of H, $r \in (0, \infty]$ and set $K_r = \{u \in H : d(u, K) < r\}$. If K_r is uniformly prox-regular, then:*
i) $\forall u \in K_r, P_{K_r}(u) \neq \emptyset$;
ii) $\forall r' \in (0, r)$, P_{K_r} *is Lipschitz continuous with constant* $\dfrac{r}{r - r'}$ *on* $K_{r'}$.

Definition 2.4. An operator $T : H \to H$ is said to be:
i) *strongly monotone*, if and only if, there exists a constant $\alpha > 0$ such that

$$\langle Tu - Tv, u - v\rangle \geq \alpha||u - v||^2, \quad \forall u, v \in H;$$

ii) *Lipschitz continuous*, if and only if, there exists a constant $\beta > 0$ such that

$$||Tu - Tv|| \leq \beta||u - v||, \quad \forall u, v \in H.$$

3. PROJECTION METHODS

In this section, we establish the equivalence between the nonconvex variational inequality (2.1) and the fixed point problem using the projection operator technique. This alternative formulation is used to discuss the existence of a solution of the problem (2.1) and to suggest some new iterative methods for solving the strongly nonlinear nonconvex variational inequality (2.1).

Using the idea and technique of Noor [21]-[28], we show that the strongly nonlinear nonconvex variational inequalities (2.1) are equivalent to the fixed point problem. To convey an idea and technique, we include its proof.

Lemma 3.1. *$u \in K_r$ is a solution of the strongly nonlinear nonconvex variational inequality* (2.1) *if and only if $u \in K_r$ satisfies the relation*

$$u = P_{K_r}[u - \rho Tu + \rho A(u)], \tag{3.1}$$

where P_{K_r} is the projection of H onto the uniformly prox-regular set K_r.

Proof. Let $u \in K_r$ be a solution of (2.1). Then, for a constant $\rho > 0$,

$$0 \in u + \rho N^P_{K_r}(u) - (u - \rho(Tu - A(u))) = (I + \rho N^P_{K_r})(u) - (u - \rho(Tu - A(u)))$$

$$\Leftrightarrow$$

$$u = (I + \rho N^P_{K_r})^{-1}[u - \rho Tu + \rho A(u)] = P_{K_r}[u - \rho Tu + \rho A(u)],$$

where we have used the well-known fact that $P_{K_r} \equiv (I + N^P_{K_r})^{-1}$. □

Lemma 3.1 implies that the strongly nonlinear nonconvex variational inequality (2.1) is equivalent to the fixed point problem (3.1). This alternative equivalent formulation is very useful from the numerical and theoretical point of views.

We rewrite the the relation (3.1) in the following form

$$F(u) = P_{K_r}[u - \rho Tu + \rho A(u)], \tag{3.2}$$

which is used to study the existence of a solution of the strongly nonlinear nonconvex variational inequality (2.1).

We now study those conditions under which the strongly nonlinear nonconvex variational inequality (2.1) has a solution and this is the main motivation of our next result.

Theorem 3.1. *Let P_{K_r} be the Lipschitz continuous operator with constant $\delta = \frac{r}{r - r'}$. Let T be strongly monotone with constant $\alpha > 0$ and Lipschitz continuous with constant $\beta > 0$. If the operator A is Lipschitz continuous with constant $\gamma > 0$, and there exists a constant ρ such that*

$$|\rho - \frac{\delta\alpha - \gamma}{\delta(\beta^2 - \gamma^2)}| < \frac{\sqrt{\delta(\delta\alpha - \gamma)^2 - (\beta^2 - \gamma^2)(\delta^2 - 1)}}{\delta(\beta^2 - \gamma^2)},$$
$$\delta\rho\gamma < 1, \quad \delta\alpha > \gamma + \sqrt{(\beta^2 - \gamma^2)(\delta^1 -)}, \tag{3.3}$$

then there exists a solution of the problem (2.1).

Proof. From Lemma 3.1, it follows that problems (3.1) and (2.1) are equivalent. Thus it is enough to show that the map $F(u)$, defined by (3.2), has a fixed point. For all $u \neq v \in K_r$, we have

$$\begin{aligned} \|F(u) - F(v)\| &= \|P_{K_r}[u - \rho Tu + \rho A(u)] - P_{K_r}[v - \rho Tv + \rho A(v)]\| \\ &\leq \delta\|u - v - \rho(Tu - Tv)\| + \delta\rho\|A(u) - A(v)\|, \end{aligned} \tag{3.4}$$

where we have used the fact that the operator P_{K_r} is a Lipschitz continuous operator with constant δ.

