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A SURVEY ON CONSTRAINED EXTREMA

OLTIN DOGARU AND MIHAI POSTOLACHE

Abstract. The aim of this article is to revisit some problems of constrained extrema and discuss their peculiarities as they appear in our works [1]-[17], as well as to indicate possible up to date state of investigations in this field. As basic tool of this work is the idea of considering the values of a function along all curves from a given family, which allows an unitary approach of various types of extrema problems (either holonomic or nonholonomic). Some open problems with comments will be stated.

The research on this topic has started with the distinguished collaboration of C. Udrişte and has been continued in the last years with the coauthors I. Ţevy, M. Ferrara, V. Dogaru and al.

1. INTRODUCTION

Consider the function $f\colon D \subseteq \mathbb{R}^p \to \mathbb{R}$, where D is an open set, and Γ a family of parametrized curves in D. For any $a \in D$, denote by $\Gamma(a)$ the family of all parametrized curves in Γ passing through the point a. For each $\alpha \in \Gamma$, consider $f_\alpha = f \circ \alpha$, obtaining a family of functions $\{f_\alpha\}_{\alpha\in\Gamma}$.

Question 1.1. Find the conditions which have to be satisfied by the families Γ and $\{f_\alpha\}_{\alpha\in\Gamma}$ such that the function f is unique determined by $\{f_\alpha\}_{\alpha\in\Gamma}$ and/or to satisfy a certain property (continuity, differentiability, convexity etc.)?

It is known that the active constraints from a constrained optimization problem lead to a completely integrable Pfaff system. For example, let us consider the extremum problem

$$\min f(x)\,,\ \text{subject to } g(x) = 0, \tag{1.1}$$

where $f, g\colon D \subseteq \mathbb{R}^p \to \mathbb{R}$ are functions of C^1-class. If $a \in D$ is a solution of this problem, then a is a critical point for f constrained by the Pfaff form $\omega = dg$, that is f satisfies the equation

$$df(x) = \lambda\omega(x)\,. \tag{1.2}$$

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If f and g are C^2-class functions, then a and λ satisfy equation (1.2); moreover, if the restriction of the quadratic form $d^2 f(a) - \lambda d^2 g(a)$ to the subspace defined by the equation $\omega(a) = 0$ is positive defined, then a is a solution of problem (1.1).

Now, consider

$$\omega(x) = \sum_{i=1}^{p} \omega_i(x) dx^i$$

Pfaff form of C^1-class. Let (a, λ) be a solution of equation (1.2). Suppose that the restriction of the quadratic form

$$d^2 f(a) - \frac{\lambda}{2} \sum_{i,j=1}^{p} \left(\frac{\partial \omega_i}{\partial x^j} + \frac{\partial \omega_j}{\partial x^i} \right) (a)\, dx^i dx^j$$

to the subspace defined by $\omega(a) = 0$ is positive definite. According to our above considerations, if ω is an exact form, that is $\omega = dg$, then a is a solution of the problem

$$\min f(x), \text{ subject to } g(x) = g(a).$$

Question 1.2. What could we say about the point a when ω is not a complete integrable form?

For our subsequent study, the definitions in the following are necessary.

Definition 1.1. Let $I \subset \mathbb{R}$ be an interval. A function $\alpha\colon I \to \mathbb{R}^p$ of C^k-class, $k \geq 1$, is called parametrized curve of C^k-class and it is denoted by α. We shall say that:

(i) α *passes through* the point $x_0 \in \mathbb{R}^p$, if exists $t_0 \in \text{Int}\, I$ such that $\alpha(t_0) = x_0$;

(ii) α is a *simple* parametrized curve, if α is injective;

(iii) α is *regular* at the point $x_0 = \alpha(t_0)$, if $\alpha'(t_0) \neq 0$;

(iv) α *has a tangent* at the point $x_0 = \alpha(t_0)$, if exists $m \in \overline{1,k}$ such that $\alpha^{(m)}(t_0) \neq 0$.

Definition 1.2. Two parametrized curves $\alpha\colon I \to \mathbb{R}^p$ and $\beta\colon J \to \mathbb{R}^p$ of C^k-class, are called *equivalent* if there exists a difeomorphism $j\colon I \to J$ of C^k-class, such that $\alpha = \beta \circ h$. We shall write $\alpha \sim \beta$.

Definition 1.3. The set $\tilde{a}$ of all parametrized curves, of C^k-class, which are equivalent to $\alpha\colon I \to \mathbb{R}^p$ is called *curve* of C^k-class. The curve $\tilde{a}$ satisfies (i)$\div$(iv) from Definition 1.1 if a representative α has these properties.

We shall see that the solution to the two questions stated above is essentially based on the idea of considering the values of a function along various curves from a given family of curves.

The rest of our work is organized as follows. Sections 2 and 3 approach Question 1.1, while Section 4 to Section 8 deal with Question 1.2.

2. CONTINUITY

Consider the function $f\colon D \subseteq \mathbb{R}^p \to \mathbb{R}$, where D is an open set. Let $a \in D$ and $\Gamma(a)$ a family of parametrized curves passing through the point a. Suppose $\Gamma(a)$ belongs to one of the following families:

- the family of all simple parametized curves of C^1-class passing through a and regular at the point a;
- the family of all C^m parametrized curves passing through a, $m \geq 1$, having a tangent at the point a.

Theorem 2.1 ([4])**.** *If for all $\alpha \in \Gamma(a)$ $(\alpha(t_0) = a)$, the function $f_\alpha = f \circ \alpha$ is continuous at the point t_0, then f is continuous at the point a, too.*

It is a surprise that Theorem 2.1 is not valid if $\Gamma(a)$ is the family of all C^2-class curves passing through the point a and regular at a.

Example 2.1. Given $g(x,y) = \left(y^3 - x^4\right)\left(y^3 - 8x^4\right)$, consider the function

$$f(x,y) = \begin{cases} 1, & \text{if } \ g(x,y) \geq 0 \\ 0, & \text{if } \ g(x,y) < 0. \end{cases}$$

It is obvious that f is not continuous at the point $a = (0,0)$. However, for all $\alpha \in \Gamma(a)$ the function $f \circ \alpha$ it is continuous.

Indeed, let $D_- = \left\{(x,y) \in \mathbb{R}^2 |\ g(x,y) < 0\right\}$. From the definition of f, it follows that $f(0,0) = 1$. By reductio ad absurdum, suppose that there exists $\alpha \in \Gamma(a)$, with $\alpha(t_0) = (0,0)$, such that $\lim\limits_{t \to t_0} \alpha(t)$ either does not exist or it is different from 1. Let $\alpha(t) = (x(t), y(t))$ be a parametrized curve. From the definition of f, it follows that there is a sequence (t_n), with $t_n \to t_0$, such that the points $(x_n, y_n) = \alpha(t_n)$ are contained within the set D_-, which will lead to a contradiction. Since $x_n \to 0$, we can suppose, for example, that $x_n > 0$, for all $n \in \mathbb{N}$, passing maybe to a subsequence. Since $(x_{n,} y_n) \in D_-$, we obtain that $x_n^{4/3} < y_n^{4/3} < 2x_n^{4/3}$, whence it follows

$$\frac{y_n}{x_n} \to 0, \quad \frac{y_n}{x_n^2} \to \infty. \tag{2.1}$$

Because the parametrized curve α is regular at $(0,0)$, two cases are possible:

• CASE 1. Locally, the image of the parametrized curve α is the graph of a function $y = \varphi(x)$ of C^2-class in a neighborhood of the origin. Applying the Taylor formula of order one at zero, $\varphi(x) = \varphi'(0)\,x + 0(x)$, we find that

$$\varphi'(0) = \lim_{x \to 0} \frac{\varphi(x)}{x} = \lim_{n \to \infty} \frac{y_n}{x_n} = 0.$$

Applying the Taylor formula of order two at zero, $y_n = \varphi(x_n) = \dfrac{1}{2}\varphi''(0)\,x_n^2 + 0\left(x_n^2\right)$, we get

$$\lim_{n \to \infty} \frac{y_n - \frac{1}{2}\varphi''(0)\,x_n^2}{x_n^2} = 0,$$

that is $\lim_{n\to\infty} \frac{y_n}{x_n^2} = \frac{\varphi''(0)}{2}$, which contradicts (2.1).

• CASE 2. Locally, the image of the parametrized curve α represents the graph of a C^2-class function $x = \psi(y)$ defined on a neighborhood of zero. By the Taylor formula of order one at zero, $\psi(y) = \psi'(0) + 0(y)$, we obtain

$$\psi'(0) = \lim_{y\to 0} \frac{\psi(y)}{y} = \lim_{n\to\infty} \frac{x_n}{y_n},$$

which is not true because from (2.1) it follows $\lim_{n\to\infty} \frac{x_n}{y_n} = \infty$.

3. Γ-FUNCTIONS

We denote by $\Gamma^0(D)$ the family of all C^0-class parametrized curves in D and by $\Gamma^1(D)$ the family of all piecewise C^1-class parametrized curves in D. Let be given $f\colon D \subset \mathbb{R}^p \to \mathbb{R}$ a C^1-class function. For each $\alpha \in \Gamma^1(D)$, we consider the function $f_\alpha = f \circ \alpha$, which is an element of $\Gamma^1(\mathbb{R})$. We produced a family $\{f_\alpha\}$ which has the properties in the following:

(1) For any $\alpha \in \Gamma^1(D)$, the functions α and f_α have the same domain of definition. Also, for a parametrized piecewise C^1 curve α, the following statements are true:

(a) the function f_α is a piecewise C^1 function;

(b) if α is a C^1 function in a neighborhood of a point t_0, then f_α is a C^1 function in the same neighborhood.

(2) If α and $\beta = \alpha \circ \varphi$ are equivalent parametrized curves, then $f_\beta = f_\alpha \circ \varphi$.

(3) If $\alpha \in \Gamma^1(D)$, $\alpha\colon I \to D$, and J is a subinterval of I, then $f_{\alpha|J} = f_\alpha|J$.

