J. Adv. Math. Stud.
Vol. **4**(2011), No. 1, 57-76
`http://journal.fairpartners.ro`

NONSMOOTH INVEX FUNCTIONS VIA UPPER DIRECTIONAL DERIVATIVE OF DINI

ŞTEFAN MITITELU AND MIHAI POSTOLACHE

Abstract. The notion of invexity is introduced by Hanson [6] în 1981, for real differentiable functions, see also [2] by Craven. After the works of Hanson and Craven, other types of differentiable functions have appeared with the intent of generalizing invex functions from different viewpoints, see [7] by Jeyakumar, [8] by Kaul and Kaul and [15], [16], [22] by Mititelu. Stipulating possible applications of these functions for programming problems, see [11] by Mishra and [24] by Nahak and Mohapatra to quote some examples, some scientists payed attention to possible extensions of this concept to nonsmooth functions. A direction is those of Craven [3] followed by Reiland [26], which extended the concept of invexity to nonsmooth functions by using the generalized directional derivative of Clarke for Lipschitz functions. Another direction is those of Mititelu, which extended the concept of invexity to nonsmooth functions by using essentially the upper directional derivative of Dini [5], [13], [16], [17], [18], [19], [22]. The goal of this work is to describe the state of the art on the question of invex functions for this case. We recall the main results on this theory with proofs when necessary, we supply up to date references, a description of some further developments and a few new results as well. For additional background material on this topic, we address the reader to the following research works: [10], [12], [14], [21], [23]

1. CLASSES AND TYPES OF INVEXITY

To study the types of simple and generalized invexity of nonsmooth real functions, we use the upper directional derivative of Dini. These types are introduced by classes (ρ-invexity, ρ-pseudoinvexity, ρ-quasiinvexity). The use of *pseudo* and *quasi* from the convexity of functions allowed the determination of all types of invex and generalized invex functions.

Let $A \subseteq \mathbb{R}^n$ be an open nonempty set, $f\colon A \to \mathbb{R}$ a real-valued function and $\rho \in \mathbb{R}$ a number.

Definition 1.1. The function f is called *ρ-invex* (shortly ρI) at the point $u \in A$ if there are the vector functions $\eta, \theta\colon A \times A \to \mathbb{R}^n$ such that

$$\forall x \in A\colon f(x) - f(u) \geq f'_+(u; \eta(x,u)) + \rho\|\theta(x,u)\|^2. \qquad (\rho\mathrm{I})$$

Received: August 12, 2010. *Revised*: November 11, 2010.
2010 *Mathematics Subject Classification*: 26B 25, 90C 25.
Key words and phrases: invexity, preinvexity, generalized convexity.

(1a) If $\rho > 0$, then f is called *strongly invex* (TI) at u;
(1b) If $\rho = 0$, then f is called *invex* (I) at u;
(1c) If $\rho < 0$, then f is called *weakly invex* (sI) at u;
(1d) If $(\forall)\, x, u \in A$, $x \neq u$: $f(x) - f(u) > f'_+(u; \eta(x,u))$ then f is called *strictly invex* (SI) u.

If u is arbitrary in A, then we obtain all global invexity types.

Theorem 1.1. *For a function f and a point $u \in A$, the following implications hold:*

a) *strongly invex* (TI) *and* $\langle x \neq u \Rightarrow \theta(x,u) \neq 0 \rangle \Rightarrow$ *strictly invex* (SI);
b) *strictly invex* (SI) $\Rightarrow$ *invex* (I) $\Rightarrow$ *weakly invex* (sI).

Theorem 1.2. *If f is ρ-convex function at the point u, then f is ρ-invex at u. Moreover, if f is strictly convex at the point u, then f is a strictly invex function at u.*

Proof. We could use the mappings $\eta(x,u) = x - u$ and $\theta(x,u) = x - u$. □

Definition 1.2. The function f is called *ρ-pseudoinvex* (ρPI) at the point $u \in A$ if there are the vector functions $\eta, \theta \colon A \times A \to \mathbb{R}^n$ such that

$$\forall x \in A\colon f'_+(u; \eta(x,u)) + \rho \|\theta(x,u)\|^2 \geq 0 \Rightarrow f(x) \geq f(u). \qquad (\rho\text{PI})$$

(2a) If $\rho > 0$, then f is called *strongly pseudoinvex* (TPI) at u;
(2b) If $\rho = 0$, then f is called *pseudoinvex* (PI) at u;
(2c) If $\rho < 0$, then f is called *weakly pseudoinvex* (sPI) at u;
(2d) If $\forall x \in A$, $x \neq u$: $f'_+(u; \eta(x,u)) \geq 0 \Rightarrow f(x) > f(u)$, then f is called *strictly pseudoinvex* (SPI) at u.

If u is arbitrary in A, then we obtain the global pseudoinvexity types.

Theorem 1.3. *For a function f, at the point u, the following implications hold:*
a) *strongly pseudoinvex* (TPI) *and injective* $\Rightarrow$ *strictly pseudoinvex* (SPI)*;*
b) *strictly pseudoinvex* (SPI) $\Rightarrow$ *pseudoinvex* (PI) $\Rightarrow$ *weakly pseudoinvex* (sPI)*.*

Theorem 1.4. *If f is ρ-pseudoconvex at the point u, then f is ρ-pseudoinvex at the point u. Moreover, if f is strictly pseudoconvex at the point u, then f is strictly pseudoinvex at the point u.*

Theorem 1.5. *If f is ρ-invex function, then f is ρ-pseudoinvex. Moreover, if f is strictly invex, then f is strictly pseudoinvex.*

Definition 1.3. The function f is called *ρ-quasiinvex* (ρQI) at the point $u \in A$ if there are the mappings $\eta, \theta \colon A \times A \to \mathbb{R}^n$ so that

$$\forall x \in A\colon f(x) \leq f(u) \Rightarrow f'_+(u; \eta(x,u)) + \rho \|\theta(x,u)\|^2 \leq 0. \qquad (\rho\text{QI})$$

(3a) If $\rho > 0$, then f is called *strongly quasiinvex* (TQI) at u;
(3b) If $\rho = 0$, then f is called *quasiinvex* (QI) at u;
(3c) If $\rho < 0$, then f is called *weakly quasiinvex* (sQI) at u;
(3d) If $\forall x \in A$, $x \neq u$: $f(x) \leq f(u) \Rightarrow f'_+(u; \eta(x,u)) < 0$, then f is called *strictly quasiinvex* (SQI) at u;

(3e) If $\forall x \in A$, $x \neq u$: $f(x) < f(u) \Rightarrow f'_+(u;\eta(x,u)) < 0$, then f is called *semi-strictly quasiinvex* (SSQI) at u.

If u is an arbitrary point in A, then we obtain global quasiinvexity types.

Theorem 1.6. *For a function f and a point $u \in A$, the following implications hold:*

a) *strongly quasiinvex* (TQI) *and* $\langle x \neq u \Rightarrow \theta(x,u) \neq 0\rangle \Rightarrow$ *strictly quasiinvex* (SQI)*;*

b) *strictly quasiinvex* (SQI) $\Rightarrow$ *semistrictly quasiinvex* (SSQI)*;*

c) *semistrictly quasiinvex* (SSQI) *and* $\langle f$ *lower semicontinuous on* A *and* $\eta(\cdot;u)$ *bounded on* $A\rangle \Rightarrow$ *quasiinvex* (QI)*;*

d) *quasiinvex* (QI) $\Rightarrow$ *weakly quasiinvex* (sQI)*.*

All direct implications between pseudoinvexity and quasiinvexity types are given by the following theorem:

Theorem 1.7. *For each function f, at the point u, we have:*

a) *strongly pseudoinvex* (TPI) $\Rightarrow$ *strongly quasiinvex* (TQI)*;*

b) *strictly pseudoinvex* (SPI) $\Rightarrow$ *strictly quasiinvex* (SQI)*;*

c) *weakly pseudoinvex* (sPI) $\Rightarrow$ *weakly quasiinvex* (sQI)*;*

d) *pseudoinvex* (P) $\Rightarrow$ *semistrictly quasiinvex* (SSQI)*.*

Proof. The proof follows by analogy to that of Theorem 1.6. □

In the hypotheses of Theorems 1.1-1.7, direct implications between the types of invexity are presented in Figure 1.

