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VERSOR FIELDS ALONG A CURVE IN A FOUR DIMENSIONAL LORENTZ SPACE

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ABSTRACT. We extend the Frenet frame and Frenet formulas from the curves in Lorentz spaces to versor fields along a curve in a Lorentz space.

INTRODUCTION

As it is well known, the geometry of versor fields along a curve is a generalization of usual theory of curves, in which the versor field is nothing but the tangent versor field to the given curve. A first systematic approach of this geometry in three Euclidian spaces is due to Radu Miron ([7], [8]). There exists a continuous interest in the study of Lorentz geometry [2], [9], [10], [11], and, in particular, of curves in Lorentz spaces [3], [4], [5], [6], [12].

In this paper, we extend the Frenet frame and formulas to versor fields in Lorentz spaces. We confine ourselves to the case $n = 4$, but it is obvious that similar Frenet formulas hold for every dimension n. The Frenet formulas introduce a set of invariants, some of them also called curvatures of the versor fields. We prove that, conversely, given such a set of invariants a versor field along a curve is determined, unique up to a Lorentz motion (the fundamental theorem).

The paper is organized as follows. In section 1 we recall some basic notions from Lorentz geometry and we fix the notations. In section 2 we discuss the case of dimension $n = 4$. We provide a complete proof of the fundamental theorem.

1. PRELIMINARIES

A pseudo-Euclidean space $\mathbb{E}^n_v$ is a real n-space $\mathbb{R}^n$ endowed with the standard flat metric given by

$$\langle\,,\rangle = -\sum_{i=1}^{v} dx_i^2 + \sum_{i=v+1}^{n} dx_i^2$$

where $(x_1, x_2, \ldots, x_n)$ is a rectangular coordinate system in $\mathbb{E}^n_v$.

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Since $\langle\ ,\ \rangle$ is an indefinite metric, recall that a vector $\vec{v} \in \mathbb{E}^n_v$ can have one of three causal characters: it can be spacelike if $\langle\vec{v},\vec{v}\rangle > 0$ or $\vec{v} = 0$, timelike if $\langle\vec{v},\vec{v}\rangle < 0$ and null (lightlike) if $\langle\vec{v},\vec{v}\rangle = 0$ and $\vec{v} \neq 0$.

Let $x(s)\colon I \subset \mathbb{R} \to \mathbb{E}^n_v$ be a regular curve in $\mathbb{E}^n_v$. It can be spacelike, timelike or null (lightlike), if all of its velocity vectors $x'(s)$ are spacelike, timelike or null (lightlike), respectively.

Recall that the pseudo-norm of any arbitrary vector $v \in \mathbb{E}^n_v$ is $\|v\| = \sqrt{|\langle v, v\rangle|}$.

For $n \geq 2$, the space $\mathbb{E}^n_1$ is called Minkowski n-space. The Minkowski 4-space $\mathbb{E}^4_1$ is called a Minkowski spacetime.

The Lorentz group is the group of all isometries of Minkovski spacetime. It may be described as the generalized orthogonal group $O(1,3)$, the matrix Lie group which preserves the quadratic form

$$(t,x,y,z) \to t^2 - x^2 - y^2 - z^2 \text{ on } \mathbb{R}^4.$$

This quadratic form is interpreted in Physics as the metric tensor of Minkovski spacetime, so this definition is a simply restatement of the fact that Lorentz transformations are precisely the linear transformations which are also isometries of Minkovski spacetime.

The Lorentz group also preserves the quadratic form

$$(t,x,y,z) \to x^2 + y^2 + z^2 - t^2$$

and it is therefore sometimes denoted $O(3,1)$.

2. VERSOR FIELDS IN $\mathbb{E}^4_1$. FUNDAMENTAL THEOREM

We study the Frenet frames and Frenet formulas in the Minkovski spacetime $\mathbb{E}^4_1$, that is $\mathbb{R}^4$ with the metric $g = -dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2$.

Let ξ be a versor field on $\mathbb{E}^4_1$. We can locally have one of the following causal characters: ξ spacelike, null or timelike. We will separately look at these three cases and construct their Frenet frame $\{\xi_1, \xi_2, \xi_3, \xi_4\}$.

Case 1. $\xi = \xi_1$ is spacelike, that is $\langle\xi_1, \xi_1\rangle = 1$.