(4) For any point x in $\mathbb{R}^n$ and each $i = \overline{1,n}$, we define the parametrized axis $\alpha_x^i(t) = x + te^i$, for all $t \in (-\varepsilon_i, \varepsilon_i)$, where $e^i = (0, \dots, 1, \dots 0)$. It is obvious that $f'_{\alpha_x^i}(0) = \frac{\partial f}{\partial x^i}(x)$. In this case, it follows that the function $h^i\colon D \to \mathbb{R}$, $h^i(x) = f'_{\alpha_x^i}(0)$ is continuous.

We shall see that these properties are sufficient to determine the function f.

Consider a family $\{f_\alpha\}$ of elements from $\Gamma^k(\mathbb{R})$, $k = \overline{0,1}$, indexed by the family $\Gamma^1(D)$. Now, we can introduce some axioms.

(A_0) If $\alpha \in \Gamma^1(D)$, then $\mathrm{dom}(\alpha) = \mathrm{dom}(\mathrm{f}_\alpha)$. In addition, if $k = 1$ and if α is a C^1 function in a neighborhood of a point $t_0 \in \mathrm{dom}(\alpha)$, then f_α is also a C^1 function in the same neighborhood.

(A_1) Axiom (A_0) is satisfied. Moreover, if $\alpha \in \Gamma^1(D)$ and φ is a change of parameter on α, then $f_{\alpha\circ\varphi} = f_\alpha \circ \varphi$.

(A_2) Axiom (A_0) is satisfied. Moreover, if $\alpha \in \Gamma^1(D)$ with $\mathrm{dom}(\alpha) = \mathrm{I}$, then $f_{\alpha|J} = f_\alpha|J$ for every subinterval J in I.

In the case $k = 1$, we can consider one more axiom, as follows.

Suppose that (A_2) is satisfied. Then, for each $i = \overline{1,n}$, we consider $h^i \colon D \to \mathbb{R}$, $h^i(x) = f'_{\alpha^i_x}(0)$, where $\alpha^i_x(t) = x + te^i$, $\forall t \in (-\varepsilon_i, \varepsilon_i)$, and $e^i = (0, \ldots, 1, \ldots 0)$. Taking into account axiom (A_2), it results that the function $f_{\alpha^i_x}$ does not depend on ε_i in a neighborhood of 0, so the number $h^i(x)$ is well defined.

(A_3) Axiom (A_2) is satisfied and, in addition, for every $i = \overline{1,n}$ and for every $\alpha \in \Gamma^1(D)$, it follows that $h^i \circ \alpha \in \Gamma^0(\mathbb{R})$.

Example 3.1. For each $\alpha \in \Gamma^1(D)$ we choose $x_0 \in \mathrm{Im}\,(\alpha)$ and $t_0 \in \mathrm{dom}(\alpha)$ such as $\alpha(t_0) = x_0$. We can easily see that the family $\{f_\alpha\}$, where $f_\alpha(t) = \displaystyle\int_{t_0}^{t} ||\alpha'(u)||du$ satisfies (A_0), but does not satisfy (A_1) and (A_2).

Example 3.2. Let $f \colon D \to \mathbb{R}$ be a C^k function, where $k = \overline{0,1}$. Now we consider $\{f_\alpha\}$, where $f_\alpha = f \circ \alpha$. It is obvious that $\{f_\alpha\}$ fulfills (A_1) and (A_2).

Example 3.3. Let us consider $\{f_\alpha\}$, where $f_\alpha(t) = t$, $t \in \mathrm{dom}(\alpha)$. Obviously, $\{f_\alpha\}$ satisfies (A_2) and (A_3), but does not satisfy (A_1).

Example 3.4. Let us consider $\{f_\alpha\}$, where f_α is defined as follows: $f_\alpha(t) = 0$ $\forall t \in \mathrm{dom}(\alpha)$, if $\mathrm{Im}\,(\alpha)$ is included in straight line and $f_\alpha(t) = 1$, $\forall t \in \mathrm{dom}(\alpha)$, otherwise. The family $\{f_\alpha\}$ satisfies (A_1), but does not satisfy (A_2).

Example 3.5. Consider the function

$$f \colon \mathbb{R}^2 \to \mathbb{R}, \quad f\,(x, y) = \begin{cases} \dfrac{x^2 y}{x^2 + y^2}, & \text{for} \quad (x, y) \neq (0, 0) \\ 0, & \text{for} \quad (x, y) = (0, 0). \end{cases}$$

The family $\{f_\alpha\}$, with $f_\alpha = f \circ \alpha$, satisfies (A_1) and (A_2) but does not satisfy (A_3), [4].

From the previous examples it follows that (A_1) and (A_2) are independent axioms and no one is equivalent to (A_0). Also, axiom (A_3) is independent with respect to (A_1) and (A_2).

Let $\mathcal{F}\,(\mathbb{R})$ be the set of all functions $\alpha \colon I \to \mathbb{R}$, where I is an interval in $\mathbb{R}$.

The results in the following ([4]) state necessary and sufficient conditions such that a function $f \colon D \to \mathbb{R}$, endowed with some properties, could be reconstitute from the family $\{f \circ \alpha\}$, with $\alpha \in \Gamma^1(D)$.

Theorem 3.1. *Let $\{f_\alpha\}$ be a family of elements from $\mathcal{F}\,(\mathbb{R})$, indexed by the family $\Gamma^1\,(D)$. There exists a unique function $f \colon D \to \mathbb{R}$ such that for every $\alpha \in \Gamma^1(D)$ it follows: $f \circ \alpha = f_\alpha$ if and only if the family $\{f_\alpha\}$ satisfies axioms (A_1) and (A_2).*

Theorem 3.2. *Let $\{f_\alpha\}$ be a family of elements from $\Gamma^0\,(\mathbb{R})$ indexed by the family $\Gamma^1\,(D)$. Then, there exists a unique continuous function $f \colon D \to \mathbb{R}$ such that for every $\alpha \in \Gamma^1(D)$ it results $f \circ \alpha = f_\alpha$ if and only if the family $\{f_\alpha\}$ satisfies axioms (A_1) and (A_2).*

Theorem 3.3. *Let $\{f_\alpha\}$ be a family of elements from $\Gamma^1(\mathbb{R})$ indexed by the family $\Gamma^1(D)$. Then, there exists a unique continuous function $f\colon D \to \mathbb{R}$ having partial derivatives such that for every $\alpha \in \Gamma^1(D)$ it results $f \circ \alpha = f_\alpha$ if and only if the family $\{f_\alpha\}$ satisfies axioms (A_1) and (A_2).*

Theorem 3.4. *Let $\{f_\alpha\}$ be a family of elements from $\Gamma^1(\mathbb{R})$. Then, there exists a unique C^1 function $f\colon D \to \mathbb{R}$ such that for every $\alpha \in \Gamma^1(D)$ it results $f \circ \alpha = f_\alpha$ if and only if the family $\{f_\alpha\}$ satisfy axioms (A_1) and (A_3).*

The above mentioned results can be completed and unified.

At the beginning, let us remark that a family $\{f_\alpha\}$ of elements from $\Gamma^k(\mathbb{R})$, $k = \overline{0,1}$, indexed by the family $\Gamma^1(D)$, could be identified with a mapping $g\colon \Gamma^1(D) \to \Gamma^k(\mathbb{R})$, with $g(\alpha) = f_\alpha$.

Definition 3.1. Let $\Gamma(D) \subseteq \Gamma^1(D)$. A mapping $g\colon \Gamma^1(D) \to \Gamma^k(\mathbb{R})$, $k = \overline{0,1}$ which satisfies axiom (A_0) is called Γ-*function.*

We consider now the following sets:

$$\mathcal{G}^0(D) = \{\, g\colon \Gamma^1(D) \to \Gamma^0(\mathbb{R}) | g \text{ satisfies } (A_1) \text{ and } (A_2)\,\},$$
$$\mathcal{G}^{1/2}(D) = \{\, g\colon \Gamma^1(D) \to \Gamma^1(\mathbb{R}) | g \text{ satisfies } (A_1) \text{ and } (A_2)\,\},$$
$$\mathcal{G}^1(D) = \{\, g\colon \Gamma^1(D) \to \Gamma^1(\mathbb{R}) | g \text{ satisfies } (A_1) \text{ and } (A_3)\,\},$$

and

$$C^{1/2}(D) = \{\, f\colon D \to \mathbb{R} | \; f \circ \alpha \text{ is } C^1 \text{ function for any simple parametrized curve } \alpha \in \Gamma^1(D)\}.$$

Remark the inclusions $\mathcal{G}^1(D) \subset \mathcal{G}^{1/2}(D) \subset \mathcal{G}^0(D)$, and underline that these function sets can be organized as real vector spaces.

To each continuous function $f\colon D \to \mathbb{R}$ we can attach the Γ-function g defined by $g(\alpha) = f \circ \alpha$, for all $\alpha \in \Gamma^1(D)$. It follows

Theorem 3.5. *The correspondence $f \to g$, above introduced, induces the following isomorphisms of vector spaces:*

$$C^0(D) \simeq \mathcal{G}^0(D), \quad C^{1/2}(D) \simeq \mathcal{G}^{1/2}(D), \quad C^1(D) \simeq \mathcal{G}^1(D).$$

4. EXTREMA SUBJECT TO A FAMILY OF CONSTRAINTS

Consider the extremum problem

$$\min f(x), \text{ subject to } x \in M, \tag{4.1}$$

where M is a subset in $\mathbb{R}^p$ with a given structure. If M is an open subset of $\mathbb{R}^p$, which coincides with the domain of f, then the extremum problem is called *unconstrained*; in any other case it is called *constrained.*

The extremum conditions (necessary and sufficient) depend on the subset M. If M is a differentiable manifold, then these conditions depend on the geometrical structure of M (for example, Euclidean, Riemannian, Finslerian, Lagrangian, Hamiltonian, linear connection, the existence of differentiable distribution etc.).