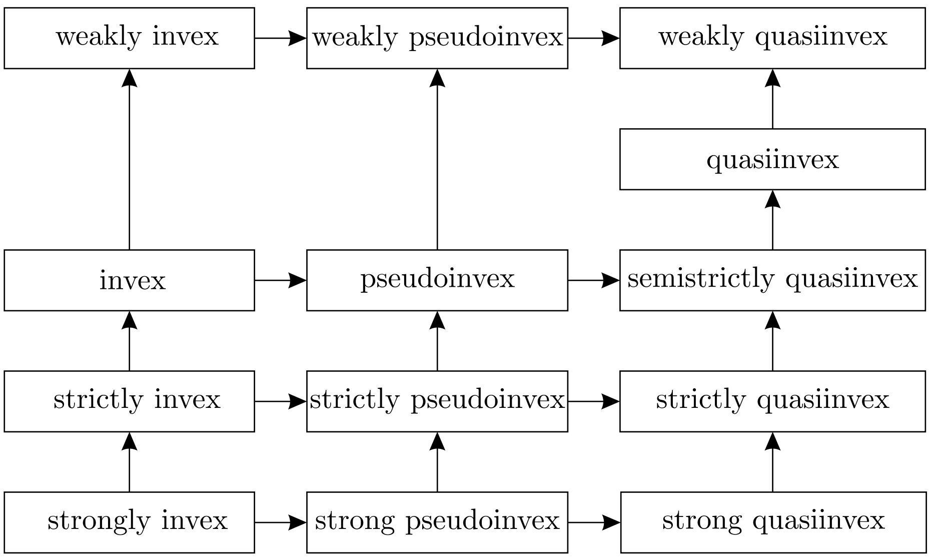

Figure 1. All implications between types of invexity

The following considerations refer to the implications which exist between the types of quasiconvexity and quasiinvexity.

Theorem 1.8. *For a function f and a point u, the following implications hold true:*
a) *ρ-quasiconvex* (ρQX) $\Rightarrow$ *ρ-quasiinvex* (ρQI)*;*
b) *strictly quasiconvex* (SQX) $\Rightarrow$ *strictly quasiinvex* (SQI)*;*
c) *semistrictly quasiconvex* (SSQX) $\Rightarrow$ *strictly quasiinvex* (SSQI).

Proof. Consider the mappings $\eta(x,u) = x - u$, and $\theta(x,u) = x - u$. □

2. CLASSES AND TYPES OF INCAVITY

For a nonsmooth real function $f\colon A \to \mathbb{R}$ and $\rho \in \mathbb{R}$, the types of simple and generalized incavity at a given point are introduced by the following definitions:

Definition 2.1. The function f is called *ρ-incave* (ρIV) at the point $u \in A$ if there are two mappings $\eta, \theta\colon A \times A \to \mathbb{R}^n$ ($\eta \neq 0$) such that

$$\forall x \in A\colon f(x) - f(u) \leq f'_+(u; \eta(x,u)) - \rho\|\theta(x,u)\|^2. \qquad (\rho\text{IV})$$

(4a) If $\rho > 0$, then f is called *strongly incave* (TIV) at u;
(4b) If $\rho = 0$, then f is called *incave* (IV) at u;
(4c) If $\rho < 0$, then f is called *weakly incave* (sIV) at u;
(4d) $\forall x \in A$, $x \neq u\colon f(x) - f(u) < f'_+(u; \eta(x,u)$, then f is called *strictly incave* (SIV) at u.

Definition 2.2. The function f is called *ρ-pseudoincave* (ρPIV) at the point $u \in A$ if there are two mappings $\eta, \theta\colon A \times A \to \mathbb{R}^n$ ($\eta \neq 0$) such that

$$\forall x \in A\colon f'_+(u; \eta(x,u)) - \rho\|\theta(x,u)\|^2 \leq 0 \Rightarrow f(x) \leq f(u). \qquad (\rho\text{PIV})$$

(5a) If $\rho > 0$, then f is called *strongly pseudoincave* (TPIV) at u;
(5b) If $\rho = 0$, then f is called *pseudoincave* (PIV) at u;
(5c) If $\rho < 0$, then f is called weakly *pseudoincave* (sPIV) at u;
(5d) If $\forall x \in A$, $x \neq u\colon f'_+(u; \eta(x,u)) - \rho\|\theta(x,u)\|^2 \leq 0 \Rightarrow f(x) < f(u)$, then f is called *strictly pseudoincave* (SPIV) at u.

Definition 2.3. A function f is called *ρ-quasiincave* (ρQIV) at the point $u \in A$ if there are the mappings $\eta, \theta\colon A \times A \to \mathbb{R}^n$ ($\eta \neq 0$) such that

$$\forall x \in A\colon f(x) \geq f(u) \Rightarrow f'_+(u; \eta(x,u)) - \rho\|\theta(x,u)\|^2 \geq 0. \qquad (\rho\text{QIV})$$

(6a) If $\rho > 0$, then f is called *strongly quasiincave* (TQIV) at u;
(6b) If $\rho = 0$, then f is called *quasiincave* (QIV) at u;
(6c) If $\rho < 0$, then f is called *weakly quasiincave* (sQIV) at u;
(6d) If $\forall x \in A$, $x \neq u\colon f(x) \geq f(u) \Rightarrow f'_+(u; \eta(x,u)) > 0$, then f is called *strictly quasiincave* (SQIV) at the point u;
(6e) If $\forall x \in A$, $x \neq u\colon f(x) > f(u) \Rightarrow f'_+(u; \eta(x,u)) > 0$, then f is called *semi-strictly quasiincave* (SSQIV) at the point u.

If u is an arbitrary point from A, then the previous three definitions characterize the global incavity types.

In general, if f is ρ-invex function, it does not mean that $-f$ is ρ-incave, but this statement is true for the generalized invex types. In order to prove this property, we need the following

Lemma 2.1. *Let u, ρ, η and θ given by Definitions 2.1-2.3 and suppose that the functions $f'_+(u;\cdot)$ and $(-f)'_+(u;\cdot)$ are finite. Then we have the properties:*

a) $f'_+(u;\eta(x,u)) \le \rho\|\theta(x,u)\|^2 \Leftrightarrow f(u+\lambda\eta(x,u)) \le f(u)+\lambda\rho\|\theta(x,u)\|^2$, $\forall\lambda\in(0,1)$;

b) $f'_+(u;\eta(x,u)) < 0 \Rightarrow f(u+\lambda\eta(x,u)) < f(u)$, $\forall\lambda\in(0,1)$;

c) $(-f)'_+(u;\eta(x,u)) \ge -f'_+(u;\eta(x,u))$.

Now we are able to prove the previous statement.

Proposition 2.1. *Let $u\in A$, $\rho\in\mathbb{R}$ and suppose that the functions $f'_+(u;\cdot)$ and $(-f)'_+(u;\cdot)$ are finite. Then, at the point u, the following implications hold:*

$$(\mathrm{i}_1)\ f \textit{ is } (\rho\mathrm{I}) \Rightarrow -f \textit{ is } (\rho\mathrm{IV}); \qquad (\mathrm{i}_2)\ f \textit{ is } (\mathrm{SI}) \Rightarrow -f \textit{ is } (\mathrm{SIV}).$$

Moreover, if f is Gâteaux differentiable at u, then the converse implications are true.

Proof. (i_1) If f is $(\rho\mathrm{I})$ at the point u, for $x\in A$, we have

$$f(x)-f(u)\ge f'_+(u;\eta(x,u))+\rho\|\theta(x,u)\|^2;$$
$$(-f)(x)-(-f)(u)\le -f'_+(u;\eta(x,u))-\rho\|\theta(x,u)\|^2$$

and according to Lemma 2.1, it follows

$$(-f)(x)-(-f)(x)\le(-f)'_+(u;\eta(x,u))-\rho\|\theta(x,u)\|^2. \tag{2.1}$$

Hence, $-f$ is $(\rho\mathrm{IV})$ at the point u.

(i_2) is proved in the same manner as (i_1), with $\rho=0$ and using the signs $>$, $<$ instead of $\ge$, $\le$.

$(\mathrm{i}_1 \Leftarrow)$ We multiply relation (2.1) by -1 and we get

$$f(x)-f(u)\ge -(-f)'_+(u;\eta(x,u))+\rho\|\theta(x,u)\|^2.$$

But $-(-f)'_+(u;\eta(x,u)) = f'_-(u;\eta(x,u))$ and f, being Gâteaux differentiable at u, we have $f'_-(u;\eta(x,u)) = f'_+(u;\eta(x,u))$.