Let C be a curve tangent to ξ_1. We shall construct its Frenet frame (C, ξ). If $\dfrac{d\xi_1}{ds} \neq 0$, then $\dfrac{d\xi_1}{ds}$ is perpendicular to ξ_1, so take ξ_2 in the direction of $\dfrac{d\xi_1}{ds}$.

Depending on the causal character of $\dfrac{d\xi_1}{ds}$ we have the following cases.

Case 1.1. $\left\langle \dfrac{d\xi_1}{ds}, \dfrac{d\xi_1}{ds} \right\rangle > 0$.

ξ_2 is then the normalized vector field corresponding to $\dfrac{d\xi_1}{ds}$. The vector field ξ_3 is in the direction of the normal component $C^\perp$ of $\dfrac{d\xi_2}{ds}$ with respect to the plane $\{\xi_1, \xi_2\}$ and can have all the causal characters.

Case 1.1.1. $\langle C^{\perp}, C^{\perp}\rangle > 0$.

Then ξ_3 is the normalized vector field $C^{\perp}$ and ξ_4 is the unique timelike unit vector field perpendicular to the 3-dimensional subspace $\{\xi_1, \xi_2, \xi_3\}$, such that the orientation of $\{\xi_1, \xi_2, \xi_3, \xi_4\}$ is the same as that of $\mathbb{E}_1^4$.

The Frenet formulas are

$$\begin{cases} \dfrac{d\xi_1}{ds} = K_1\xi_2; \\ \dfrac{d\xi_2}{ds} = -K_1\xi_1 + K_2\xi_3; \\ \dfrac{d\xi_3}{ds} = -K_2\xi_2 + K_3\xi_4; \\ \dfrac{d\xi_4}{ds} = K_3\xi_3. \end{cases} \tag{2.1}$$

In this case, we are able to prove the converse theorem, as follows.

Theorem 2.1 (THE FUNDAMENTAL THEOREM). *Consider the differentiable real functions* $a_1, a_2, a_3, K_1, K_2, K_3\colon [0, L] \to \mathbb{R}$ *of* C^k*-class,* $k \geq 1$*. Then, there exists a versor field* ξ *tangent to a curve* C *such that* K_1*,* K_2*,* K_3 *are the curvatures and* a_1*,* a_2*,* a_3 *are the components of* ξ *in the Frenet frame associated to the versor field* ξ*.*

Proof. Let us consider (2.1) as a system of differential equations with $\xi_1, \xi_2, \xi_3, \xi_4$ unknown and K_1, K_2, K_3 given.

We shall suppose the initial conditions $\xi_i(0) = \xi_i^0, i = \overline{1,4}$.

The system has an unique solution. It represents the Frenet formulas for a versor field if the functions $\xi_1, \xi_2, \xi_3, \xi_4$ form a positively oriented orthonormal basis.

We choose the initial conditions such that the vectors $\{\xi_i^0\}_{i=\overline{1,4}}$ form a positively oriented.orthonormal basis.

Consequently,

$$\xi_1^2(0) = 1, \xi_2^2(0) = 1, \xi_3^2(0) = 1, \xi_4^2(0) = -1, \langle \xi_i(0), \xi_j(0)\rangle = 0, i \neq j. \tag{2.2}$$

In order that the solutions $\xi_1, \xi_2, \xi_3, \xi_4$ of system (2.1), with the initial conditions (2.2) form an orthonormal basis we must have the identities:

$$\begin{aligned} &b_1(s) = \xi_1^2(s) = 1,\ b_2(s) = \xi_2^2(s) = 1, \\ &b_3(s) = \xi_3^2(s) = 1,\ b_4(s) = \xi_4^2(s) = -1, \\ &b_5(s) = \langle \xi_1(s), \xi_2(s)\rangle = 0,\ b_6(s) = \langle \xi_1(s), \xi_3(s)\rangle = 0, \\ &b_7(s) = \langle \xi_1(s), \xi_4(s)\rangle = 0,\ b_8(s) = \langle \xi_2(s), \xi_3(s)\rangle = 0, \\ &b_9(s) = \langle \xi_2(s), \xi_4(s)\rangle = 0,\ b_{10}(s) = \langle \xi_3(s), \xi_4(s)\rangle = 0. \end{aligned} \tag{2.3}$$

These relations hold for s=0.

We will express the derivatives of the functions $b_i,\ i = \overline{1,10}$.