Since the operator T is strongly monotone with constant $\alpha > 0$ and Lipschitz continuous with constant $\beta > 0$, it follows that

$$\begin{aligned} ||u - v - \rho(Tu - Tv)||^2 &\leq ||u - v||^2 - 2\rho\langle Tu - Tv, u - v\rangle + \rho^2||Tu - Tv||^2 \\ &\leq (1 - 2\rho\alpha + \rho^2\beta^2)||u - v||^2. \end{aligned} \tag{3.5}$$

From (3.5), (3.4) and using the Lipschitz continuity of the operator A with constant $\gamma > 0$, we have

$$\|F(u) - F(v)\| \leq \delta\{\gamma\rho + \sqrt{1 - 2\alpha\rho + \beta^2\rho^2)}\}\|u - v\| = \theta\|u - v\|,$$

where

$$\theta = \delta\{\gamma\rho + \sqrt{1 - 2\alpha\rho + \beta^2\rho^2}\}. \tag{3.6}$$

From (3.3), it follows that $\theta < 1$, which implies that the map $F(u)$ defined by (3.2), has a fixed point, which is the unique solution of (2.1). □

This fixed point formulation (3.1) is used to suggest the following iterative method for solving the nonconvex variational inequality (2.1).

Algorithm 3.1. For a given $u_0 \in K_r$, find the approximate solution u_{n+1} by the iterative scheme

$$u_{n+1} = P_{K_r}[u_n - \rho Tu_n + \rho A(u_n)], \quad n = 0, 1, 2, \ldots$$

In this paper, we suggest and analyze the following one-step iterative method for solving the nonconvex variational inequalities (2.1).

Algorithm 3.2. For a given $u_0 \in K_r$, find the approximate solution u_{n+1} by the iterative schemes

$$u_{n+1} = (1 - \alpha_n)x_n + \alpha_n P_{K_r}[u_n - \rho Tu_n + \rho A(u_n)], \quad n = 0, 1, 2, \ldots, \tag{3.7}$$

where $\alpha_n \in [0, 1]$, $\forall n \geq 0$. Clearly for $\alpha_n = 1$, Algorithm 3.2 reduces to Algorithm 3.1.

It is worth mentioning that if $r = \infty$, then the nonconvex set set K_r reduces to a convex set K. Consequently Algorithms 3.1-3.2 collapse to the following algorithms for solving the classical variational inequalities (2.2).

We now consider the convergence analysis of Algorithm 3.1 and this is the main motivation of our next result.

Theorem 3.2. *Let P_{K_r} be the Lipschitz continuous operator with constant $\delta = \dfrac{r}{r - r'}$. Let the operator $T : H \longrightarrow H$ be strongly monotone with constants $\alpha > 0$ and Lipschitz continuous with constants with $\beta > 0$. If* (3.3) *holds, $\alpha_n \in [0, 1]$, $\forall n \geq 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$, then the approximate solution u_n obtained from Algorithm* 3.1 *converges to a solution $u \in K_r$ satisfying the strongly nonlinear nonconvex variational inequality* (2.1).

Proof. Let $u \in K_r$ be a solution of the nonconvex variational inequality (2.1). Then, using Lemma 3.1, we have

$$u = (1 - \alpha_n)u + \alpha_n P_{K_r}[u - \rho Tu + \rho A(u)], \tag{3.8}$$

where $0 \leq \alpha_n \leq 1$ are constants.

From (3.4), (3.5), (3.7), (3.8) and using the Lipschitz continuity of the projection P_{K_r} with constant δ, we have

$$\begin{aligned}
\|u_{n+1} - u\| &= \|(1 - \alpha_n)(u_n - u) + \alpha_n\{P_{K_r}[u_n - \rho Tu_n + \rho A(u_n)] \\
&\qquad - P_{K_r}[u - \rho Tu + \rho A(u)]\}\| \\
&\leq (1 - \alpha_n)\|u_n - u\| + \alpha_n \|P_{K_r}[u_n - \rho(Tu_n - A(u_n))] \\
&\qquad - P_{K_r}[u - \rho Tu + \rho A(u)]\| \\
&\leq (1 - \alpha_n)\|u_n - u\| + \alpha_n \delta \|u_n - u - \rho(Tu_n - Tu) + \rho(A(u_n) - A(u))\|
\end{aligned}$$

$$\begin{aligned}
&\leq (1-\alpha_n)\|u_n - u\| + \alpha_n \delta\{\gamma\rho + \sqrt{1-2\alpha\rho+\beta^2\rho^2}\}\|u_n - u\| \\
&\leq (1-\alpha_n)\|u_n - u\| + \alpha_n \theta \|u_n - u\| \\
&= \{1 - (\alpha_n(1-\theta)\}\|u_n - u\| \\
&\leq \prod_{i=0}^{n} [1 - \alpha_i(1-\theta)]\, \|u_0 - u\|,
\end{aligned}$$

where $\theta = \delta\{\gamma\rho + \sqrt{1-2\alpha\rho+\beta^2\rho^2}\}$. From (3.3), it follows that $\theta < 1$. Since $\sum_{n=0}^{\infty} \alpha_n$ diverges and $1-\theta > 0$, we have $\lim_{n\to\infty} \left\{ \prod_{i=0}^{n} [1-(1-\theta)\alpha_i] \right\} = 0$. Consequently the sequence $\{u_n\}$ convergences strongly to u. This completes the proof. □

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