$(\mathrm{i}_2 \Leftarrow)$ is proved similarly to $(\mathrm{i}_1 \Leftarrow)$, where $\rho=0$. □

Proposition 2.2. *Let $u\in A$, $\rho\in\mathbb{R}$ and suppose that the functions $f'_+(u;\cdot)$ and $(-f)'_+(u;\cdot)$ are finite. Then, at the point u,*

$$(\mathrm{p}_1)\ f \textit{ is } (\rho\mathrm{PI}) \Leftrightarrow -f \textit{ is } (\rho\mathrm{PIV}); \qquad (\mathrm{p}_2)\ f \textit{ is } (\mathrm{SPI}) \Leftrightarrow -f \textit{ is } (\mathrm{SPIV}).$$

Proof. (p_1) Suppose that f is $(\rho\mathrm{PI})$ at the point u with respect to the mapping η. According to Definition 1.2, the following implications hold:

$$x\in A\colon f'_+(u;\eta(x,u))\ge-\rho\|\theta(x,u)\|^2 \Rightarrow f(x)\ge f(u);$$
$$x\in A\colon f(x)<f(u)\Rightarrow f'_+(u;\eta(x,u))<-\rho\|\theta(x,u)\|^2;$$
$$x\in A\colon (-f)(x)>(-f)(u)\Rightarrow -f'_+(u;\eta(x,u))>\rho\|\theta(x,u)\|^2$$

and according to Lemma 2.1 c), we get the inequality

$$x \in A\colon (-f)(x) > (-f)(u) \Rightarrow (-f)'_+(u;\eta(x,u)) > \rho\|\theta(x,u)\|^2.$$

In the same manner, we obtain

$$x \in A\colon (-f)'_+(u;\eta(x,u)) \leq \rho\|\theta(x,u)\|^2 \Rightarrow (-f)(x) \leq (-f)(u), \tag{2.2}$$

that is $-f$ is (ρPIV) at the point u.

Conversely, suppose that $-f$ is (ρPIV) at the point u with respect to the mappings $\eta, \theta\colon A \times A \to \mathbb{R}$. Considering Lemma 2.1 a), then implication (2.2), becomes

$$\forall\lambda \in (0,1)\colon (-f)(u+\lambda\eta(x,u)) \leq (-f)(u) + \lambda\rho\|\theta(x,u)\| \Rightarrow (-f)(x) \leq (-f)(u),$$

or, multiplying by -1:

$$\forall\lambda \in (0,1)\colon f(u+\lambda\eta(x,u)) \geq f(u) - \lambda\rho\|\theta(x,u)\|^2 \Rightarrow f(x) \geq f(u).$$

From here we get

$$x \in A\colon f'_+(u;\eta(x,u)) \geq -\rho\|\theta(x,u)\|^2 \Rightarrow f(x) \geq f(u),$$

that is f is (ρPI) at the point u.

(p_2) is proved in a similar manner as point (p_1). For *Necessity*, we use Lemma 2.1 c), and, for *Sufficiency*, Lemma 2.1 a), with $\rho = 0$. $\square$

Proposition 2.3. *Let $u \in A$, $\rho \in \mathbb{R}$. Suppose that the functions $f'_+(u;\cdot)$ and $(-f)'_+(u;\cdot)$ are finite. Then, at the point u, we obtain the relations:*

(q_1) f *is* (ρQI) $\Rightarrow -f$ *is* (ρQIV).

Moreover, if f is Gâteaux differentiable at the point u, then the converse is true.

(q_2) f *is* (SQI) $\Leftrightarrow -f$ *is* (SQIV);
(q_3) f *is* (SSQI) $\Leftrightarrow -f$ *is* (SSQIV) *at the point u.*

Proof. (q_1) Suppose that f is (ρQI) function at the point u with respect to η and θ. Then relation (ρQI) holds, and by multiplying by -1 and using Lemma 2.1 c), we get

$$x \in A\colon (-f)(x) \geq (-f)(u) \Rightarrow (-f)'_+(u;\eta(x,u)) \geq \rho\|\theta(x,u)\|^2. \tag{2.3}$$

Therefore, $-f$ is (ρQIV) function at the point u.

Conversely, if $-f$ is (ρQIV) at the point u, relation (2.3) holds, and it is equivalent to the implication

$$x \in A\colon (-f)'_+(u;\eta(x,u)) < \rho\|\theta(x,u)\|^2 \Rightarrow (-f)(x) < (-f)(u).$$

Using Lemma 2.1 b) at the left-hand of this implication, we obtain

$$x \in A,\ \lambda \in (0,1)\colon (-f)(u+\lambda\eta(x,u)) < (-f)(u) + \lambda\rho\|\theta(x,u)\|^2 \Rightarrow -f(x) < -f(u).$$

By multiplying by -1 this relation, it follows

$$x \in A\colon f'_+(u;\eta(x,u)) > -\rho\|\theta(x,u)\|^2 \Rightarrow f(x) > f(u),$$

that is, $x \in A\colon f(x) \leq f(u) \Rightarrow f'_+(u;\eta(x,u)) \leq -\rho\|\theta(x,u)\|^2$.

Therefore, f is (ρQI) at the point u.

(q_2) If f is (SQI) at the point u, we get

$$x \in A,\ x \neq u\colon f(x) \leq f(u) \Rightarrow f'_+(u;\eta(x,u)) < 0.$$

Multiplying this relation by -1 and using Lemma 2.1 c), we obtain

$$x \in A,\ x \neq u\colon (-f)(x) \geq (-f)(u) \Rightarrow (-f)'_+(u;\eta(x,u)) > 0, \tag{2.4}$$

that is $-f$ is (SQIV) function at the point u.

Conversely, $-f$ is (SQIV) at the point u, hence relation (2.4) holds, equivalent to $x \in A,\ x \neq u\colon (-f)'_+(u;\eta(x,u)) \leq 0 \Rightarrow -f(x) < -f(u)$.

At the left-hand side of this implication we use Lemma 2.1 a), with $\rho = 0$ and we obtain:

$$x \in A,\ x \neq u,\ \lambda \in (0,1)\colon (-f)(u + \lambda\eta(x,u)) \leq (-f)(u) \Rightarrow -f(x) < -f(u);$$
$$x \in A,\ x \neq u,\ \lambda \in (0,1)\colon f'_+(u;\eta(x,u)) \geq 0 \Rightarrow f(x) > f(u),$$

from where it follows $x \in A,\ x \neq u\colon f(x) \leq f(u) \Rightarrow f'_+(u;\eta(x,u)) < 0$.

Therefore f is (SQI) at the point u.

(q_3) Is proved in a similar manner as (q_2). □

3. MINIMA FOR NONSMOOTH INVEX FUNCTIONS

Consider the mappings $f\colon A \times A \to \mathbb{R}$ and $\eta\colon A \times A \to \mathbb{R}^n$.

Theorem 3.1. *If the nonsmooth function f is [strictly] quasiinvex, then any [strictly] local minimum point (if such point does exist), is a [strictly] global minimum (shortly* (PML) $\Rightarrow$ (PMG)*).*

Proof. We shall prove the strictly version. Let $x^1, x^2 \in A$ be two points of local minimum of this function and suppose that $f(x^2) \leq f(x^1)$. We assume that f is quasiinvex with respect to a mapping η and let $V_1 \subset A$ be a neighborhood of the point of x^1. There is $\varepsilon > 0$, small enough, such that $x^1 + \varepsilon\eta(x^2, x^1) \in V_1$. Obviously, $f(x^1 + \varepsilon\eta(x^2, x^1)) > f(x^1)$, and we get

$$f'_+(x^1;\eta(x^2,x^1)) \geq \frac{f(x^1 + \varepsilon\eta(x^2,x^1)) - f(x^1)}{\varepsilon} > 0,$$

which leads to a contradiction.

For the weakly version, we consider that a function f has two subdomains of local minimum, S_1 and S_2. Let $x^1 \in S_1$, $x^2 \in S_2$. Without restricting the generality, we could consider $x^1 \in \operatorname{Fr} S_1$ and x^2 in S_2, such that direction $\eta(x^1, x^2)$ gets outside of S_1 through the point x^1. Then, if V_1 is a neighborhood of S_1, there is $\varepsilon > 0$ such that $x^1 + \varepsilon\eta(x^2, x^1) \in V_1 \setminus S_1$, case in which $f(x^1 + \varepsilon\eta(x^2, x^1)) > f(x^1)$, and the proof continues as in the strictly version. □

Theorem 3.1 praises that if f is a [strictly] invex, [strictly] pseudoinvex or [strictly] quasiinvex function, then (if there exists) any [strict] local minimum point is a [strict] global minimum point.

Definition 3.1. [Komlósi [9]] A point $x^\circ \in A$ is called *stationary point* of the function f if $f'_+(x^\circ; v) \geq 0$, $\forall v \in \mathbb{R}^n$.

Theorem 3.2. *The nonsmooth function f is pseudoinvex if and only if each one of its stationary point is a global minimum,* (PS) $\Rightarrow$ (PMG).

Proof. NECESSITY. Suppose that f is pseudoinvex with respect to the mapping η and let u be a critical point of f from A. Then $f'_+(u;v) \geq 0$, $\forall v \in \mathbb{R}^n$ and considering, in particular, $v = \eta(x,u)$, $\forall x \in A$, we obtain $f'_+(u;\eta(x,u)) \geq 0$, $\forall x \in A$. But f is pseudoinvex function with respect to η and then, according to Definition 1.2 (2b), it follows the inequality $f(x) \geq f(x^\circ)$, $\forall x \in A$. Therefore, u is a global minimum point.

