

GENERALIZED ZETA FUNCTIONS

TIAN-XIAO HE

Dedicated to Professor Leetsch C. Hsu on the Occasion of his 90th Birthday

ABSTRACT. We present here a wide class of generalized zeta function in terms of the generalized Möbius functions and its properties.

1. INTRODUCTION

For any integer $z \in \mathbb{C}$, a Fleck-type generalized Möbius function (cf. [3] or [7]) of order z can be defined by

$$\mu_z(n) := \prod_p (-1)^{e_p(n)} \binom{z}{e_p(n)} \quad (1.1)$$

for any $n \in \mathbb{N}$, where p runs through all the prime divisors of n , and $e_p(n) = \text{ord}_p(n)$ denotes the highest power k of p such that p^k divides n . Obviously, $\mu_1(n) = \mu(n)$, $n \in \mathbb{N}$, is the classical Möbius function: $\mu(1) = 1$; if n is not square free then $\mu(n) = 0$; if n is square free and if q is the number of distinct primes dividing n , then $\mu(n) = (-1)^q$. In addition, (1.1) implies

$$\mu_0 = \prod_{p|n} (-1)^{e_p(n)} \binom{0}{e_p(n)} = \begin{cases} 1, & n = 1, \\ 0, & n > 1, \end{cases}$$

and

$$\mu_{-1} = \prod_p (-1)^{e_p(n)} \binom{-1}{e_p(n)} = \prod_{p|n} \frac{(e_p(n))!}{(e_p(n))!} = 1.$$

It is easy to verify that for each complex number α , μ_α is a multiplicative function, but is not complete multiplicative except μ_0 , which is complete multiplicative. The generalized zeta function, denoted by ξ_z , is defined accordingly by

$$\xi_z(s) = 1 / \sum_{n \geq 1} \frac{\mu_z(n)}{n^s}, \quad (1.2)$$

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where $z \in \mathbb{C}$. Hence, $\xi_1 = \xi$ the classical zeta function. And $\xi_0 = 1$.

Remark 1.1. $\xi_z(s)$ defined in (1.2) can be extended to \mathbb{C} . However, throughout this note, we do not consider the zero points of $\xi_1(s)$ in its domain, i.e., the points at which $\sum_{n \geq 1} \frac{\mu_1(n)}{n^s}$ diverges to infinite.

In this note, we will show that the set of functions ξ_α ($\alpha \in \mathbb{C}$) forms an Abelian group with the Dirichlet series multiplication followed by a number of applications.

2. GENERALIZED ZETA FUNCTION GROUP

We now recall the definition of the Dirichlet product (or convolution) of two arithmetic functions f and g (cf. [1], [2]).

Definition 2.1. Given two arithmetic functions f and g , the Dirichlet (convolution) product $f * g$ is again an arithmetic function which is defined by

$$(f * g)(n) := \sum_{d|n} f(d)g\left(\frac{n}{d}\right) = \sum_{d|n} f\left(\frac{n}{d}\right)g(d), \quad (2.1)$$

where the summations are taken over all positive divisors d of n .

Definition 2.2. Denote $M := \{\mu_z : z \in \mathbb{C}\}$, where \mathbb{C} denotes the set of complex numbers. We call M the set of generalized Möbius functions of complex order. The set, denoted by N , of the corresponding nonzero generalized zeta functions of complex order is defined by $N := \{\xi_z : z \in \mathbb{C}\}$, where ξ_z are presented in (1.2).

From [2] and [5], $(M, *)$ forms an Abelian group with identity element μ_0 under the operation $* : M \times M \mapsto M$, $\mu_\alpha * \mu_\beta = \mu_{\alpha+\beta}$, where $\alpha, \beta \in \mathbb{C}$.

Lemma 2.1. For any given $\alpha, \beta \in \mathbb{C}$, we define $\cdot : N \times N \mapsto N$ by

$$\xi_\alpha \cdot \xi_\beta = \xi_\gamma$$

for some $\gamma \in \mathbb{C}$ if $\frac{1}{\xi_\alpha} * \frac{1}{\xi_\beta} = \frac{1}{\xi_\gamma}$, where $*$ is the regular Dirichlet product of Dirichlet series (cf. (2.1) and [8] for more details). Thus, we have

$$\xi_\alpha \cdot \xi_\beta = \xi_{\alpha+\beta}.$$

Proof. By writing $1/\xi_\alpha(s) = \sum_{n \geq 1} \frac{\mu_\alpha(n)}{n^s}$, and $1/\xi_\beta(s) = \sum_{n \geq 1} \frac{\mu_\beta(n)}{n^s}$, we obtain

$$\begin{aligned} 1/(\xi_\alpha \xi_\beta)(s) &= \sum_{n \geq 1} \left(\sum_{d|n} \mu_\alpha(d) \mu_\beta\left(\frac{n}{d}\right) \right) / n^s = \sum_{n \geq 1} (\mu_\alpha * \mu_\beta) / n^s \\ &= \sum_{n \geq 1} \frac{\mu_{\alpha+\beta}}{n^s} = 1/\xi_{\alpha+\beta}(s), \end{aligned}$$

which completes the proof. \square

We now ready to show (N, \cdot) is an Abelian group.