We obtain the following system of differential equations with b_i, $i = \overline{1,10}$, unknown.

$$\begin{cases} \dfrac{db_1}{ds} = 2K_1 b_5(s) \\ \dfrac{db_2}{ds} = -2K_1 b_5(s) + 2K_2 b_8(s) \\ \dfrac{db_3}{ds} = -2K_2 b_8(s) + 2K_3 b_{10}(s) \\ \dfrac{db_4}{ds} = 2K_3 b_{10}(s) \\ \dfrac{db_5}{ds} = K_1 b_2(s) - K_1 b_1(s) + K_2 b_6(s) \\ \dfrac{db_6}{ds} = K_1 b_8(s) - K_2 b_5(s) + K_3 b_7(s) \\ \dfrac{db_7}{ds} = K_1 b_9(s) + K_2 b_6(s) \\ \dfrac{db_8}{ds} = -K_1 b_6(s) + K_2 b_3(s) - K_2 b_2(s) + K_3 b_9(s) \\ \dfrac{db_9}{ds} = -K_1 b_7(s) + K_2 b_{10}(s) + K_3 b_8(s) \\ \dfrac{db_{10}}{ds} = -K_2 b_9(s) + K_3 b_4(s) + K_3 b_3(s) \end{cases}$$

By a direct computation, we observe that $(1, 1, 1, -1, 0, 0, 0, 0, 0, 0)$ are solutions of this system, which satisfy the initial conditions (2.2).

Using the uniqueness of the solution with the initial given conditions, it follows that identities (2.3) hold for every s.

If a_1, a_2, a_3, a_4 are given, we consider the differential equation

$$\frac{d\vec{r}}{ds} = a_1 \xi_1(s) + a_2 \xi_2(s) + a_3 \xi_3(s) + a_4 \xi_4(s),$$

where $\xi_1, \xi_2, \xi_3, \xi_4$ were calculated above.

The previous equation determines the curve C which is tangent to the versor field $\xi = \xi_1$. □

Having in view the background we have just introduced, in the following, we continue the discussion started above.

Case 1.1.2. $\left\langle C^{\perp}, C^{\perp} \right\rangle < 0$.

Then ξ_3 is the timelike normalized vector field $C^{\perp}$ and ξ_4 is the unique spacelike unit vector field perpendicular to the 3-dimensional subspace $\{\xi_1, \xi_2, \xi_3\}$, such that the orientation of $\{\xi_1, \xi_2, \xi_3, \xi_4\}$ is the same as that of $\mathbb{E}_1^4$.

The Frenet formulas then become

$$\begin{cases} \dfrac{d\xi_1}{ds} = K_1\xi_2; \\ \dfrac{d\xi_2}{ds} = -K_1\xi_1 + K_2\xi_3; \\ \dfrac{d\xi_3}{ds} = K_2\xi_2 + K_3\xi_4; \\ \dfrac{d\xi_4}{ds} = -K_3\xi_3. \end{cases}$$

Case 1.1.3. $\langle C^{\perp}, C^{\perp} \rangle = 0$.

Then ξ_3 is the vector field $C^{\perp}$ and ξ_4 is the unique null vector field perpendicular to the plane $\{\xi_1, \xi_2\}$, such that $\langle \xi_3, \xi_4 \rangle = 1$.

The Frenet formulas then become

$$\begin{cases} \dfrac{d\xi_1}{ds} = K_1\xi_2; \\ \dfrac{d\xi_2}{ds} = -K_1\xi_1 + K_2\xi_3; \\ \dfrac{d\xi_3}{ds} = K_3\xi_3; \\ \dfrac{d\xi_4}{ds} = -K_2\xi_2 - K_3\xi_4. \end{cases}$$

Such a vector field is called *partially null vector field* and lies in a three dimensional subspace, as we see from the Frenet formulas. By making a *null rotation*, that is a map from one null tetrad to another null tetrad, that keeps ξ_1 and ξ_2 fixed, we can make $K_3 = 0$. Thus ξ_3 is a constant null vector.

Working out this case, we make the following null rotation

$$\overline{\xi_1} = \xi_1; \quad \overline{\xi_2} = \xi_2; \quad \overline{\xi_3} = \frac{1}{a}\xi_3; \quad \overline{\xi_4} = a\xi_4.$$

The Frenet formulas then become

$$\begin{cases} \dfrac{d\overline{\xi_1}}{ds} = \overline{K_1\xi_2}; \\ \dfrac{d\overline{\xi_2}}{ds} = -\overline{K_1\xi_1} + \overline{K_2\xi_3}; \\ \dfrac{d\overline{\xi_3}}{ds} = \overline{K_3\xi_3}; \\ \dfrac{d\overline{\xi_4}}{ds} = -\overline{K_2\xi_2} - \overline{K_3\xi_4}. \end{cases}$$

with $\overline{K_1} = K_1$, $\overline{K_2} = aK_2$ and $K_3 = \dfrac{-\dot{a}}{a} + K_3$.