SUFFICIENCY. There are two cases.

(i) Suppose that u is a stationary point of the function f. By hypothesis, we have

$$f'_+(x^\circ;v) \geq 0,\ \forall v \in \mathbb{R}^n \Rightarrow f(x) \geq f(u).$$

In particular, for $v = \eta(x,u)$, where $\forall x \in A$, this implication becomes

$$\forall x \in A\colon f'_+(x^\circ;\eta(x,u)) \geq 0 \Rightarrow f(x) \geq f(u).$$

Hence, f is a pseudoinvex function at the point u.

(ii) u is not a stationary point of f. Then there is a vector $\bar{v} \in \mathbb{R}^n$ such that $f'_+(u;\bar{v}) < 0$. Consider a function η, such that $\eta(\cdot,u)$ is a surjective function. Then there is $\bar{x} \in A$ such that $\bar{v} = \eta(\bar{x},u)$ and it follows the inequality $f'_+(u;\eta(\bar{x},u)) < 0$. In the definition of pseudoinvexity of f at the point u with respect to η, this relation is implied by the inequality $f(\bar{x}) < f(u)$, that is

$$f(\bar{x}) < f(u) \Rightarrow f'_+(u;\eta(\bar{x},u)) < 0. \tag{3.1}$$

Suppose the contrary, that f is not a pseudoinvex function at the point u. Then $f(\bar{x}) \geq f(u)$. Consequently,

$$f'_+(u;\eta(\bar{x},u)) < 0 \Rightarrow f(\bar{x}) \geq f(u). \tag{3.2}$$

Because relations (3.1) and (3.2) are contradictory, our assumption is false. Therefore f is pseudoinvex at the point u. □

From Theorem 3.2 it is clear that the nonsmooth function f is invex if and only if each of its its stationary points is a global minimum point [1], [4].

Theorem 3.3. *If f is a semistrictly quasiinvex function, then every stationary point is a global minimum point, but its domain of global minimum is not connected.*

Proof. Suppose that $u \in A$ is a critical point and f is semistrictly quasiinvex at the point u with respect to η. Then $f'_+(u;\eta(x,u)) \geq 0$, $\forall x \in a$. If f is semistrictly quasiinvex function at the point u with respect to η, according to Definition 1.3 (3e), we have

$$\forall x \in A,\ x \neq u\colon f'_+(u;\eta(x,u)) \geq 0 \Rightarrow f(x) \geq f(u).$$

Therefore, $f(x) \geq f(u)$, $\forall x \in A$ that is u is a global minimum point.

If we refer to the domain, we may consider the function

$$f\colon \mathbb{R} \to \mathbb{R}, \quad f(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0, & \text{if } x \neq 0 \end{cases}$$

which is a semistrictly quasiconvex function, therefore is semistrictly quasiinvex, with $\eta\colon A \times A \to \mathbb{R}$ and $\eta(x,u) = x - u$. Subintervals $(-\infty, 0)$ and $(0, \infty)$ are subdomains of global minimum of this function. But the global minimum domain $(-\infty, 0) \cup (0, \infty)$ is not connected. □

Theorem 3.4. *Let $u \in A$ be a non-stationary point of f, at which the directional function, $f'_+(u;\cdot)$, is finite. Then the following implications hold: f quasiinvex [strictly quasiconvex or semi-strictly quasiinvex] at $u \Rightarrow f$ pseudoinvex at u.*

Because of properties (PML) $\Rightarrow$ (PMG) and (PS) $\Rightarrow$ (PMG), the [generalized] invex functions replaced in optimization theory on those [generalized] convex.

4. CLASSES AND TYPES OF PREINVEX FUNCTIONS

In 1988 Weir and Mond [28] have introduced the notion of preinvex function, and, in 1991, Pini [25] has introduced prepseudoinvex and prequasiinvex functions, which generalize preinvex functions. These functions are defined without the derivative f'_+. We refer to a function $f\colon X \to \mathbb{R}$, where X is a nonempty set within $\mathbb{R}^n$ and to a vector function $\eta\colon X \times X \to \mathbb{R}^n$.

Definition 4.1. [Weir and Mond [28]] The function f is called *preinvex* (pI) on X if there is a vector function η such that

$$(\forall)\, x, u \in X,\ \forall \lambda \in A\colon f(u + \lambda\eta(x,u)) \le \lambda f(x) + (1-\lambda) f(u). \tag{pI}$$

Example 4.1. The function $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = x$, is preinvex with respect to the function $\eta\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$, $\eta(x,u) = -x^2 - u^2 + x - u$.

Indeed, relation (pI) becomes $-\lambda(x^2 + u^2) \le 0$.

Example 4.2. $f\colon \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = -x^2 + 2y$, is a preinvex function with respect to $\eta\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$, $\eta(x,y,u,v) = \left(u, \dfrac{1}{2}x^2 - y + v\right)$.

Denote $\eta = (\eta_1, \eta_2)$, and for $(x,y), (u,v) \in \mathbb{R}^2$, relation (pI) becomes

$$f((u,v) + \lambda(\eta_1, \eta_2)) \le \lambda f(x,y) + (1-\lambda) f(u,v),$$

that is, $(\lambda + 3)u^2 \ge 0$.

Definition 4.2. [Pini [25]] A function f is called *prepseudoinvex* on X, if there is a vector function η and a strictly positive function $b\colon X \times X \to \mathbb{R}_+ \setminus \{0\}$, such that

$$(\forall)\, x, u \in X,\ \forall \lambda \in (0,1)\colon f(x) < f(u) \Rightarrow f(u + \lambda\eta(x,u)) \le f(u) + \lambda(\lambda - 1) b(x,u). \tag{pPi}$$

Definition 4.3. [Mohan and Neogy [23], Pini [25]] A function f is called *prequasiinvex* (pQI) on X if there is a vector function η such that

$$(\forall)\, x, u \in X,\ \forall \lambda \in [0,1]\colon f(x) \le f(u) \Rightarrow f(u + \lambda\eta(x,u)) \le f(u). \tag{pQI}$$

Comparing Definition 4.1 to Definitions 3.1 and 4.2, we remark that the definition of prepseudoinvexity does not obey the same autonomous criterion of generalization, as in Definitions 1.1-1.3, but uses an external element, a function b.

But keeping this criterion to prepseudoinvex functions is confirmed by the following theorem.

Theorem 4.1. *A function f is prepseudoinvex on X if and only if*

$$(\forall)\, x, u \in X,\ \forall\lambda \in (0,1)\colon f(u+\lambda\eta(x,u)) \geq f(u) \Rightarrow f(x) \geq f(u). \tag{4.1}$$

For necessity, we suppose that f and $\eta(\cdot, u)$ are continuous functions.

Proof. Necessity. Relation (pPI) is fulfilled, and it follows that $(\forall)\, x, u \in X$, $x \neq u$, $\lambda \in (0,1)\colon f(x) < f(u) \Rightarrow$

$$f(u+\lambda\eta(x,u)) \leq f(u) + \lambda(\lambda-1)b(x,u) < f(u),$$

or more

$$(\forall)\, x, u \in X,\ x \neq u,\ \lambda \in (0,1)\colon f(u+\lambda\eta(x,u)) \geq f(u) \Rightarrow f(x) \geq f(u).$$

Using the continuity of f and of $\eta(\cdot, u)$, if $x \to u$, from the previous implication we get

$$\forall\lambda \in (0,1)\colon f(u+\lambda\eta(u,u)) \geq f(u) \Rightarrow f(u) \geq f(u).$$

Finally, the two situations give relation (4.1).