Theorem 2.1. *Let \cdot be the operation define in Lemma 2.1. Then (N, \cdot) is an Abelian group with identity element $\xi_0 = 1$.*

Proof. From Lemma 2.1, we see that N is closed respect to the operation \cdot . Moreover, for any α and $\beta \in \mathbb{C}$, we have

$$\xi_\alpha \cdot \xi_\beta = \xi_{\alpha+\beta} = \xi_\beta \cdot \xi_\alpha.$$

And for any α, β and $\gamma \in \mathbb{C}$,

$$\begin{aligned} (\xi_\alpha \cdot \xi_\beta) \cdot \xi_\gamma &= \xi_{\alpha+\beta} \cdot \xi_\gamma \\ &= 1/ \sum_{n \geq 1} \left(\sum_{d|n} \mu_{\alpha+\beta}(d) \mu_\gamma\left(\frac{n}{d}\right) \right) / n^s = 1/ \sum_{n \geq 1} \frac{\mu_{\alpha+\beta+\gamma}}{n^s} = \xi_{\alpha+\beta+\gamma}(s). \end{aligned}$$

Similarly, $\xi_\alpha \cdot (\xi_\beta \cdot \xi_\gamma) = \xi_{\alpha+\beta+\gamma}$. Thus, $(\xi_\alpha \cdot \xi_\beta) \cdot \xi_\gamma = \xi_\alpha \cdot (\xi_\beta \cdot \xi_\gamma)$. It is also easy to check $\xi_\alpha \cdot 1 = 1 \cdot \xi_\alpha = \xi_\alpha$ and $\xi_\alpha \cdot \xi_{-\alpha} = \xi_{-\alpha} \cdot \xi_\alpha = 1$. Therefore, the theorem is proved. \square

From Theorem 2.1 and Equation (1.2) we have

Corollary 2.1. *For all $\alpha \in \mathbb{Z}$,*

$$\xi_\alpha(s) = (\xi(s))^\alpha,$$

where $(\xi(s))^\alpha := \xi(s)(\xi(s))^{\alpha-1}$.

Theorem 2.2. *Group $(M, *)$ and (N, \cdot) are isomorphic.*

Proof. Mapping $\phi : M \mapsto N$ is defined by

$$\phi(\mu_\alpha) := \xi_\alpha = 1/ \sum_{n \geq 1} \frac{\mu_\alpha(n)}{n^s},$$

where $\alpha \in \mathbb{C}$. It is easy to verify that the mapping is one-to-one and onto. In addition, for any $\alpha, \beta \in \mathbb{C}$,

$$\phi(\mu_\alpha * \mu_\beta) = \phi(\mu_{\alpha+\beta}) = \xi_{\alpha+\beta} = \xi_\alpha \cdot \xi_\beta = \phi(\mu_\alpha) \cdot \phi(\mu_\beta).$$

This completes the proof. \square

3. SOME RESULTS FROM GENERALIZED ZETA FUNCTION GROUP

A series $\sum_{n \geq 1} a_n n^{-s}$ is called an arithmetic Dirichlet series if all of its coefficients $a_n = a(n)$ are arithmetic functions.

Theorem 3.1. (GENERALIZED ZETA INVERSION FORMULAE) *For any $\alpha \in \mathbb{C}$ and Dirichlet series f and g ,*

$$f = \xi_\alpha g \Leftrightarrow g = \xi_{-\alpha} f.$$

Moreover, if both $f = \sum_{n \geq 1} f_n n^{-s}$ and $g = \sum_{n \geq 1} g_n n^{-s}$ are arithmetic Dirichlet series, then for any $n \in \mathbb{N}$

$$f(n) = \sum_{d|n} \mu_\alpha \left(\frac{n}{d} \right) g(d) \Leftrightarrow g(n) = \sum_{d|n} \mu_{-\alpha} \left(\frac{n}{d} \right) f(d).$$

Proposition 3.1. *For any $n \in \mathbb{N}$ and $\alpha \in \mathbb{C}$,*

$$\sum_{d|n} \mu_{\alpha-1}(d) = \mu_\alpha(n). \quad (3.1)$$

Proof. From Corollary 2.1

$$\begin{aligned} \sum_{n \geq 1} \frac{\mu_\alpha(n)}{n^s} &= \frac{1}{\xi_\alpha(s)} = \frac{1}{\xi_{\alpha-1}(s)} \cdot \sum_{n \geq 1} \frac{\mu_{-1}(n)}{n^s} \\ &= \sum_{n \geq 1} \frac{\mu_{\alpha-1}(n)}{n^s} \cdot \sum_{n \geq 1} \frac{\mu_{-1}(n)}{n^s} = \sum_{n \geq 1} \sum_{d|n} \mu_{\alpha-1}(d) \mu_{-1} \left(\frac{n}{d} \right) / n^s, \end{aligned}$$

which leads (3.1) by applying the Dirichlet series multiplication and noting that $\mu_{-1} \equiv 1$. \square