Thus, we may choose $a(s)$ such that $\overline{K_3} = 0$. This means that $\dfrac{d\overline{\xi_3}}{ds} = 0$, or, in other words, $\overline{\xi_3}$ is a constant null vector.

There are only two curvatures in this case, the second curvature is determined only up to a constant factor.

Case 1.2. $\left\langle \frac{d\xi_1}{ds}, \frac{d\xi_1}{ds} \right\rangle < 0$. ξ_2 is then the normalized timelike vector field corresponding to $\frac{d\xi_1}{ds}$. The unit vector field ξ_3 is in the direction of the normal component of $\frac{d\xi_2}{ds}$ and must be spacelike. ξ_4 is the unique spacelike unit vector field perpendicular to the 3-dimensional subspace $\{\xi_1, \xi_2, \xi_3\}$, such that the orientation of $\{\xi_1, \xi_2, \xi_3, \xi_4\}$ is the same as that of E_1^4.

The Frenet formulas are

$$\begin{cases} \frac{d\xi_1}{ds} = K_1\xi_2; \\ \frac{d\xi_2}{ds} = K_1\xi_1 + K_2\xi_3; \\ \frac{d\xi_3}{ds} = K_2\xi_2 + K_3\xi_4; \\ \frac{d\xi_4}{ds} = -K_3\xi_3. \end{cases}$$

Case 1.3. $\left\langle \frac{d\xi_1}{ds}, \frac{d\xi_1}{ds} \right\rangle = 0$. Such a vector field will be called *pseudo null vector field.*

ξ_2 is the vector field $\frac{d\xi_1}{ds}$ if $\frac{d\xi_1}{ds} \neq 0$. $\frac{d^2\xi_1}{ds^2}$ can be a spacelike vector field or a null vector field. If $\frac{d^2\xi_1}{ds^2}$ is spacelike, then let ξ_3 be the normalized vector field $\frac{d^2\xi_1}{ds^2}$. ξ_4 is the unique null vector field perpendicular to the subspace $\{\xi_1, \xi_3\}$, such that $\langle \xi_2, \xi_4 \rangle = 1$.

The Frenet system, known as the Standard Tetrad then becomes

$$\begin{cases} \frac{d\xi_1}{ds} = K_1\xi_2; \\ \frac{d\xi_2}{ds} = K_2\xi_3; \\ \frac{d\xi_3}{ds} = K_3\xi_2 - K_2\xi_4; \\ \frac{d\xi_4}{ds} = -K_1\xi_1 - K_3\xi_3, \end{cases}$$

where the curvature K_1 can only take two values: 0, when we have a straight line, or 1, in all other cases.

Case 2. $\xi = \xi_1$ is timelike, that is $\langle \xi_1, \xi_1 \rangle = -1$. ξ_1 is the timelike unit tangent vector field of (C, ξ). $\frac{d\xi_1}{ds}$ is perpendicular to ξ_1, so ξ_2 is the spacelike normalized vector field $\frac{d\xi_1}{ds}$. ξ_3 is a spacelike unit vector field in the direction of the normal component of $\frac{d\xi_2}{ds}$ with respect to the plane $\{\xi_1, \xi_2\}$. ξ_4 is the unique spacelike unit vector field perpendicular to $\{\xi_1, \xi_2, \xi_3\}$ and such that the orientation of $\{\xi_1, \xi_2, \xi_3, \xi_4\}$ corresponds with that of $\mathbb{E}^4_1$.

The Frenet formulas are

$$\begin{cases} \frac{d\xi_1}{ds} = K_1\xi_2; \\ \frac{d\xi_2}{ds} = K_1\xi_1 + K_2\xi_3; \\ \frac{d\xi_3}{ds} = -K_2\xi_2 + K_3\xi_4; \\ \frac{d\xi_4}{ds} = -K_3\xi_3. \end{cases}$$

Case 3. $\xi = \xi_1$ is null.