Sometimes, M is a union of family of subsets (a plane as union of straight lines, an integral manifold of a Pfaff equation as union of integral curves, and so on.

The following problems appear:

(P1) Let D be an open set in $\mathbb{R}^p$ and $\{A_i\}_{i\in\mathcal{I}}$ a family of subsets of D having a joint point $a \in A_i$, for all $i \in \mathcal{I}$. Suppose a is a minimum local point for each restriction $f|_{A_i}$ of a function $f\colon D \to \mathbb{R}$ to the set A_i, $i \in \mathcal{I}$. Is a local minimum of f? A special case is when the subsets A_i are curves passing through a.

(P2) Let $f\colon D \subset \mathbb{R}^p \to \mathbb{R}$ and $\alpha_i\colon I_i \subset \mathbb{R} \to D$, $i \in \mathcal{I}$, be a family of parametrized curves. We are interested to study the connections between the extrema of $f \circ \alpha_i$, the extrema of restrictions $f|_{\alpha_i(I_i)}$, and the extrema of f. Of course, we can state a more general question using a family of topological spaces $\{I_i\}_{i\in\mathcal{I}}$ and a family of functions $\alpha_i\colon I_i \to D$.

4.1. **Extrema constrained by a family of subsets.** Let us discuss problem (P1). In general, the answer is negative.

Example 4.1. The function of Peano $f\colon \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = (x - y^2)(2x - y^2)$, has a minimum at $(0, 0)$ along any straight line passing through the origin, but the origin $(0, 0)$ is not a local minimum of f.

In the following, an affirmative answer to problem (P1) is given.

We say that the subset $A \subset \mathbb{R}^p$ is *dense at the point* $a \in A$ if $\bar{A}$ (the closure of A) is a closed neighborhood of a in $\mathbb{R}^p$, [1].

Proposition 4.1. *Let $f\colon D \subset \mathbb{R}^p \to \mathbb{R}$ be a continuous function, where D is an open set, and $A \subset D$ a dense set at the point $a \in A$. If a is a local minimum of the restriction $f|_A$, then a is a local minimum of f.*

We underline that, in the above proposition, the function f must be continuous on a neighborhood of a, not at a only.

Example 4.2. The function

$$f\colon \mathbb{R}^2 \to \mathbb{R}, \quad f(x, y) = \begin{cases} y^2, & \text{for } \ y \neq 0 \\ -x, & \text{for } \ y = 0 \end{cases}$$

is continuous at the point $a = (0, 0)$ only. Let $A = \{(x, y) \mid y \neq 0\} \cup \{(0, 0)\}$. Then A is dense at a and this point is a minimum for $f|_A$. However, a is not a minimum point for f.

Now, we can study problem (P2). For this purpose, we shall consider a function $f\colon D \to \mathbb{R}$, where D is an open set from $\mathbb{R}^p$.

Definition 4.1. Let $f\colon D \to \mathbb{R}$, $a \in D$ and $\alpha\colon I \to D$ a parametrized curve passing through a. We say that:

(i) a is a *minimum point of* f *constrained by* α if for all $t_0 \in I$, with $\alpha(t_0) = a$, the point t_0 is a local minimum for $f \circ \alpha$, that is there exists a neighborhood $I_{t_0} \subset I$ of t_0 such that

$$f(a) = f(\alpha(t_0)) \leq f(\alpha(t)),\ \forall t \in I_{t_0},$$

(ii) a is a minimum point of f *constrained by the curve* $\tilde{a}$ if there exists a neighborhood V of a such that

$$f(a) \leq f(x),\ \forall x \in V \cap \alpha(I),$$

that is a is a local minimum point for the restriction of f at $\alpha(I)$.

Remark 4.1. If a is a minimum point of f constrained by the parametrized curve α, then for any parametrized curve β equivalent to α, the point a is a minimum point of f constrained by β. Also, Definition 4.1 (ii) is correct, that is it does not depend on the representative β of $\tilde{\alpha}$.

Remark 4.2. If a is a minimum point of f constrained by the curve $\tilde{a}$, then a is a minimum point constrained by the parametrized curve α. The converse is not true, even if α is a simple curve (see, example below). However, when $\alpha\colon I \to D$ is a simple regular parametrized curve, and I is a compact set, the two notions are identical.

Example 4.3. Let be given the function $f\colon D \to \mathbb{R}$, $f(x, y) = x^2 - y^2$, and the simple curve $\tilde{a}$ with $\alpha\colon\ (-1, \infty) \to \mathbb{R}$, $\alpha(t) = \left(\dfrac{3t}{t^3+1}, \dfrac{3t^2}{t^3+1}\right)$, (a branch of Descartes folium). Then $\varphi(t) = f(\alpha(t)) = \dfrac{9\left(t^2 - t^4\right)}{\left(t^3+1\right)^2}$. Since $\alpha(0) = (0, 0)$ and $t_0 = 0$ is local minimum point for the function φ, it follows that $(0, 0)$ is a minimum point for f subject to the parametrized curve α. On other hand, $\lim\limits_{t\to\infty} \alpha(t) = (0, 0)$ and $\varphi(t) < 0$ for $t > 1$. Therefore, in any neighborhood from $\mathbb{R}^2$ of $(0, 0)$ there are points on the curve $\tilde{a}$ where the function f has negative values, that is $(0, 0)$ is not a minimum point of f subject to the curve $\tilde{a}$.

Definition 4.2. Let $\Gamma(a)$ be a family of curves (parametrized curves), passing through a. We say that a is a minimum point of f *constrained by the family* $\Gamma(a)$ if a is a minimum point of f constrained by each curve (parametrized curve) of the family $\Gamma(a)$.

Definition 4.3. According to the definition given above, a is a minimum point of f constrained by the family of curves $\Gamma(a)$, if for any curve $\tilde{a} \in \Gamma(a)$ there exists a neighborhood V of a such that $f(a) \leq f(x)$, for all $x \in V \cap \mathrm{Im}\tilde{a}$. If this neighborhood does not depend on the curve $\tilde{a}$, we shall say that a is a minimum point of f *uniformly constrained by the family of curves* $\Gamma(a)$.

It is obvious that a local minimum point of f is a minimum point uniformly constrained by the family passing through the point.

As it is well known, within the open sets in $\mathbb{R}^p$ there are dense curves of C^∞-class (for example, Lissajous type curves), and from Proposition 4.1 we obtain [1]

Proposition 4.2. *Let $f\colon D \to \mathbb{R}$ be a continuous function. The point $a \in D$ is a minimum point of f if and only if it is a minimum point of f constrained by the family curves of C^∞-class passing through a.*

If in Proposition 4.2 we modify the family of curves, we can cancel the continuity of f, and obtain

Theorem 4.1. *Let $f\colon D \to \mathbb{R}$ be an arbitrary function. The following statements are equivalent:*

1) *$a \in D$ is a minimum point of f;*

2) *a is a minimum point of f uniformly constrained by the family of curves of C^2-class passing through a, regular at the point a;*

3) *a is minimum point of f constrained by the family of curves of C^2-class passing through a, regular at a.*

A version of Theorem 4.1 is

Theorem 4.2. *Let $\Gamma(a)$ be the family of all curves passing through the point a such that for any point $x \in D$ exists a curve from the family containing both points. Then a is a local minimum for f if and only if a is a minimum for f uniformly constrained by the family $\Gamma(a)$.*

Proof. By reductio ad absurdum, suppose that a is a minimum point for f, uniformly constrained by the family $\Gamma(a)$, but is not a local minimum for f. So, in any neighborhood of a there exists a point x such that $f(a) > f(x)$. Using the supposition on the family $\Gamma(a)$, we obtain a curve from the family containing both a and x, which contradicts the fact that a is a minimum point for f uniformly constrained by the family $\Gamma(a)$. □

Corollary 4.1. *Let $f\colon D \to \mathbb{R}$ an arbitrary function. Then a is a local minimum for f if and only if a is a minimum point uniformly constrained by the family of all straight lines passing through a.*

Proof. Since D is an open set, there exists an open ball $B_r(a)$, contained in D. Next, Theorem 4.2 can be applied for $f|_{B_r(a)}$. □

The following example shows that not any minimum point constrained by a family of curves is a minimum point uniformly constrained by the family.

Example 4.4. Consider the function $f(x,y) = (x - y^2)(2x - y^2)$ and $\alpha = (0,0)$. Denote by $\Gamma(a)$ the family of all straight lines passing through the point a. As we have shown in Example 4.1, a is a minimum point constrained by the family of curves $\Gamma(a)$. However, a cannot be a minimum point uniformly constrained by $\Gamma(a)$, since according to Theorem 4.2, a should be a locally minimum point for f, which is a contradiction.

Now, we give a study of Problem (P2), stated at the beginning of this section.

4.2. **Extrema constrained by a family of parametrized curves.** In contrast with the notion of extremum constrained by a family of curves, the notion of extremum constrained by a family of parametrized curve is much more interesting, because it holds [1]

Proposition 4.3. *If $\alpha\colon I \to \mathbb{R}^p$ is a parametrized curve of C^k-class, passing through $a = \alpha(t_0)$, having tangent at this point, then exists a neighborhood $I_{t_0} \subset I$ of t_0 such that $\alpha(I_{t_0})$ is not a neighborhood in $\mathbb{R}^p$.*

To preserve the geometric background, in the following any family $\Gamma(a)$ of parametrized curves will be considered having the properties:
(1) if $\alpha \in \Gamma(a)$, then $\beta \in \Gamma(a)$, for all parametrized curve β, equivalent to α;
(2) if $\alpha \in \Gamma(a)$, then $\alpha|_J \in \Gamma(a)$, for any interval J contained in dom(α).

With these assumptions, we obtain

Proposition 4.4. *Let $\Gamma(a)$ be a family of parametrized curves passing through a. Then a is a minimum point for the function f constrained by $\Gamma(a)$ if and only if for all $\alpha \in \Gamma(a)$ with $\alpha(t_0) = \alpha$ it follows $f(\alpha(t_0)) \leq f(\alpha(t))$, for all $t \in [t_0, t_0 + \varepsilon)$.*

Now, we emphasize a condition required by a family of parametrized curves $\Gamma(a)$ such that a result alike Proposition 4.2 holds.