Sufficiency. Relation (4.1) is true. That is,

$$x, u \in X,\ \forall\lambda \in (0,1)\colon f(x) < f(u) \Rightarrow f(u+\lambda\eta(x,u)) < f(u). \tag{4.2}$$

Denote

$$d(\lambda) = f(u+\lambda\eta(x,u)) - f(u) < 0$$

and, for $\lambda_0 \in (0,1)$, we consider

$$d_0(x,u) = f(u+\lambda\eta(x,u)) - f(u) < 0.$$

There is a natural number, $N(x,u)$, such that $N(x,u)d(\lambda) \geq |d_0(x,u)|$. For any $\lambda \in (0,1)$, we get:

$$\begin{aligned} &-N(x,u)d(\lambda) \geq |d_0(x,u)| > \lambda(1-\lambda)|d_0(x,u)|,\\ &N(x,u)d(\lambda) < \lambda(\lambda-1)|d_0(x,u)|\\ &d(\lambda) < \lambda(\lambda-1)\frac{|d_0(x,u)|}{N(x,u)}. \end{aligned} \tag{4.3}$$

Denote $\beta(x,u) = \dfrac{|d_0(x,u)|}{N(x,u)} > 0$ and relation (4.3) becomes

$$f(u+\lambda\eta(x,u)) - f(u) < \lambda(\lambda-1)\beta(x,u).$$

Therefore,

$$\forall\lambda \in (0,1)\colon f(u+\lambda\eta(x,u)) - f(u) < 0 \Rightarrow f(u+\lambda\eta(x,u)) - f(u) < \lambda(\lambda-1)\beta(x,u). \tag{4.4}$$

Combining relations (4.2) and (4.4), it follows

$$(\forall)\, x, u \in X,\ \forall\lambda \in (0,1)\colon f(x) < f(u) \Rightarrow f(u+\lambda\eta(x,u)) \leq f(u) + \lambda(\lambda-1)b(x,u).$$

Hence, f is a prepseudoinvex function on X in the sense of Pini. $\square$

Example 4. Function $f\colon \left(0, \dfrac{\pi}{2}\right) \to \mathbb{R}$, $f(x) = \sin^2 x$, is prepseudoinvex with respect to

$$\eta(x,u) = \begin{cases} \cos x(\sin x - \sin u), & \text{if } x \geq u \\ 0, & \text{if } x < u. \end{cases}$$

Indeed, for $x \geq u$, relation (4.1) leads to $\eta(x,u) \geq 0$, which implies $\sin^2 x \geq \sin^2 x$.

Example 5. Function $f\colon \left(0, \dfrac{\pi}{2}\right) \to \mathbb{R}$, $f(x) = \sin x$, is prequasiinvex with respect to

$$\eta(x,u) = \begin{cases} \cos x(\sin x - \sin u), & \text{if } x > u \\ 0, & \text{if } x \leq u. \end{cases}$$

The inequality $f(u+\lambda\eta(x,u)) > f(u) \Rightarrow f(x) > f(u)$ is proved, equivalent to (pQI).

Theorem 4.2. *Between the notions of preinvexity, prepseudoinvexity and prequasi-invexity there are the following implications:*

$$\textit{preinvexity} \Rightarrow \textit{prepseudoinvexity} \Rightarrow \textit{prequasiinvexity}.$$

In the second implication we assume that f and $\eta(u,\cdot)$, $\forall u \in A$, are continuous functions.

Proof. The first implication follows easily, starting with relation (pI). For the second implication, we begin with the hypothesis of Definition 4.1, from which we get $f(u) = \max\{f(x), f(u)\}$, and from Definition 4.1, with $x \neq u$, from which we obtain

$$\lambda \in (0,1)\colon f(u + \lambda\eta(x,u)) \leq f(u) + \lambda(\lambda - 1)b(x,u) \leq f(u) = \max\{f(x), f(u)\},$$

because $b(x,u) > 0$. This relation still remains true for $x = u$, $\lambda = 0$ and for $\lambda = 1$, case in which we take into account the fact that f and $\eta(u,\cdot)$ are continuous functions. $\square$

In 1997, Mititelu [13] introduced the types of preinvex functions, but using the name of invex functions and classified them using the following definitions:

Definition 4.4. A (Preinvexities)**.** A function f is called *ρ-preinvex* (ρpI) at the point $u \in X$ if there are the vector functions $\eta, \theta\colon X \times X \to \mathbb{R}^n$ such that

$$\forall x \in X,\ \forall \lambda \in [0,1]\colon f(u + \lambda\eta(x,u)) \leq \lambda f(x) + (1-\lambda)f(u) - \rho\lambda\|\theta(x,u)\|^2. \quad (\rho\text{pI})$$

(1a′) If $\rho > 0$, f is called *strongly preinvex* function (TpI) at u;
(1b′) If $\rho = 0$, f is called *preinvex* function (pI) at u;
(1c′) If $\rho < 0$, f is called *weakly preinvex* function (spI) at u;
(1d′) If $\forall x \in X$, $x \neq u$, $\forall \lambda \in (0,1)\colon f(u + \lambda\eta(x,u)) < \lambda f(x) + (1-\lambda)f(u)$, f is called *strictly invex* function (SpI) at u.

B (Preincavities)**.** f is called *ρ-preincave* function at the point u if $-f$ is ρ-preinvex function at the point u. If $\rho > 0$, $\rho = 0$ or $\rho < 0$, then f is called *strongly preincave*, *preincave* or *weakly preincave* at the point u. The function f is called *strictly preincave* at the point u if $-f$ is strictly preinvex function at u.

C**.** The function f is *preinvex* [preincave] etc. on X if it is preinvex [preincave] etc. at each point of X.

Definition 4.5. A (Prepseudoinvexities)**.** A function f is called *ρ-prepseudoinvex* (ρpPI) at the point $u\in X$ if there are vector functions $\eta,\theta: X\times X\to\mathbb{R}^n$ such that

$$\forall x \in X,\ \forall\lambda \in (0,1)\colon f(u+\lambda\eta(x,u))+\rho\lambda\|\theta(x,u)\|^2 \geq f(u) \Rightarrow f(x) \geq f(u). \quad (\rho\text{pPI})$$

(2a′) If $\rho > 0$, f is called *strongly prepseudoinvex* function (TpPI) at u;
(2b′) If $\rho = 0$, f is called *prepseudoinvex* function (pPI) at u;
(2c′) If $\rho < 0$, f is called *weakly prepseudoinvex* function (spPI) at u;
(2d′) If $\forall x \in X$, $x \neq u$, $\forall\lambda \in (0,1)\colon f(u+\lambda\eta(x,u)) \geq f(u) \Rightarrow f(x) > f(u)$, f is called *strictly prepseudoinvex* function (SpPI) at the point u.

B (Prepseudoincavities)**.** f is called *ρ-prepseudoincave* function at the point u if $-f$ is ρ-prepseudoinvex function at u. If $\rho > 0$, $\rho = 0$ or $\rho < 0$, then f is called *strongly prepseudoincave*, *prepseudoincave* or *weakly prepseudoincave* at u. The function f is called *strictly prepseudoincave* at the point u if $-f$ is strictly prepseudoinvex at u.

C**.** A function f is called *prepseudoinvex* [prepseudoincave] on X etc. if it is prepseudoinvex [prepseudoincave] at each point of X etc..

Definition 4.6. A (Prequasiinvexities)**.** A function f is called *ρ-prequasiinvex* (ρpQI) at the point $u \in X$ if there are the vector functions $\eta,\theta\colon X \times X \to \mathbb{R}^n$ such that

$$\forall x \in X,\ \forall\lambda \in (0,1)\colon f(x) \leq f(u) \Rightarrow f(u+\lambda\eta(x,u))+\rho\lambda\|\theta(x,u)\|^2 \leq f(u). \quad (\rho\text{pQI})$$

(3a′) If $\rho > 0$, f is called *strongly prequasiinvex* function (TpQI) at u;
(3b′) If $\rho = 0$, f is called *prequasiinvex function* (pQI) at u;
(3c′) If $\rho < 0$, f is called *weakly prequasiinvex* function (spQI) at u;
(3d′) If $\forall x \in X$, $x \neq u$, $\forall\lambda \in (0,1)\colon f(x) \leq f(u) \Rightarrow f(u+\lambda\eta(x,u)) < f(x)$, f is called *strictly prequasiinvex* function (SpQI) at u;
(3e′) $\forall x \in X$, $x \neq u$, $\forall\lambda \in (0,1)\colon f(x) < f(u) \Rightarrow f(u+\lambda\eta(x,u)) < f(x)$, f is called *semistrictly prequasiinvex* function (SSpQI) at u.

B (Prequasiincavities)**.** f is called *ρ-prequasiincave* function at the point u if $-f$ is ρ-prequasiinvex at u. If $\rho > 0$, $\rho = 0$ or $\rho < 0$, then f is called *strongly prequasiincave*, *prequasiincave* or *weakly prequasiincave* at u. f is called strictly *prequasiincave* at the point u, if $-f$ is strictly prequasiinvex at u. f is called *semistrictly prequasiincave* at the point u if $-f$ is semistrictly prequasiinvex at u.

C**.** f is *prequasiinvex* [prequasiincave] function on X etc. if it is prequasiinvex [prequasiincave] at each point of X etc..

Weir and Mond [19] proved that, for a preinvex function, each local minimum point is a global minimum point. But this property is common to other generalized preinvex functions, as the following theorem shows.