Proposition 3.2. *Let $f = \sum_{n \geq 1} c_n n^{-s}$, and let all c_n be completely multiplicative functions. For any fixed positive integer α ,*

$$f^{\alpha-1} \sum_{n \geq 1} \frac{\mu_\alpha(n) c_n}{n^{-s}} = \sum_{n \geq 1} \frac{\mu_0(n) c_n}{n^{-s}}. \quad (3.2)$$

Proof. This follows easily from Proposition 3.1 and mathematical induction on α . In fact, first we have

$$\begin{aligned} f \sum_{n \geq 1} \frac{\mu_\alpha(n)c_n}{n^{-s}} &= \sum_{n \geq 1} \frac{((\mu_\alpha c) * c)(n)}{n^{-s}} \\ &= \sum_{n \geq 1} \frac{c(n) \sum_{d|n} \mu_\alpha(d)}{n^{-s}} = \sum_{n \geq 1} \frac{c(n)\mu_{\alpha-1}(n)}{n^{-s}}. \end{aligned}$$

Secondly, using mathematical induction on α we obtain (3.2). \square

It is known (cf., for examples, [4] and [6]) that for any fixed integer $\alpha \geq 1$

$$\xi_1^\alpha(s) = \left(\sum_{n \geq 1} \mu(n)n^{-s} \right)^\alpha = \sum_{n \geq 1} r_\alpha(n)n^{-s}, \quad (3.3)$$

where

$$r_\alpha(n) = \sum_{n_1 n_2 \cdots n_\alpha = n} 1$$

is the number of ways that n can be written as a product of α fixed factors, so that $r_\alpha(n)$ is clearly a multiplicative function of n . In particular, $r_2(n)$ denotes the number of positive divisors of n .

Theorem 3.2. (CHARACTERISTIC OF GENERALIZED MÖBIUS FUNCTIONS) *For any fixed integer $\alpha \geq 1$, the inverse of μ_α in the group $(M, *)$ is r_α ; or equivalently, $r_\alpha = \mu_{-\alpha}$. Namely, for all integers $n \geq 1$*

$$(r_\alpha * \mu_\alpha)(n) = \sum_{d|n} r_\alpha(d)\mu_\alpha(n/d) = \delta_{n,1}, \quad (3.4)$$

where $\delta_{n,1} = 1$ if $n = 1$ and 0 otherwise.

Proof. Multiplying $\xi_1^\alpha(s)$ shown in (3.3) with

$$\frac{1}{\xi_\alpha(s)} = \frac{1}{\xi_1^\alpha(s)} = \sum_{n \geq 1} \frac{\mu_\alpha(n)}{n^s}$$

yields

$$\sum_{n \geq 1} \frac{(r_\alpha * \mu_\alpha)(n)}{n^s} = 1,$$

which leads $r_\alpha(1)\mu_\alpha(1) = 1$ and

$$(r_\alpha * \mu_\alpha)(n) = \sum_{d|n} r_\alpha(d)\mu_\alpha(n/d) = 0$$

for $n \geq 2$, completing the proof. \square

Denote $F_\alpha(s) = \sum_{n \geq 1} r_\alpha(n)n^{-s}$. Then we obtain

$$F_\alpha(s)\xi_{1-\alpha}(s) = \sum_{n \geq 1} \sum_{d|n} r_\alpha(d)\mu_{1-\alpha}\left(\frac{n}{d}\right) / n^s = \xi(s)$$

and

$$F_\alpha(s)\xi_{-\alpha}(s) = (\xi_1)^\alpha(\xi_1)^{-\alpha} = \xi_1^0 = \xi_0(s) = 1,$$

i.e., identities (3.4) and $\sum_{d|n} r_\alpha(d)\mu_{1-\alpha}\left(\frac{n}{d}\right) = 1$. In particular, for $r_2 = \sum_{d|n} 1$, the

number of positive divisors of n , from $\mu_{-2}(n) = r_2(n)$ we obtain

$$F_2(s)\xi_{-1}(s) = 1 \quad \text{and} \quad F_2(s)\xi_{-2}(s) = \xi_0(s) = 1,$$

i.e.,

$$\sum_{d|n} r_2(d)\mu\left(\frac{n}{d}\right) = 1 \quad \text{and} \quad \sum_{d|n} r_2(d)\mu_2\left(\frac{n}{d}\right) = \delta_{n,1}.$$

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Illinois Wesleyan University
Department of Mathematics and Computer Science
P. O. Box 2900, Bloomington, IL 61702-2900, USA
E-mail address: the@iwu.edu