Suppose a a point in $\mathbb{R}^p$ and $S(a)$ a family of sequences with elements from $\mathbb{R}^p$, convergent to a.

Definition 4.4. A parametrized curve α passing though a $(\alpha(t_0) = a)$ is said to be *subordinate to the sequence* $(x_n) \in S(a)$, if there exist a subsequence (x_{n_k}) and a decreasing sequence of real numbers t_k, $t_k \to t_0$ such that $\alpha(t_k) = x_{n_k}$, $\forall k \in \mathbb{N}^*$.

Definition 4.5. Let $\Gamma(a)$ be a family of parametrized curves passing through the point $a \in \mathbb{R}^p$. The family $\Gamma(a)$ is said to be $S(a)$-*subordinate* if for any $(x_n) \in S(a)$ there exists $\alpha \in \Gamma(a)$ subordinate to the sequence (x_n).

Theorem 4.3. *Let $f\colon D \to \mathbb{R}$ and a a point in D. Consider $C(a)$ the family of all sequences of distinct elements from the open set D converging to a. Assume that $\Gamma(a)$ is $C(a)$-subordinate family. Then, a is a local minimum for f if and only if a is a minimum for f constrained by $\Gamma(a)$.*

In this context, we can study the general extremum problem (4.1), where M is a submanifold in D, defined by a set of equalities and/or inequalities.

Let $a \in M$ and $\Gamma_M(a)$ the set of all parametrized curves α, $(\alpha(t_0) = a)$, such that $\alpha(t) \in M$, for all $t \in [t_0, \varepsilon)$. Let $C(a)$ be the family of all sequences (x_n) of distinct points from M, with $x_n \to a$.

Theorem 4.4. *Let $f\colon D \to \mathbb{R}$. Assume that $\Gamma_M(a)$ is $C(a)$-subordinate family. Then, a is a local minimum point for f constrained by M if and only if a is a minimum point for f constrained by $\Gamma_M(a)$.*

Definition 4.6. Let $\Gamma(a)$ a family of parametrized curves passing through a. The family $\Gamma(a)$ is said to be *optimal* if for any function $f\colon D \to \mathbb{R}$, a is a local minimum (maximum) point for f (constrained by M) if and only if a is a minimum (maximum) point for f constrained by $\Gamma(a)$.

The results in the above show that any family of parametrized curves which is $C(a)$-subordinate is optimal. It is obvious that both properties of a family of parametrized curves to be optimal or $C(a)$-subordinate also hold for all families that include it. Therefore, it is useful to find families of curves with these properties with as "few" elements as possible. On other hand, the optimality property of a family is not preserved for all its subfamilies (see Example 4.5, below).

Open problem 4.1. Are there minimal elements with respect to the inclusion in the class of all optimal families or in the class of all $C(a)$-subordinate families?

Open problem 4.2. Are there optimal families which are not $C(a)$-subordinate?

The following items contain examples of $C(a)$-subordinate family of curves [3].

1) $\Gamma^1(a)$: the family of all parametrized curves passing through the point a, of C^1-class, regular at a.

2) $\Gamma^m(a)$: the family of all C^m-class parametrized curves passing through the point a having a tangent at a.

3) Let $g = (g^1, \ldots, g^s)\colon D \to \mathbb{R}^s$ of C^1-class and $a \in D$. Furthermore, let $\operatorname{rank}\left[\dfrac{\partial g^i}{\partial x^j}(a)\right] = s$ and $g(a) = 0$. Let $C_g(a)$ be the family of all sequences (x_n) of distinct elements of D, such that $g(x_n) \geq 0$ $(g(x_n) = 0)$ and $x_n \to a$. We denote by $\Gamma_g^m(a)$ the family of all parametrized curves $\alpha \in \Gamma^m(a)$ $(\alpha(t_0) = a)$, such that $g(\alpha(t)) \geq 0$ $(g(\alpha(t)) = 0)$, for all $t \in [t_0, t_0 + \varepsilon)$. Then, the family $\Gamma_g^m(a)$ is a $C_g(a)$-subordinate family.

Having in mind these examples, we can state various results such that:

Theorem 4.5. *Let $f\colon D \to \mathbb{R}$ and $a \in D$. Then a is a local minimum (maximum) point for f if and only if a is a minimum (maximum) point for f constrained by $\Gamma^m(a)$.*

Theorem 4.6. *Let $f\colon D \to \mathbb{R}$ and $a \in D$. Let $g = (g^1, \ldots, g^s)\colon D \to \mathbb{R}^s$ be a vector function of C^1-class with* $\operatorname{rank}\left[\dfrac{\partial g^i}{\partial x^j}(a)\right] = s$, *such that $g(a) = 0$. Then, a is a local minimum (maximum) point for f constrained by $g \geq 0$ $(g = 0)$ if and only if a is a minimum point constrained by $\Gamma_g^m(a)$.*

Example 4.5. In $\mathbb{R}^2$, consider the family $\Gamma^m(0,0)$, with $m \geq 2$. We have established that this family is optimal. Let us define the subfamily $\Phi^m(0,0)$ of all parametrized curves $\alpha \in \Gamma^m(0,0)$ $(\alpha(0) = (0,0))$ for which $\min\{i \mid \alpha^{(i)}(0) \neq 0\} \leq m-1$. Also, consider the function

$$f\colon D \to \mathbb{R}, \quad f(x,y) = \left(y^m - x^{m+1}\right)\left(y^m - 2^m x^{m+1}\right).$$

This function is of C^{∞}-class, but the point $(0,0)$ is not a local minimum for f. However, $(0,0)$ is a local minimum point for f constrained by the family $\Phi^m(0,0)$, that is the subfamily $\Phi^m(0,0)$ is not optimal.

We underline that for obtaining the above results we considered f be an arbitrary function, with no contribution to the development of these results. That is why, we suppose that we can preserve the conclusions of these results if we associate some properties to f, but restricting the family $\Gamma(a)$.

Also, in relation with the above results, the following problem arises: is it possible that a is an extremum point for f constrained by any parametrized curve from the optimal family but with different kind on at least two curves from the family? In the following, we shall see that this fact is not possible.

Theorem 4.7. *Let $f\colon D \to \mathbb{R}$ be a continuous function and let $a \in D$. Let $\Gamma(a)$ be a $C(a)$-subordinate family of parametrized curves. Then, the point a is a strict extremum point of f if and only if a is a strict extremum point of f constrained by each $\alpha \in \Gamma(a)$.*

Proof. We prove that a is a strict extremum point of f constrained by $\Gamma(a)$ if a is a strict extremum point of f constrained by each $\alpha \in \Gamma(a)$; in accordance with Theorem 4.3 it follows that a is a strict extremum point of f. Suppose, by reductio ad absurdum, that there exist two parametrized curves $\alpha\colon I \to D$ and $\beta\colon J \to D$ in $\Gamma(a)$ $(a = \alpha(t_0) = \beta(u_0))$, such that a is a point of strict minimum for f constrained by α, and in the same time a is a point of strict maximum for f constrained by β. Hence, we can find two sequences of real numbers (t_n) and (u_n), $t_n \to t_0$ and $u_n \to u_0$, such that $f(\alpha(t_n)) > f(a)$ and $f(\beta(u_n)) < f(a)$. Since the sequences $\alpha(t_n)$ and $\beta(t_n)$ have a common limit a, it follows $||\alpha(t_n) - a|| < \dfrac{1}{n}$ and $||\beta(t_n) - a|| < \dfrac{1}{n}$, for all $n > N$. We select a curve arc situated in the interior of a ball of radius $\dfrac{1}{n}$ and center a, joining the points $\alpha(t_n)$ and $\beta(t_n)$ and avoiding the center a. From the continuity of the function f, the curve arc contains a point x_n for which $f(x_n) = f(a)$. Since $\Gamma(a)$ is a $C(a)$-subordinate family, we get a parametrized curve γ in $\Gamma(a)$, a subsequence (x_{n_k}) of (x_n) and a sequence of real numbers (q_k) such that $q_k \to 0$, $\gamma(0) = a$ and $\gamma(q_k) = x_{n_k}$. It follows $f(\gamma(q_k)) = f(a)$, for all $k \in \mathbb{N}$, that is the point a cannot be a strict extremum point of f constrained by γ. This result is a contradiction. □

Corollary 4.2. *Let $f\colon D \to \mathbb{R}$ be a continuous function and let $a \in D$. The point a is a strict extremum point of f if and only if a is a strict extremum point of f constrained by each $\alpha \in \Gamma^m(a)$.*

Since the above results are still valid when $\mathbb{R}^p$ is replaced by a finite dimensional differentiable manifold, Theorem 4.6 could be enhanced.

Theorem 4.8. *Let $f\colon D \to \mathbb{R}$ be a continuous function and $a \in D$. Consider $g = (g^1, \ldots, g^s)\colon D \to \mathbb{R}^s$ of C^1-class with* rank $\left[\dfrac{\partial g^i}{\partial x^j}(a)\right] = s$, *such that $g(a) = 0$. Then, a is a strict extremum point of f constrained by $g = 0$ if and only if a is a strict extremum point constrained by each $\alpha \in \Gamma_g^m(a)$.*

The previous Corollary remains also true when we renounce to the continuity of the function f, but we modify the family of parametrized curves.