Theorem 4.3 (Mititelu [13])**.** *Let u be a local [strict] minimum point of the function f in X. If f is [strictly] preinvex, [strictly] prepseudoinvex or [strictly] prequasiinvex at the point u, then u is a [strictly] global minimum point of function f.*

Proposition 4.1 ([13]). *Let f be a semistrictly prequasiinvex function at the point $u \in A$. If u is a local minimum point of f, then u is an absolut minimum point of f (the subdomain of f is not connected).*

In the following, we deal with operations with preinvex functions.

Proposition 4.2. *If functions $f_1, \dots, f_m \colon X \to \mathbb{R}$ are η-preinvex [η-prepseudoinvex, η-prequasiinvex], then function $c_1 f_1 + \dots + c_m f_m$ ($c_1 > 0, \dots, c_m > 0$) is η-preinvex [η-prepseudoinvex, η-prequasiinvex].*

Proposition 4.3. *If $(f_i)_{i \in I}$ is a family of preinvex real functions on X, then $\sup\limits_{i \in I} f_i$ is a preinvex function on X.*

Proposition 4.4 ([27]). *If f is η-preincave function on X such that $f(x) > 0$ for each $x \in X$, then $\dfrac{1}{f}$ is η-preinvex function on X.*

Theorem 4.4. *If f and g are two positive η-preinvex functions on X, then fg is (-1)-preinvex function on X with respect to functions $\eta, \theta \colon X \times X \to \mathbb{R}^n$.*

Theorem 4.5. *Let us consider $X \subseteq \mathbb{R}^n$ an invex set, $C \subseteq \mathbb{R}^m$ a convex set, the vector function $u = (u_1, \dots, u_m) \colon X \to C$, a scalar function $f \colon C \to \mathbb{R}$ and composition function $F = f \circ u \colon X \to \mathbb{R}$. If $f(u_1, \dots, u_m)$ is a convex [strictly convex] function and in addition, f is increasing [strictly increasing] function with respect to each invex [strictly invex] component, u_i, or f is decreasing [strictly decreasing] with respect to each incave [strictly incave] component u_j, then F is invex [strictly invex] on X.*

Conversely, if F is invex [strictly invex] function and u is an affine vector function, then f is invex [strictly invex] function on C.

5. INVEXITY AND PREINVEXITY ON OPEN SETS

For real functions defined on open sets, we establish connections between the notions of invexity and preinvexity. Therefore, in the following, we refer to a function $f \colon A \to \mathbb{R}$, where A is an open nonempty set, and also to vector functions $\eta, \theta \colon A \times A \to \mathbb{R}^n$. Let $u \in A$ and $\rho \in \mathbb{R}$. The following theorems claim that the notions of simple and generalized preinvexity with respect to open sets are respectively equivalent. The origin of these equivalences consists in some equivalences of convexity (Theorem 2.41 [19]) and quasiconvexity (Theorem 3.9 [19]).

Theorem 5.1. *Suppose that f'_+ is a finite function. Then the nonsmooth function f is η-invex if and only if f is η-preinvex.*

Proof. NECESSITY. Suppose that f is η-invex function at the point $u \in A$. Then, according to Definition 1.1 (1b), we get the inequality

$$\forall x \in A \colon f(x) - f(u) \geq f'_+(u; \eta(x, u)). \tag{5.1}$$

Function $f'_+(u;\cdot)$ is positively homogeneous and then, after a multiplication by an arbitrary scalar $t \geq 0$, relation (5.1) becomes

$$x \in A\colon tf(x) - tf(u) \geq f'_+(u; t\eta(x,u));$$
$$x \in A\colon tf(x) - tf(u) \geq \limsup_{\mu \searrow 0} \frac{f(u + \mu t\eta(x,u)) - f(u)}{\mu}.$$

For any $t \geq 0$ there is $\bar{\mu} > 0$ such that

$$x \in A\colon tf(x) - tf(u) \geq \frac{f(u + \bar{\mu} t\eta(x,u)) - tf(u)}{\bar{\mu}}.$$

It follows

$$\forall x \in A,\ \forall t \geq 0\colon \bar{\mu} tf(x) - \bar{\mu} tf(u) \geq f(u + \bar{\mu} t\eta(x,u)) - f(u).$$

Denoting $\lambda = \bar{\mu} t$ and taking $t = \dfrac{1}{\mu}$, this inequality becomes the definition of preinvexity of f at u with respect to η.

Sufficiency. Function f is η-preinvex at the point u. According to Definition 4.1, from relation (pI) it follows the inequality

$$\frac{f(u + \lambda\eta(x,u)) - f(u)}{\lambda} \leq f(x) - f(u)$$

and takin the upper limit at the left hand side after $\lambda \searrow 0$, we get

$$f'_+(u; \eta(x,u)) \leq f(x) - f(u),$$

for any $x \in A$. Therefore f is η-invex at the point u. □

Theorem 5.2. *Suppose that the derivative f'_+ is finite. Then the nonsmooth function f is η-pseudoinvex if and only if it is η-prepseudoinvex.*

Proof. Necessity. Suppose that f is η-pseudoinvex at the point $u \in A$. Then

$$\forall x \in A\colon f'_+(u; \eta(x,u)) \geq 0 \Rightarrow f(x) \geq f(u),$$

or equivalently,

$$\forall x \in A\colon f(x) < f(u) \Rightarrow f'_+(u; \eta(x,u)) < 0.$$

According to Lemma 2.1 b), we obtain

$$\forall \lambda \in (0,1)\colon f(u + \lambda\eta(x,u)) < f(u).$$

By transitivity, we get the implication

$$\forall x \in A,\ \forall \lambda \in (0,1)\colon f(x) < f(u) \Rightarrow f(u + \lambda\eta(x,u)) < f(u),$$

that is,

$$\forall x \in A,\ \forall \lambda \in (0,1)\colon f(u + \lambda\eta(x,u)) \geq f(u) \Rightarrow f(x) \geq f(u).$$

Therefore, f is η-prepseudoinvex function at the point u.

Sufficiency. We admit now that f is η-prepseudoinvex function at the point $u \in A$. Then, according to Definition 4.2′ (2b′) we have

$$\forall x \in A,\ \forall \lambda \in (0,1)\colon f(u + \lambda\eta(x,u)) \geq f(u) \Rightarrow f(x) \geq f(u).$$

From the foregoing implication, we obtain

$$\forall x \in A\colon \frac{f(u+\lambda\eta(x,u)) - f(u)}{\lambda} \geq 0 \Rightarrow f(x) \geq f(u)$$

and taking the upper limit $\lambda \searrow 0$ from the fraction, this relation becomes

$$\forall x \in A\colon f'_+(u;\eta(x,u)) \geq 0 \Rightarrow f(x) \geq f(u).$$

Therefore, function f is η-pseudoinvex at the point u. □

Theorem 5.3. *Suppose that the derivative f'_+ is a finite function. Then f is η-quasiinvex function if and only if it is η-prequasiinvex.*

Proof. Necessity. Assume that f is η-quasiinvex function at the point $u \in A$. Then, according to Definition 1.3, we get

$$\forall x \in A\colon f(x) \leq f(u) \Rightarrow f'_+(u;\eta(x,u)) \leq 0$$

and according to Lemma 2.1 a), with $\rho = 0$, it follows

$$f'_+(u;\eta(x,u)) \leq 0 \Rightarrow f(u+\lambda\eta(x,u)) \leq f(u),\ \forall\lambda \in [0,1].$$

By transitivity, we obtain

$$\forall x \in A,\ \forall\lambda \in [0,1]\colon f(x) \leq f(u) \Rightarrow f(u+\lambda\eta(x,u)) \leq f(u), \tag{5.2}$$

that is f is η-prequasiinvex at the point u.

Sufficiency. If f is η-prequasiinvex at the point u, relation (5.2) holds. Then

$$\forall x \in A,\ \lambda > 0\colon f(x) \leq f(u) \Rightarrow \frac{f(u+\lambda\eta(x,u)) - f(u)}{\lambda} \leq 0,$$

and $\limsup\limits_{\lambda\searrow 0} \dfrac{f(u+\lambda\eta(x,u)) - f(u)}{\lambda} \leq 0$. Therefore, we obtain

$$\forall x \in A\colon f(x) \leq f(u) \Rightarrow f'_+(u;\eta(x,u)) \leq 0,$$

that is f is η-quasiinvex at the point u. □

Having in view the equivalence of the notions of invexity and preinvexity, to different levels of generalizations on open sets, according to Theorems 4.2-4.4, it is natural that the invex function and its generalizations to be overtaken by the definitions of preinvex functions and of generalized preinvex functions with respect to open sets, because these one are not defined with derivatives, an advantage in computations. Moreover, the definition of (generalized) invex functions, without the derivative f'_+, offers them a wider area of applications (we mean the cases when f'_+ may be infinite).