If we exclude straight lines, then $\left\langle \frac{d\xi_1}{ds}, \frac{d\xi_1}{ds} \right\rangle = 1$, so take $\xi_2 = \frac{d\xi_1}{ds}$.

Let ξ_3 be the normal component of $\frac{d^2\xi_1}{ds^2}$ with respect to the plane $\{\xi_1, \xi_2\}$. From $\left\langle \xi_1, \frac{d^2\xi_1}{ds^2} \right\rangle = -1$ and $\left\langle \frac{d\xi_1}{ds}, \frac{d^2\xi_1}{ds^2} \right\rangle = 0$; we get that $\langle \xi_3, \xi_3 \rangle$ must be equal to 0 and ξ_3 is completely determined by $\langle \xi_1, \xi_3 \rangle = 1$. ξ_4 is the unique spacelike unit vector field perpendicular to $\{\xi_1, \xi_2, \xi_3\}$ and such that the orientation of $\{\xi_1, \xi_2, \xi_3, \xi_4\}$ corresponds with that of $\mathbb{E}^4_1$.

The Frenet formulas are

$$\begin{cases} \frac{d\xi_1}{ds} = K_1\xi_2; \\ \frac{d\xi_2}{ds} = K_2\xi_1 - K_1\xi_3; \\ \frac{d\xi_3}{ds} = -K_2\xi_2 + K_3\xi_4; \\ \frac{d\xi_4}{ds} = -K_3\xi_1, \end{cases}$$

where the curvature K_1 can only take two values: 0, when we have a straight line, or 1, in all other cases.

This tetrad $\{\xi_1, \xi_2, \xi_3, \xi_4\}$ is called the *Cartan tetrad.*

3. CONCLUSIONS AND FURTHER DEVELOPMENTS

The Frenet equations were first established for curves in the 3-dimensional Euclidean space $\mathbb{E}^3$. Inductively, they were naturally extended to curves in the Euclidean space $\mathbb{E}^n$.

For Frenet formulas for curves in the Minkowski space we refer to [12].

We would like to point-out that our study of Frenet formulas for versor fields along a curve in the Minkowski spacetime is more general and, as it can be seen in the previous sections, we had to consider causal characters of the versor fields. In principle, the same techniques can be used for the study of versor fields along a curve in any pseudo-Euclidean space $\mathbb{E}^n_\nu$, but the number of cases to be considered is much larger. Also the fundamental theorem can be stated similarly when ξ is timelike or null, respectively. Moreover, we think that the fundamental theorem could be generalized to higher dimensional spacetimes.

On the other hand, the Frenet equations were also extended for curves in a pseudo-Finsler space (see [1]).

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REFERENCES

[1] A. Bejancu and H.R. Farran: *Geometry of pseudo-Finsler submanifolds*, Kluwer Acad. Publ, 2000.

[2] S. M. Carroll: *Lecture Notes on General Relativity*, `arxiv:gr-qc/9712019v1`.

[3] J. B. Formiga and C. Romero: *On the differential geometry of curves in Minkowski space*, `arxiv:gr-qc/0601002v1`.

[4] K. Ilarslan and E. Nesovic: *Some characterizations of null, pseudo null and partially null rectifying curves in Minkowski space-time*, Taiwanese J. Math., **12**(2008), 1035-1044.

[5] K. Ilarslan, E. Nesovic and M. Petrovic-Torgasev: *Some characterizations of rectifying curves in the Minkowski* 3*-space*, Novi Sad J. Math., **33**(2003), 23-32.

[6] J.-I. Inoguchi and S. Lee: *Null curves in Minkowski* 3*-space,* Int. Electron. J. Geom., **1**(2008), 40-83.

[7] R. Miron: *Geometria configuraţiilor Myller*, Editura Tehnică, Bucureşti, 1966 (Romanian).

[8] R. Miron: *The Geometry of Myller Configurations. Applications to Theory of Surfaces and Nonholonomic Manifolds*, Editura Academiei Române, Bucureşti, 2010 (Romanian).

[9] Gh. Munteanu and V. Balan: *Lecţii de teoria relativităţii*, Editura Bren, Bucureşti, 2000 (Romanian).

[10] B. O'Neill: *Semi-Riemannian Geometry*, Academic Press, New York, 1983.

[11] S. Sternberg: *Semi-Riemann Geometry and General Relativity*, 2003.

[12] J. Walrave: Curves and surfaces in Minkowski space, Ph. D. Thesis, K. U. Leuven, Faculty of Science, K. U. Leuven, 1995.

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