Theorem 4.9. *Let $f\colon D \to \mathbb{R}$ and $a \in D$. The point a is a strict extremum point of the function f if and only if it is a strict extremum point of f constrained by each $\alpha \in \Gamma^1(a)$.*

Theorems 4.7 and 4.9 are not valid if we replace "strict extremum point" with "extremum point". More precisely, the point a can be extremum point for a function f constrained by any parametrized curve from the family, without being extremum point for f, even if the function f is of C^∞-class. To have an example, we fix

$$D_1\colon y^2 - xy < 0,\ D_2\colon x^2 - xy \le 0,\ D_3\colon x^2 + xy < 0,\ D_4\colon y^2 + xy \le 0$$

in $\mathbb{R}^2$. Then we define the C^∞-class function

$$f\colon \mathbb{R}^2 \to \mathbb{R},\ f(x,y) = \begin{cases} \exp\dfrac{1}{y^2 - xy}, & \text{if} \quad (x,y) \in D_1 \\ 0, & \text{if} \quad (x,y) \in D_2 \cup D_4 \\ -\exp\dfrac{1}{x^2 + xy}, & \text{if} \quad (x,y) \in D_3. \end{cases} \tag{4.2}$$

The point $a = (0,0)$ is not an extremum point of the function f. On the other hand, since the subsets $D_i, i = \overline{1,4}$, are star-shaped at the point a, it follows that any parametrized curve α of $\Gamma^1(a)$, passing through a, remains, in a neighborhood of the point a, in one of the sets $D_1 \cup D_2 \cup D_4$ or $D_2 \cup D_3 \cup D_4$. In this way, the point a is an extremum point (non-strict) for f constrained by any $\alpha \in \Gamma^1(a)$.

4.3. **Lateral extrema and convexity.** In some optimization problems, the requirement of convexity may be too strong and not essential, and convexity at a point may be all that is needed. In the following, we introduce sufficient conditions of extremum, by using the notions of lateral extremum and of convexity at a point.

Let $\varphi\colon I \subseteq \mathbb{R} \to \mathbb{R}$, where I is an interval.

Definition 4.7. Let $t_0 \in I$. We say that the function φ is *strictly convex at* t_0 if there exists a convex neighborhood $I_{t_0} \subseteq I$ of t_0, such that

$$\varphi(ut + (1-u)t_0) < u\varphi(t) + (1-u)\varphi(t_0)$$

for all $t \in I_{t_0} \backslash \{t_0\}$ and $\forall u \in (0,1)$.

If φ is strictly convex in a convex neighborhood I_{t_0} of t_0, that is

$$\varphi(ut_2 + (1-u)t_1) < u\varphi(t_2) + (1-u)\varphi(t_1)$$

for $\forall t_1, t_2 \in I_{t_0}$ and $\forall u \in (0,1)$, then φ is strictly convex at t_0. The converse is not true. For this it is sufficient to consider the function

$$\varphi\colon \mathbb{R} \to \mathbb{R}, \quad \varphi(t) = \begin{cases} t^2, & \text{if} \quad t \in (-\infty, 0] \cup [1, +\infty) \\ \dfrac{1}{n} t^2, & \text{if} \quad t \in \left[\dfrac{1}{n+1}, \dfrac{1}{n}\right), \ n \in \mathbb{N}^*, \end{cases}$$

which is strictly convex at $t_0 = 0$, but is not strictly convex in any neighborhood of t_0, because there exist points of discontinuity in any neighborhood of t_0.

Definition 4.8. We say that $t_0 \in I$ is a *strict right-hand minimum (maximum) point* of the function φ if $\varphi(t_0) < \varphi(t)$ $(\varphi(t_0) > \varphi(t))$, for all $t \in (t_0, t_0 + \varepsilon)$.

Similar, the notion of *strict left-hand minimum (maximum) point* is introduced.

We say that t_0 is a *lateral extremum point* of φ, if t_0 is a left-hand or a right-hand extremum point of φ.

Lemma 4.1. *Suppose that the mapping $\varphi\colon I \subset \mathbb{R} \to \mathbb{R}$ is strictly convex or strictly concave at $t_0 \in I$. Then t_0 is both a strict right-hand extremum point of φ and a strict left-hand extremum point of φ. Moreover, if φ is strictly convex (strictly concave) in a convex neighborhood of t_0, then t_0 cannot be a local maximum (minimum) point.*

Proof. For example, let us suppose that φ is strictly convex at the point t_0. Let $I_{t_0} \subset I$ be the neighborhood from Definition 4.7. There are two possible cases.

a) For all $t \in I_{t_0} \cap (t_0, \infty)$, we have $\varphi(t_0) < \varphi(t)$, that is t_0 is a strict right-hand minimum point of φ.

b) In the neighborhood I_{t_0} there exists $t_1 > t_0$ such that $\varphi(t_0) \geq \varphi(t_1)$. Then for all $t \in (t_0, t_1)$, we get $t = ut_1 + (1-u)\, t_0$, $u \in (0,1)$. Therefore, we obtain $\varphi(t) < u\varphi(t_1) + (1-u)\,\varphi(t_0) \leq \varphi(t_0)$, that is t_0 is a strict right-hand maximum point of φ.

If φ is strictly convex in a neighborhood I_{t_0}, let us suppose, by reductio ad absurdum, that t_0 is a local maximum point of φ. Therefore, there exist t_1, $t_2 \in I_{t_0}$ such that $t_1 < t_0 < t_2$, $\varphi(t_1) \leq \varphi(t_0)$ and $\varphi(t_2) \leq \varphi(t_0)$. Because $t_0 = ut_2 + (1-u)\, t_1$, $u \in (0,1)$, we get $\varphi(t_0) < u\varphi(t_2) + (1-u)\,\varphi(t_1) \leq u\varphi(t_0) + (1-u)\,\varphi(t_0) = \varphi(t_0)$, which is a contradiction.

In a similar manner, the case when φ is strict concave can be studied. $\square$

Let $f\colon D \subseteq \mathbb{R}^p \to \mathbb{R}$, where D is an open set. Let $a \in D$ and $\Gamma(a)$ a family of parametrized curves passing through a.

Definition 4.9. The point $a \in D$ is *a strict lateral extremum point (strict right-hand extremum point)* of the function f *constrained by* $\alpha \in \Gamma(a)$ $(\alpha(t_0) = a)$, if t_0 is a strict lateral extremum point of $f \circ \alpha$.

Definition 4.10. We say that the function f is *strictly convex at a with respect to the parametrized curve* $\alpha \in \Gamma(a)$ $(\alpha(t_0) = a)$, if the function $f \circ \alpha$ is strictly convex at t_0. We say that f is strictly convex at a with respect to the family $\Gamma(a)$ if f is strictly convex at a with respect to each $\alpha \in \Gamma(a)$.

The result in the following is a refinement of those in Theorems 4.7 and 4.9.

Theorem 4.10. *Let $f\colon D \to \mathbb{R}$ and $a \in D$. Suppose that for each $\alpha \in \Gamma^1(a)$ the point a is a strict extremum point of f constrained by α if the image of α is contained within a straight line or a strict right-hand extremum point of f constrained by α, otherwise. Then a is a strict extremum point of f.*

Proof. We shall prove that a is a strict extremum point of f constrained by α, for all $\alpha \in \Gamma^1(a)$. Applying Theorem 4.9, we get the conclusion. Suppose, per absurdum, that there exists $\alpha \in \Gamma^1(a)$ $(\alpha(0) = 0)$ such that a is not a strict extremum point of f constrained by α. Then, there exist two sequences of real numbers (t_n) and (u_n) which converge to 0 such that, for example $t_n > 0$, $u_n < 0$, $f(\alpha(t_n)) > 0$ and $f(\alpha(u_n)) < 0$, for all $n \in \mathbb{N}$. Let $\beta \in \Gamma^1(a)$, $\beta(t) = \alpha'(0)\,t$. By hypothesis, a is a strict extremum point of f constrained by β, because the image of β is contained within a straight line. We get $f(\beta(t_n)) > 0$, for all $n \in \mathbb{N}$, since in the opposite case, according to Lemma 3.3 from [6], applied to the parametrized curves α and β, we would find a curve γ such that a is not a strict right-hand extremum point of f constrained by γ. Applying the same reasoning for $\alpha_1(t) = \alpha(-t)$ and $\beta_1(t) = \beta(-t)$, we obtain $f(\beta_1(-u_n)) < 0$, for all $n \in \mathbb{N}$, that is $f(\beta(u_n)) < 0$, for all $n \in \mathbb{N}$, which contradicts the fact that a is a strict extremum point of f constrained by β. □

From this theorem and Lemma 4.1, we get

Theorem 4.11 ([6]). *Let $f\colon D \to \mathbb{R}$ and $a \in D$ such that:*

1) *a is a strict extremum point of f constrained by each straight line passing through the point a;*

2) *For all $\alpha \in \Gamma^1(a)$, it follows that f or $-f$ is strictly convex at the point a constrained by α.*

Then a is a strict extremum point of f.

Moreover, if we consider Theorem 2.1, we obtain

Theorem 4.12. *Let $f\colon D \to \mathbb{R}$ and $a \in D$ such that:*

1) *a is a strict extremum point of the function f constrained by each straight line passing through the point a;*

2) *For all $\alpha \in \Gamma^1(a)$, it follows that f is strictly convex at the point a constrained by α.*

Then the function f is continuous at a and a is a strict minimum point of f.

The following result is a refinement of Corollary 4.1.