6. PROPERTIES FROM PREINVEXITY TO INVEXITY

Some of the properties of [generalized] preinvex functions on open sets transfer, respectively, to [generalized] invex functions. In the following, we shall refer to definitions, minimum points and algebraic operations.

a) Definitions of invex functions on open sets, without derivative f'_+

We address to the same function $f\colon A \to \mathbb{R}$, where A is a nonempty open set from $\mathbb{R}^n$ and to mappings $\eta, \theta\colon A \times A \to \mathbb{R}^n$. First we introduce:

Proposition 6.1. *Let $\rho \in \mathbb{R}$ and assume that the derivative f'_+ is finite. Then f is ρ-invex function with respect to η if and only if there is a mapping θ such that*

$$(\forall)\, x, u \in A,\ \forall \lambda \in [0,1]\colon f(u + \lambda\eta(x,u)) \leq \lambda f(x) + (1-\lambda) f(u) - \lambda\rho\|\theta(x,u)\|^2.$$

Proof. Similar to the one of Theorem 4.2. □

Proposition 4.2, Lemma 2.1, Definition 1.1-1.3 and Theorems 4.2-4.4 allow us to redefine strongly, strictly and weakly types of the notions of invexity, pseudoinvexity and quasiinvexity without derivative f'_+, using these definitions:

Definition 6.1 (Invexities)**.** A function f is called *ρ-invex* (ρI$'$) at the point $u \in A$ if there are two vector functions η and θ such that

$$\forall x \in A,\ \forall \lambda \in [0,1]\colon f(u + \lambda\eta(x,u)) \leq \lambda f(x) + (1-\lambda) f(u) - \rho\lambda\|\theta(x,u)\|^2. \quad (\rho\mathrm{I}')$$

(1a$''$) If $\rho > 0$, then f is called *strongly invex* (TI$'$) at u;
(1b$''$) If $\rho = 0$, then f is called *invex* (I$'$) at u;
(1c$''$) If $\rho < 0$, then f is called *weakly invex* (sI$'$) at u;
(1d$''$) If $\forall x \in A$, $x \neq u$, $\forall \lambda \in (0,1)\colon f(u+\lambda\eta(x,u)) < \lambda f(x) + (1-\lambda) f(u)$, then f is called *strictly invex* (SI$'$) at u.

Definition 6.2 (Pseudoinvexities)**.** A function f is called *ρ-pseudoinvex* (ρPI$'$) at the point $u \in A$ if there are two vector functions η and θ such that

$$\forall x \in A,\ \forall \lambda \in (0,1)\colon f(u + \lambda\eta(x,u)) + \rho\|\theta(x,u)\|^2 \geq f(u) \Rightarrow f(x) \geq f(u). \quad (\rho\mathrm{PI}')$$

(2a$''$) If $\rho > 0$, then f is called *strongly pseudoinvex* (TPI$'$) at u;
(2b$''$) If $\rho = 0$, then f is called *pseudoinvex* function (PI$'$) at u;
(2c$''$) If $\rho < 0$, then f is called *weakly pseudoinvex* (sPI$'$) at u;
(2d$''$) If $\forall x \in A$, $x \neq u$, $\forall \lambda \in (0,1)\colon f(u + \lambda\eta(x,u)) \geq f(u) \Rightarrow f(x) \geq f(u)$, then f is called *strictly pseudoinvex* (SPI$'$) at u.

Definition 6.3 (Quasiinvexites)**.** A function f is called *ρ-quasiinvex* (ρQI$'$) at the point $u \in A$ if there are two vector functions η and θ such that

$$\forall x \in A,\ \forall \lambda \in [0,1]\colon f(x) \leq f(u) \Rightarrow f(u + \lambda\eta(x,u)) + \rho\lambda\|\theta(x,u)\|^2 \leq f(u). \quad (\rho\mathrm{QI})$$

(3a$''$) If $\rho > 0$, then f is called *strongly quasiinvex* (TQI$'$) at u;
(3b$''$) If $\rho = 0$, then f is called *quasiinvex* (QI$'$) at u;
(3c$''$) If $\rho < 0$, then f is called *weakly quasiinvex* (sQI') at u;

(3d$''$) If $\forall x \in A$, $x \neq u$, $\forall \lambda \in (0,1)$: $f(x) \leq f(u) \Rightarrow f(u+\lambda\eta(x,u)) < f(u)$, then f is called *strictly quasiinvex* (SQI$'$) at u;

(3e$''$) $\forall x \in A$, $x \neq u$, $\forall \lambda \in (0,1)$: $f(x) < f(u) \Rightarrow f(u+\lambda\eta(x,u)) < f(u)$, then f is called *semistrictly quasiinvex* (SSQI$'$) at u.

In the same manner, following Definitions 6.1-6.3, we define the types and classes of incave functions, without derivative f'_+, at the point u or on the domain A.

From Theorem 4.3 of equivalence of the notions of pseudoinvexity and prepseudoinvexity on open domanis, and Theorem 3.2 referring to the stationary points of pseudoinvex functions, the next result is obtained.

Proposition 6.2. *For a prepseudoinvex function with open domain, every stationary point is a global minimum point.*

b) Operations with ρ-invex and ρ-incave functions on open sets

Propositions 4.2-4.4 and Theorems 4.4-4.5 hold true for invex [incave], of course, on open sets. In the following, we present other results.

Let be given A be an open set in $\mathbb{R}^n$ and the functions $f, g\colon A \to \mathbb{R}$.

Theorem 6.1. *Let the functions f and g, ρ-invex on set A with respect to mappings η, θ_1 and η, θ_2. If, f and g are positive functions on A and $\rho > 0$, then the product function fg is*

1) (-1)*-invex on A with respect to η and $\bar{\theta}_1$, where*

$$\|\bar{\theta}_1(x,u)\|^2 = f(x)g(u) + f(u)g(x) + \rho^2\|\theta_1(x,u)\|^2\|\theta_2(x,u)\|^2,$$

or

2) $(-\rho)$*-invex on X with respect to η and $\bar{\theta}_2$, where*

$$\|\bar{\theta}_2(x,u)\|^2 = \frac{1}{\rho}[f(x)g(u) + f(u)g(x)] + \rho\|\theta_1(x,u)\|^2\|\theta_2(x,u)\|^2.$$

Proof. For $x, u \in A$ and $\lambda \in [0,1]$ the following relations hold

$$\begin{aligned} f(u+\lambda\eta(x,u)) &\leq \lambda f(x) + (1-\lambda)f(u) - \rho\|\theta_1(x,u)\|^2, \\ g(u+\lambda\eta(x,u)) &\leq \lambda g(x) + (1-\lambda)g(u) - \rho\|\theta_2(x,u)\|^2, \end{aligned}$$

which, multiplied side by side led to the inequality

$$\begin{aligned} (fg)(u+\lambda\eta(x,u)) \leq{}& [\lambda f(x) + (1-\lambda)][\lambda g(x) + (1-\lambda)g(u)] \\ & -\lambda\rho\|\theta_2\|^2[\lambda f(x) + (1-\lambda)f(u)] - \lambda\rho\|\theta_1\|^2[\lambda g(x) + (1-\lambda)g(u)] \\ & +\lambda^2\rho^2\|\theta_1\|^2\|\theta_2\|^2 \\ \leq{}& \lambda(fg)(x) + (1-\lambda)(fg)(u) + f(x)g(u) + f(u)g(x) + \rho^2\|\theta_1\|^2\|\theta_2\|^2. \end{aligned}$$

From the latter form of the right-hand member of these inequality follows 1) and 2). We have denoted θ_1 instead $\theta_1(x,u)$ etc. $\square$

Theorem 6.2. *Let us consider two functions f and g positive definite on A. Suppose that f is ρ-invex with respect to mappings η and θ_1, and g is ρ-incave with respect to*

η and θ_2. Then the quotient function $\dfrac{f}{g}$ is ρ-invex with respect to η and a mapping θ, which can be determined.

7. INVEX FUNCTIONS ON ARBITRARY SETS

Let $A \subseteq \mathbb{R}^n$ be an open set, $x \in A$ and $v \in \mathbb{R}^n$.

Proposition 7.1. *Suppose that f is a continuous function on a neighborhood of x and f'_+ is finite on a neighborhood (x, v). Then f'_+ is a upper semicontinuous function at point (x, v).*

Consider now a function $f\colon D \to \mathbb{R}$, where D is an nonempty open set in $\mathbb{R}^n$ and let X be a nonempty set included in D. If f'_+ is a finite function on $X \times \mathbb{R}^n$ and following the definition of invexity of f on X then, comparing to the invexity of functions on open sets (see relation (5.1), it follows that we must study the invexity of f on the boundary of X and this refers to η (we denote by $\mathrm{Fr}\, X$ the boundary of X).