Theorem 4.13. *Let $f\colon D \to \mathbb{R}$ an arbitrary function and $a \in D$. Let $\Gamma(a)$ be the familiy of all straight lines passing through a. Suppose f is strictly convex (strictly concave) in a neighborhood of a and a is an extremum point uniformly constrained by the family $\Gamma(a)$, not having the same nature for all curves of the family. Then a is a strict local minimum (maximum) point for f.*

Proof. For example, suppose that f is strict convex in a neighborhood V of a. Also, we can suppose that for each parametrized straight line α passing through a ($\alpha(t_0) = a$), either $f(\alpha(t_0)) \leq f(a(t))$, for all $t \in \alpha^{-1}(V)$, or $f(\alpha(t_0)) \geq f(a(t))$, for all $t \in \alpha^{-1}(V)$. For each such a parametrized straight line, α, let us consider $\varphi_\alpha = f \circ \alpha$. From hypothesis, it follows that φ_α is strict convex in a neighborhood of t_0. According to Lemma 4.1, it follows that t_0 cannot be a local maximum. Then, $f(\alpha(t_0)) \leq f(a(t))$, for all $t \in \alpha^{-1}(V)$, that is a is a minimum point uniformly constrained by the family of all straight lines passing through a. From Corollary 4.1 and having in mind that f is strictly convex in a convex neighborhood of a, we obtain the conclusion. □

A version of the preceding result is

Theorem 4.14. *Let $f \colon D \to \mathbb{R}$ be a C^1-class function and $a \in D$ and $\Gamma(a)$ be the family of all parametrized straight lines passing through a. Suppose that f is strictly convex (concave) in a convex neighborhood of a and a is an extremum point constrained by any $\alpha \in \Gamma(a)$, not necessarily having the same nature for all curves of the family. Then a is a strict local minimum (maximum) point for f.*

Proof. Let $\alpha \in \Gamma(a)$, $\alpha(t_0) = a$ and $\varphi_\alpha = f \circ \alpha$. Since t_0 is a local extremum point for φ_α, it results $\varphi'_\alpha(t_0) = 0$. But α is an arbitrary parametrized straight line, so $\nabla f(a) = 0$. The function f is strictly convex (concave) in a neighborhood of a, therefore a is a strict local minimum (maximum) point for f. □

4.4. **Convexity of hypersurfaces along curves.** Let Σ be a hypersurface defined by an equation of the form $F(x) = 0$, when F is of class at least C^1. Let $a \in \Sigma$ and $\Gamma_*(a)$ a family of parametrized curves passing through a and contained in Σ.

Definition 4.11. We say that Σ *is (strictly) convex at a along a parametrized curve* $\alpha \in \Gamma_*(a)$ if a is a (strict) local extremum point for

$$f(x) = \sum_{i=1}^{p} \frac{\partial F}{\partial x^i}(a)\left(x^i - a^i\right),$$

that is α (strictly) stays around a on the same side of the hyperplane tangent at a.

We say that the hypersurface Σ *is (strictly) convex at a with respect to* $\Gamma_F(a)$ if a is a (strict) extremum point for the above function, constrained by $\Gamma_F(a)$, that is all the parametrized curves in the family $\Gamma_F(a)$ (strictly) stay around a on the same side of the hyperplane tangent at a.

We say that Σ *is (strictly) convex at a* if the hypersurface Σ (strictly) remains around a on the same side of the tangent hyperplane to Σ at a.

Let $\Gamma_F^m(a)$ be the family of all parametrized curves of C^m-class ($m \geq 1$) passing through a, contained in Σ and having tangent at a. From Theorem 4.6, we obtain

Theorem 4.15. *The hypersurface Σ is (strictly) convex at a if and only if Σ is (strictly) convex at a with respect to $\Gamma_F^m(a)$.*

It is interesting to note that the above Theorem is no more valid in the case in which the family $\Gamma_F^m(a)$ is replaced, for example, by the family $\Gamma_*^2(a)$ of all parametrized curves of C^2-class passing through a, contained in Σ and regular at a.

Example 4.6. In $\mathbb{R}^3$ let us consider the C^∞ surface $\Sigma\colon z - g(x,y) = 0$, where $g(x,y) = \left(y^3 - x^4\right)\left(y^3 - 8x^4\right)$. The tangent plane to Σ at $a = (0,0,0)$ is the plane $z = 0$. Because the point $(0,0)$ is not a local extremum point of g, it results that Σ is not convex at a. However, Σ is convex with respect to $\Gamma_*^2(a)$. Indeed, let $\alpha(t) = (x(t), y(t), z(t))$ be a parametrized curve of class C^2 contained in Σ and passing through a and regular at a. Because α is contained in Σ, we have $z(t) = g(x(t), y(t))$, and further we get $z'(t_0) = 0$. So α is regular at $a = (0,0,0)$ if and only if $\beta(t) = (x(t), y(t))$ is regular at $(0,0)$. Thus, the convexity of Σ at the point a with respect to $\Gamma_*^2(a)$ comes to the fact that the point $(0,0)$ is an extremum point of g constrained by the family of all parametrized curves of class C^2 passing through $(0,0)$ and regular at this point. It can be proved [1] that $(0,0)$ is a minimum point of g with respect to any such parametrized curve.

Also the previous theorem does not hold true in the case in which Γ_a^* is the family of all analytic parametrized curves passing through a and contained in Σ.

Example 4.7. Consider in $\mathbb{R}^3$ the surface $\Sigma : z = y(y - g(x))$, of class C^∞, where

$$g(x) = \begin{cases} \exp\left(-\dfrac{1}{x^2}\right), & \text{for} \quad x > 0 \\ 0, & \text{for} \quad x \leq 0. \end{cases}$$

We apply the same reasoning as in the previous example: because the point $(0,0)$ is not an extremum point of $f(x,y) = y(y - g(x))$, one gets that Σ is not convex at $a = (0,0,0)$; on the other side, it can be proved [3] that the point $(0,0)$ is a minimum point of f restricted by the family of all analytic and plane parametrized curves passing through $(0,0)$, that is Σ is convex at $a = (0,0,0)$ with respect to Γ_a^*.

Using Theorem 4.8, we can obtain a refinement of the above result.

Theorem 4.16. *The hypersurface $\Sigma\colon F(x) = 0$ is strictly convex at a if and only if Σ is strictly convex at a along any parametrized curve in $\Gamma_F^m(a)$.*

Reformulating:

Theorem 4.17. *The following statements are equivalent:*

1) *Each parametrized curve in the family $\Gamma_F^m(a)$ remains, around a, strictly on the same side of the tangent hyperplane to Σ at a.*

2) *All the parametrized curves in the family $\Gamma_F^m(a)$ remains, around a, strictly on the same side of the tangent hyperplane to Σ at a.*

3) *The hypersurface Σ remains, around a, on the same side of the tangent hyperplane to Σ at a.*

Remark that the previous Theorem is not valid if we replace "strictly convex" with "convex". More precisely, a hypersurface can be convex at the point a along each parametrized curve $\alpha \in \Gamma_a^*$, without being convex at a. As an example, it is enough to consider the surface $\Sigma\colon z = f(x, y)$, where f is defined by (4.2). The plane tangent to Σ at the point $a = (0, 0, 0)$ has the equation $z = 0$. The properties of the function f show that this surface does not remain, round the point a, on the same side of the tangent plane, though each parametrized curve $\alpha \in \Gamma_F^m(a)$ stays, around the point a, on the same side of the tangent plane. In other words, we have a surface which crosses the tangent plane at a, but no curve in the surface crosses the tangent plane.

5. EXTREMA WITH NOHOLONOMIC CONSTRAINTS

In the following, we shall deal with Question 1.2.

Let $D \subseteq \mathbb{R}^p$ be an open set and let

$$\omega^j(x) = \sum_{i=1}^{p} \omega_i^j(x)\, dx^i = 0,\ j = \overline{1, s} < p \tag{5.1}$$

a Pfaff system on D, where ω_i^j are C^1-class functions such that

$$\operatorname{rank}\ \left[\omega_i^j(x)\right] = s,\ \forall x \in D.$$

Let I be an interval in $\mathbb{R}^m$, $m = \overline{1, p - s}$. A C^2-class regular function $r\colon I \to D$, $r = (x^1, \ldots, x^p)$, $x^i = x^i(u)$, $u = (u^1, \ldots, u^m) \in I$, $i = \overline{1, p}$, for which we have

$$\sum_{i=1}^{p} \omega_i^j(r(u)) \frac{\partial x^i}{\partial u^k} = 0,\ \forall u \in I,\ k = \overline{1, m},\ j = \overline{1, s}$$

is called an *integral manifold* of dimension m of the Pfaff system (5.1). We say that the integral manifold $r\colon I \to D$ passes through the point $a \in D$ if there exists $u_0 \in I$ such that $r(u_0) = a$. For $m = 1$, the integral manifold is called *integral curve.*

The Pfaff system (5.1) is called *completely integrable (holonomic system)* if through the point $a \in D$ passes an integral manifold whose dimension is $p - s$ (*maximal* integral manifold). If the Pfaff system (5.1) is not completely integrable (*nonholonomic system*), then for any point $a \in D$ does exist some integral curves at a only and it is possible that integral manifolds of maximum dimension do not exist at certain points.

Definition 5.1. Let $f\colon D \to \mathbb{R}$ be a C^1-class function. We say that $a \in D$ is a minimum (maximum) point of f *uniformly constrained* by the Pfaff system (5.1) if there exists a neighborhood $V_a \subseteq D$ of a such that for every integral curve $\alpha\colon I \to D$ which passes through a

$$f(x) \geq f(a)\ \ (f(x) \leq f(a)),\ \forall x \in V_a \cap \alpha(I).$$

We say that $a \in D$ is a minimum (maximum) point of f *constrained* by the Pfaff system (5.1), if for every integral curve $\alpha\colon I \to D$ which passes through a $(\alpha(t_0) = a)$ the point t_0 is a local minimum (maximum) point of $f \circ \alpha$.

Within this framework, we defined an extremum problem, either with *holonomic* constraints or *nonholonomic* constraints of equality type, if the Pfaff system (5.1) is completely integrable or not, respectively.

It is clear that any extremum point uniformly constrained by a Pfaff system is an extremum point constrained by the same Pfaff system. Also, a is a minimum (maximum) point of f constrained by the Pfaff system (5.1) if and only if for every integral curve $\alpha\colon I \to D$, which passes through a, $(\alpha(t_0) = a)$ we have $f(\alpha(t)) \geq f(a)$ $(f(\alpha(t)) \leq f(a))$, for all $t \in [t_0, t_0 + \varepsilon)$.

As a consequence of Theorem 4.1, we have

Theorem 5.1. *Consider the complete integrable Pfaff system*

$$dg^j = 0,\ j = \overline{1,s}, \tag{5.2}$$

with g^j functions of C^2-class. Let $a \in D$ such that rank $\left[\frac{\partial g^i}{\partial x^j}(a)\right] = s$ and $f\colon D \to \mathbb{R}$ an arbitrary function. Then the following statements are equivalent:

1) *a is a local extremum point of f constrained by*

$$g^j(x) = g^j(a), \quad j = \overline{1,s}.$$

2) *a is extremum point of f uniformly constrained by the Pfaff system* (5.2).