Definition 7.1 ([17])**.** *A vector function $\tilde{\eta}\colon X \times X \to \mathbb{R}^n$ is called continuous prolongation on $X \times X$ of vector function $\eta\colon \mathrm{ri}X \times \mathrm{ri}X \to \mathbb{R}^n$ if $\tilde{\eta}(x, u) = \eta(x, u)$, $\forall (x, u) \in \mathrm{ri}X \times \mathrm{ri}X$ and for each $y, b \in (\mathrm{Fr}\, X) \cap X$ and $x, u \in \mathrm{ri}X$ the upper limit $\displaystyle\lim_{(x,u)\to(y,b)} \eta(x, u)$ exists and is equals to $\tilde{\eta}(y, b)$.*

Theorem 7.1. *Let D be an open set in $\mathbb{R}^n$, $X \subset D$ a nonempty set and two functions $f\colon D \to \mathbb{R}$ and $\eta\colon \mathrm{ri}X \times \mathrm{ri}X \to \mathbb{R}^n$. More than this, suppose that the following conditions are satisfied:*
1) *f is continuous on X;*
2) *$\tilde{\eta}\colon X \times X \to \mathbb{R}^n$ is a continuous prolongation of η on $X \times X$;*
3) *f is invex on $\mathrm{ri}X$ with respect to η.*

Then f is invex on X with respect to $\tilde{\eta}$.

Proof. Because f is η-invex function on $\mathrm{ri}X$ it follows

$$(\forall)\, x, u \in \mathrm{ri}X\colon f(x) - f(u) \geq f'_+(u; \eta(x, u)). \tag{7.1}$$

Let $y, b \in (\mathrm{Fr}\, X) \cap X$. According to Proposition 7.1, f'_+ is upper semicontinuous at the point (x, u). In relation (7.1) we take the upper limit after $(x, u) \to (y, b)$ and it results

$$f(y) - f(b) \geq \lim_{(x,u)\to(y,b)} f'_+(u; \eta(x, u)) = f'_+(b, \lim_{(x,u)\to(y,b)} \eta(x, u)). \tag{7.2}$$

But $\tilde{\eta}$ is an extension of η on X and from relation (7.2) we obtain

$$f(y) - f(b) \geq f'_+(b; \tilde{\eta}(y, b)).$$

This relation holds for each $(y, b) \in X \times X$. Therefore, f is invex on X with respect to $\tilde{\eta}$. □

Theorem 7.2 ([17]). *Let D be a nonempty open set in $\mathbb{R}^n$, $X \subset D$ a nonempty subset and the functions $f\colon D \to \mathbb{R}$ and $\eta\colon X \times X \to \mathbb{R}^n$. Suppose satisfied conditions* 1) *and* 2) *of Theorem* 7.1*, and also the following condition:*

4) f is pseudoinvex on ri$X$ *with respect to η.*

Then f is pseudoinvex on X with respect to $\tilde{\eta}$.

Proof. It is similar to those of Theorem 7.1. □

Theorem 7.3. *Let D be a nonempty open set in $\mathbb{R}^n$, $X \subset D$ a nonempty subset and the functions $f\colon D \to \mathbb{R}$ and $\eta\colon X \times X \to \mathbb{R}^n$. Suppose satisfied conditions* 1) *and* 2) *of Theorem* 7.1*, and also the following condition:*

5) f is quasiinvex on ri$X$ *with respect to η.*

Then f is quasiinvex on X with respect to $\tilde{\eta}$.

Using the sample given by Definitions 1.1-1.3 for the kinds and classes of invexity on open sets we may define the kinds and classes of an η-invex function f on arbitrary set X in the case when X is included into an open set D, on which f'_+ is finite, f is uniformly continuous on X, and η is continuous on $X \times X$.

Open problem 7.1. Could there be found an exhaustive theory for investigation optimality conditions and duality theorems for nonlinear single(multi)-objective optimization problems involving nonsmooth invex functions as they are introduced in this study ?

REFERENCES

[1] A. Ben-Israel and B. Mond: *What is invexity ?*, J. Austral. Math. Soc., Ser. B, **28**(1986), No. 1, 1-9.

[2] B. D. Craven: *Invex function and constrained local minima*, Bull. Austral. Math. Soc., **24**(1981), 357-366.

[3] B. D. Craven: *Nondifferentiable optimization by smooth approximation*, Optimization, **1**(1986), 3-17.

[4] D. D. Craven and B. M. Glover: *Invex function and duality*, J. Austral. Math. Soc., Ser. A, **39**(1985), 1-20.

[5] G. Giorgi and Şt. Mititelu: *Convexités généralisés et propriétés*, Rev. Roumaine Math. Pures Appl., **38**(1993), No. 2, 125-142.

[6] M. A. Hanson: *On sufficiency of the Kuhn-Tucker conditions*, J. Math. Anal. Appl., **80**(1981), 545-550.

[7] V. Jeyakumar: *Strong and weak invexity in mathematical programming*, Math. Oper. Res., **55**(1985), 109-125.

[8] R. N. Kaul and S. Kaul: *Optimality criteria in nonlinear programming involving nonconvex functions*, J. Math. Anal. Appl., **105**(1985), 104-112.

[9] S. Komlósi: *Some properties of nondifferentiable pseudoconvex functions*, Math. Programming, **26**(1983), 232-237.

[10] D. H. Martin: *The essence of invexity*, J. Optim. Theory Appl., **47**(1985), 65-76.

[11] S. K. Mishra: *Non-differentiable higher order symmetric duality in mathematical programming with generalized invexity*, Eur. J. Oper. Res., **167**(2005), 28-34.

[12] Şt. Mititelu: *Invex sets*, Stud. Cerc. Mat., **46**(1994), No. 5, 529-532.

[13] Şt. Mititelu: *Generalized invexities and global minimum properties*, Balkan J. Geom. Appl., **2**(1997), No. 1, 61-72.

[14] Şt. Mititelu: *An equivalency condition of the invexity and preinvexity notions*, Proc. 2-nd Internat. Coll. Math. Engng. Num. Phys. (MENP-2), April 22-27, 2002, Univ. Politehnica of Bucharest, BSG Proceedings 8, 124-131, Geometry Balkan Press, 2003.

[15] Şt. Mititelu: *Invex functions*, Rev. Roumaine Math. Pures Appl., **49**(2004), No. 5-6, 529-544.

[16] Şt. Mititelu: *Nonsmooth invex functions. The preinvex case*, Math. Reports, **8(58)** (2006), No. 4, 435-452.

[17] Şt. Mititelu: *Invex sets and nonsmooth invex functions*, Rev. Roumaine Math. Pures Appl., **52**(2007), No. 6, 665-672.

[18] Şt. Mititelu: *Extension in invexity theory*, J. Adv. Math. Stud., **1**(2008), No. 1-2, 87-94.

[19] Şt. Mititelu: *Generalized Convexities*, Geometry Balkan Press, Bucharest, 2009 (in Romanian).

[20] Şt. Mititelu and I. M. Stancu-Minasian: *Invexity at a point: generalizations and clasification*, Bull. Austral. Math. Soc., **48**(1993), 117-126.

[21] Şt. Mititelu: *Invex sets and preinvex functions*, J. Adv. Math. Stud., **2**(2009), No. 2, 41-52.

[22] Şt. Mititelu and I. M. Stancu-Minasian: *On quasiinvexity and semistrict quasiinvexity at a point*, Stud. Cerc. Mat., **47**(1995), No. 3-4, 313-316.

[23] S. R. Mohan and S. K. Neogy: *On invex sets and preinvex functions*, J. Math. Anal. Appl., **189**(1995), 901-909.

[24] C. Nohak and R. N. Mohapatra: *Nonsmooth* ρ*-*(η,θ)*-invexity in multiobjective programming problems*, Optim. Lett., DOI 10.1007/s11590-010-0239-1.

[25] R. Pini: *Invexity and generalized convexity*, Optimization, **22**(1991), 513-525.

[26] T. W. Reiland: *Nonsmooth invexity*, Bull. Austral. Math. Soc., **42**(1990), 437-446.

[27] S. K. Suneja and M. K. Srivastava: *Optimality and duality in nondifferentiable multiobjective optimization involving d-type I and related functions*, J. Math. Anal. Appl., **208**(1997), No. 2, 465-479.

[28] T. Weir and B. Mond: *Preinvex functions in multiple objective optimization*, J. Math. Anal. Appl., **136**(1988), No. 1, 29-38.

Technical University of Civil Engineering
Bd. Lacul Tei, No. 124, 020396 *Bucharest, Romania*
E-mail address: st_mititelu@yahoo.com

University "Politehnica" of Bucharest
Faculty of Applied Sciences
Splaiul Independenţei, No. 313, 060042 *Bucharest, Romania*
E-mail address: mihai@mathem.pub.ro