A version of this result follows from Theorem 4.8.

Theorem 5.2. *Consider the complete integrable Pfaff system* (5.2), *with g^j functions of C^2-class. Let $a \in D$ such that* rank $\left[\frac{\partial g^i}{\partial x^j}(a)\right] = s$ *and $f\colon D \to \mathbb{R}$ a continuous function. Then the following statements are equivalent:*

1) *a is a local strict extremum point of f constrained by*

$$g^j(x) = g^j(a), \quad j = \overline{1,s}.$$

2) *a is strict extremum point of f uniformly constrained by the Pfaff system* (5.2).

3) *a is strict extremum point of f constrained by the Pfaff system* (5.2).

4) *a is strict extremum point of f constrained by each integral curve which passes through a, having no the same nature for all these curves.*

This approach allows to consider an extremum problem constrained by equations as a particular case of extremum problem constrained by a Pfaff system. Remark that an extremum problem constrained by the completely integrable Pfaff system (5.2) is equivalent to a family of constrained extrema problems having as constraints systems of equations of the form $g^j(x) = c_j$, $j = \overline{1,s}$. Using this framework, we can study the evolution of constrained extrema points of a function f on the maximal integrable manifolds of the Pfaff system (5.2).

In what follows, we shall show that there exist constrained extrema by a Pfaff system which are not uniformly constrained.

Lemma 5.1 ([14]). *Let be given*

$$\eta\,(x)=\sum_{i=1}^{p}\eta_i\,(x)\,dx^i$$

a Pfaff form on D and $g\colon D\to\mathbb{R}$ a C^1-class function without critical points. Then, any integral curve of the Pfaff equation $dg+g\eta=0$ (not necessary completely integrable) which passes through a point of the integral hypersurface $M\colon g=0$ is contained in M.

Example 5.1. The Pfaff equation $\omega=dz+z(zdx-dy)=0$ is not completely integrable. According to the above lemma, the family of all integrable curves of Pfaff equation $\omega=0$, passing through a point of the plane $z=0$, can be identified with the family of all parametrized curves of C^2-class contained within this plane. Consider $f\,(x,y,z)=\left(y^2-x^3\right)\left(y^2-4x^3\right)+z$. Then the point $(0,0,0)$ is an extremum point for f uniformly constrained by $\omega=0$ if and only if the point $(0,0)$ is local extremum for the function $h\,(x,y)=\left(y^2-x^3\right)\left(y^2-4x^3\right)$. Having in mind that $(0,0)$ is not a local extremum for the function h, it follows that $(0,0,0)$ is not extremum point for f uniformly constrained by $\omega=0$. On other hand, according to Example 4.5, the case $m=2$, it follows that the point $(0,0)$ is a minimum point constrained by the family of all parametrized curves of C^2-class contained into the plane $z=0$, that is the point $(0,0,0)$ is a minimum point constrained by $\omega=0$.

The first answer to Question 1.2 is given in the following two theorems.

Theorem 5.3 ([15]). *If $a\in D$ is an extremum point for the C^1-class function $f\colon D\to\mathbb{R}$ constrained by the Pfaff system* (5.1)*, then there exist $\lambda_1,\dots,\lambda_s$ in $\mathbb{R}$ such that*

$$df\,(a)=\sum_{k=1}^{s}\lambda_k\omega^k\,(a)\,,$$

that is a is a critical point of f constrained by the Pfaff system (5.1)*.*

Theorem 5.4 ([15]). *If $a\in D$ is a critical point of the C^2-class function $f\colon D\to\mathbb{R}$ constrained by the Pfaff system* (5.1) *and the quadratic form*

$$d^2f\,(a)-\frac{1}{2}\sum_{k=1}^{s}\lambda_k\sum_{i,j=1}^{p}\left(\frac{\partial\omega_i^k}{\partial x^j}+\frac{\partial\omega_j^k}{\partial x^i}\right)(a)\,dx^idx^j$$

constrained by

$$\sum_{i=1}^{p}\omega_i^k\,(a)\,dx^i=0,\ \ k=\overline{1,s}$$

is positive (negative) definite, then a is a minimum (maximum) point f constrained by the Pfaff system (5.1)*.*

5.1. **Convexity of nonholonomic hypersurfaces.** The idea to consider constrained extremals by a family of integral curves of a Pfaff system allows to approach the problem of convexity for a *nonholonomic hypersurface* using a method similar to those in §§4.4.

We refer to hypersurfaces defined by a Pfaff equation

$$\omega\left(x\right)=\sum_{i=1}^{p}\omega_{i}\left(x\right)dx^{i}=0 \tag{5.3}$$

on $D\subseteq\mathbb{R}^{p}$, where $\omega_{i}\colon D\to\mathbb{R}$ are of C^{1}-class and $\text{rank}[\omega_{i}\left(x\right)]=1$, $\forall x\in D$.

If (5.3) is completely integrable, two cases are possible:

a) There exists $F\colon D\to\mathbb{R}$ of C^{2}-class, with $dF=\omega$, that is ω is an exact form;

b) There exists $F\colon D\to\mathbb{R}$ of C^{2}-class and $\mu\colon D\to\mathbb{R}$ of class C^{1}, with $dF=\mu\omega$, that is equation (5.3) admits an integrant factor μ.

In both cases in the above, an integral hypersurface implicitly defined by equation $F\left(x\right)=F\left(a\right)$ passes through each point $a\in D$. In this manner, a completely integrable equation defines a family of hypersurfaces of the form $F\left(x\right)=c$. All the integral curves passing through the point a are contained in the same "level set" $F\left(x\right)=F\left(a\right)$. Two points ($x_{1}$ and x_{2}) can be connected through an integral curve if and only if they are contained in the same "level set" ($F\left(x_{1}\right)=F\left(x_{2}\right)$).

On the contrary, if equation (5.3) is not completely integrable, we are sure only about the existence of integral curves passing through a point a. The images of integral curves can be very scattered. Also, it is shown that, in certain conditions, any two points can be connected through an integral curve.

Definition 5.2. Let Σ_{a} be the union of all images of integral curves of equation (5.3) passing through the point $a\in D$ and $\Sigma=\left\{\Sigma_{a}\right\}_{a\in D}$. Then, the pair (Σ,D) will be called *hypersurface attached to the Pfaff equation* (5.3).

In the case of the completely integrable equation, (Σ,D) will be called *holonomic hypersurface* (attached to a Pfaff equation), since it can be organized as union after c of the family of hypersurfaces of the form $F\left(x\right)=c$. In the case in which the equation (5.3) is not completely integrable, (Σ,D) is called *nonholonomic hypersurface.*

Because Σ_{a} contains the images of all integral curves passing through a, it follows that we can get information about Σ_{a} by studying these integral curves, an indispensable method when (Σ,D) is a nonholonomic hypersurface.

To the hypersurface (Σ,D) we can attach the hyperplane

$$H_{a}=\left\{x\in\mathbb{R}^{p}|\sum_{i=1}^{p}\omega_{i}\left(a\right)\left(x^{i}-a^{i}\right)=0\right\},$$

called the *tangent hyperplane* (Σ,D) at a and the *tangent space* (Σ,D) at a,

$$T_{a}\left(\Sigma,D\right)=\left\{v\in\mathbb{R}^{p}|\sum_{i=1}^{n}\omega_{i}\left(a\right)v^{i}=0\right\}.$$

Let ξ be the unit vector field on D, given by $\frac{1}{||\omega||}(\omega_1, \ldots, \omega_n)$, where we considered $||\omega|| = \sqrt{\omega_1^2 + \cdots + \omega_n^2}$. We built the Weingarten application

$$S_a \colon T_a(\Sigma, D) \to T_a(\Sigma, D), \quad S_a(v) = -D_v \xi,$$

which, in the nonholonomic case, is no more symmetric (self-adjoint). It induces the quadratic form

$$\Omega_a(v) = -\frac{1}{2||\omega||} \sum_{i,j=1}^{p} \left(\frac{\partial \omega_i}{\partial x^j} + \frac{\partial \omega_j}{\partial x^i} \right)(a)\, v^i v^j,$$

called *the second fundamental form* of (Σ, D) at a.

Definition 5.3. The hypersurface (Σ, D) is called *(strictly) convex at* a if a is a (strict) extremum point for the linear approximation

$$f_{\omega(a)} \colon D \to \mathbb{R}, \;\; f_{\omega(a)}(x) = \sum_{i=1}^{p} \omega_i(a) \left(x^i - a \right),$$

constrained by the equation $\omega = 0$.

It is obvious that at a point a in which (Σ, D) is (strict) convex, all the integral curves passing through a stay all around a, (strictly) on the same side of the tangent hyperplane H_a.

Theorem 5.5 ([10])**.** *If (Σ, D) is convex at the point a, then the quadratic form Ω_a is (either positively or negatively) semidefined.*

Theorem 5.6 ([10])**.** *If the quadratic form Ω_a is (either positively or negatively) defined then (Σ, D) is strictly convex at a.*

In the case of (Σ, D) holonomic, we find known results again, having however, the advantage that the study of holonomic surface (Σ, D) is equivalent to the study of a family of hypersurfaces $F(x) = c$. More accurately, we can state

Theorem 5.7. *Suppose equation* (5.3) *is completely integrable. Let $F(x) = F(a)$ be the maximal integral manifold of equation* (5.3) *passing through the point a. Then, the holonomic hypersurface (Σ, D) attached to equation* (5.3) *is (strictly) convex at a if and only if the hypersurface $F(x) = F(a)$ is (strictly) convex at a, that is it stays all around a on the same side of the tangent plane at a.*

6. EXTREMUM CONSTRAINED BY A SELECTOR OF CURVES

Theorem 5.3 and Theorem 5.4 show that the method of Lagrange multipliers is still valid when we suppose nonholonomic constraints of equality type, introduced by Definition 5.1. The background of these results is the selection of a family of parametrized curves for each point in D determining the extrema of the function f. We shall see that this idea allows us to unify all types of extrema and, much more, to state a generalization, keeping theorems of Karush-Kuhn-Tucker.

Let D be an open set of $\mathbb{R}^p$. For each point $x \in D$, we denote by $\Gamma(x)$ a family of parametrized curves $\alpha\colon I \to D$, which pases through the point x.

Definition 6.1. Let $\mathcal{P}(\Gamma(x))$ be the power set of $\Gamma(x)$. Any function

$$\hat{\Gamma}\colon D \to \bigcup_{x\in D} \mathcal{P}(\Gamma(x)),\ \hat{\Gamma}(x) \subseteq \Gamma(x),$$

is called *selector of curves* on D. The elements of $\hat{\Gamma}(x)$ are called *admissible curves* at the point x.

Definition 6.2. Let $f\colon D \to \mathbb{R}^p$ be a function and $\hat{\Gamma}(x)$ a selector of curves on D. If

$$f(\alpha(t)) \geq f(a), \forall t \in [t_0, t_0+\varepsilon), \forall \alpha \in \hat{\Gamma}(a), \alpha(t_0) = a,$$

then $a \in D$ is called a *minimum point of f constrained by the selector* $\hat{\Gamma}$.

In the following, we shall reformulate unconstrained/constrained extrema problems using a selector of curves. Also, using various selectors of curves, we shall obtain new types of constrained extrema.

6.1. **Unconstrained extrema.** Let be given the selector of curves

$$\hat{\Gamma}(x) = \Gamma^m(x),\ \forall x \in D,$$

where $\Gamma^m(x)$ is a family of curves considered in §§4.2.

Theorem 4.9 and Corollary 4.2 can be rewritten as

Theorem 6.1. *The point $a \in D$ is a local minimum (maximum) point of a function f if and only if it is a minimum (maximum) point of the function f constrained by the selector $\hat{\Gamma}$.*

Theorem 6.2. *The point $a \in D$ is a strict local extremum point of a continuous function f if and only if it is a strict extremum point of the function f constrained by each prametrized curve of the selector $\hat{\Gamma}$.*

6.2. **Constrained extrema.** Let $g^i\colon D \to \mathbb{R}$, $i = \overline{1,s}$, $s < p$, be C^1-class functions. These functions can be used to create *equality constraints (equations)* or *inequality constraints (inequations)* on points.

- Equalities. The equations $g^i = 0$ introduce the *partial selectors* on D

$$\hat{\Gamma}^i(x) = \begin{cases} \{\alpha \in \Gamma^m(x) \,|\, g^i(\mathrm{Im}\alpha) = 0\}, & \text{for } x \in D \text{ with } g^i(x) = 0 \\ \emptyset, & \text{for } x \in D \text{ with } g^i(x) \neq 0. \end{cases}$$

These produce the general selector

$$\hat{\Gamma}(x) = \bigcap_{i=1}^{s} \hat{\Gamma}^i(x).$$

Now, we can reformulate Theorem 4.6, in the case of constrained equalities.

Theorem 6.3. *Suppose the C^1-class functions g^i satisfy* $\operatorname{rank}\left[\dfrac{\partial g^i}{\partial x^j}(a)\right] = s$ *at a point $a \in D$, and $g^i(a) = 0$, for all $i = \overline{1,s}$. Then a is a minimum (maximum) point of a function $f\colon D \to \mathbb{R}$ constrained by $g^i(x) = 0$, for all $i = \overline{1,s}$ if and only if a is a minimum (maximum) point of the function f constrained by the selector $\hat{\Gamma}$.*

Obviously, we can replace the above selector $\hat{\Gamma}$ by those produced by the partial selectors

$$\hat{\Gamma}^i(x) = \begin{cases} \{\alpha \in \Gamma^m(x) \,|\, g^i(\alpha(t)) = 0,\ \forall t \in [t_0, t_0 + \varepsilon)\}, & \text{if } g^i(x) = 0 \\ \emptyset, & \text{if } g^i(x) \neq 0. \end{cases}$$

• Inequalities. The inequations $g^i(x) \geq 0$ define the partial selectors

$$\hat{\Gamma}^i(x) = \begin{cases} \{\alpha \in \Gamma^m(x) \,|\, g^i(\alpha(t)) \geq 0,\ \forall t \in [t_0, t_0 + \varepsilon)\}, & \text{for } x \in D \text{ with } g^i(x) = 0 \\ \Gamma^m(x), & \text{for } x \in D \text{ with } g^i(x) > 0 \\ \emptyset, & \text{for } x \in D \text{ with } g^i(x) < 0. \end{cases}$$

The general selector is

$$\hat{\Gamma}(x) = \bigcap_{i=1}^{s} \hat{\Gamma}^i(x).$$

Now we can reformulate Theorem 4.6, the case when the constraints are inequalities.

Theorem 6.4. *Suppose the C^1-class functions g^i satisfy* $\operatorname{rank}\left[\dfrac{\partial g^i}{\partial x^j}(a)\right] = s$ *at a point $a \in D$, and $g^i(a) \geq 0$, for all $i = \overline{1,s}$. Then a is a minimum (maximum) point of a function $f\colon D \to \mathbb{R}$ constrained by $g^i(x) \geq 0$, for all $i = \overline{1,s}$ if and only if a is a minimum (maximum) point of the function f constrained by the above selector $\hat{\Gamma}$.*

6.3. **Extrema constrained by a Pfaff system.** Consider the Pfaff system (5.1) where ω_i^j are C^1-class functions such that

$$\operatorname{rank}\left[\omega_i^j(x)\right] = s,\ \forall x \in D.$$

In the following, we shall denote by $\Gamma(x)$ the family of all regular parametrized curves of C^2-class passing through x.

• Pfaff equality constraints. The Pfaff equations generate the partial selectors

$$\hat{\Gamma}^j(x) = \{\alpha \in \Gamma(x) \,|\, \alpha \text{ is an integral curve of the Pfaff equation } \omega^j(x) = 0\},$$

which produce the general selector (associated to the Pfaff system)

$$\hat{\Gamma}(x) = \bigcap_{j=1}^{s} \hat{\Gamma}^j(x).$$

Using this approach, we recover the notion of constrained extremum by a Pfaff system (Definition 5.1).

• Pfaff inequality constraints. The primitives of the Pfaff forms ω^j define the partial selectors

$$\hat{\Gamma}^j(x) = \{\alpha \in \Gamma(x) \mid \int_{t_0}^{t} \left\langle \omega^j(\alpha(u)), \alpha'(u) \right\rangle du \geq 0, \forall t \in [t_0, t_0 + \varepsilon)\},$$

where $\alpha(t_0) = x$. Using this approach, we obtain a new selector associated to the Pfaff system

$$\hat{\Gamma}(x) = \bigcap_{j=1}^{s} \hat{\Gamma}^j(x).$$

Definition 6.3. $a \in D$ is a *minimum (maximum) point* of the function $f\colon D \to \mathbb{R}$ *constrained by* $\omega^j \geq 0$, $j = \overline{1, s}$ if a is a minimum (maximum) point of the function f constrained by the previous selector.

Theorem 6.5. *Let $f\colon D \to \mathbb{R}$ a C^1-class function and $a \in D$ an extremum point of the function f constrained by $\omega^j \geq 0$, $j = \overline{1, s}$. Then there exist $\lambda_1 \geq 0, \ldots, \lambda_s \geq 0$ such that*

$$df(a) = \sum_{j=1}^{s} \lambda_j \omega^j(a).$$

Theorem 6.6. *Let $f\colon D \to \mathbb{R}$ a C^2-class function and $a \in D$. Suppose there exist $\lambda_1 \geq 0, \ldots, \lambda_s \geq 0$ such that:*

1) $df(a) = \sum_{j=1}^{s} \lambda_j \omega^j(a)$;

2) *the restriction of the quadratic form*

$$d^2 f(a) - \frac{1}{2} \sum_{k=1}^{s} \lambda_k \sum_{i,j=1}^{p} \left(\frac{\partial \omega_i^k}{\partial x^j} + \frac{\partial \omega_j^k}{\partial x^i} \right)(a)\, dx^i dx^j$$

to the space

$$\sum_{i=1}^{p} \omega_i^k(a)\, dx^i = 0, \;\; k \in J' = \{j = \overline{1, s} | \lambda_j > 0\}$$

is positive definite.

Then a is a minimum point of f constrained by $\omega^j \geq 0$, $j = \overline{1, s}$.

In the case of this selector of *Pfaff inequality* type, if $\Gamma(x)$ becomes the family of all parametrized curves of C^2-class passing through x having tangent at this point, then Theorem 6.4 can be stated as

Theorem 6.7. *Suppose the C^1-class functions g^i satisfy* rank$\left[\frac{\partial g^i}{\partial x^j}(a)\right] = s$ *at a point $a \in D$. Then a is a minimum (maximum) point of a function $f\colon D \to \mathbb{R}$ constrained by $g^i(x) \geq g^i(a)$, $\forall i = \overline{1, s}$ if and only if a is a minimum (maximum) point of the function f constrained by $dg^i \geq 0$, $\forall i = \overline{1, s}$.*

In the case of classical constraints, the system of the C^1-class functions g^j, $j=\overline{1,s}$, induces two types of constraints: *point constraints*, defined by $g^j(x)=0$ or $g^j(x)\geq 0$, $j=\overline{1,s}$, and *velocity constraints*, defined by the subspace $dg^j(x)=0$, $j=\overline{1,s}$ of the tangent space T_xD; the latter constraints does contribute also in establishing the nature of critical points. The points described by $g^j(x)\geq 0$ split in two types: *interior* points and *boundary* points.

The constraints on points and on velocities, which are correlated by the functions g^j, induce a selector of curves, whose contribution arises from the quality of a point to be interior or boundary point.

In the case of extremum points with Pfaff equalities or inequalities, it appears only the velocity constraints. Here, each point could be an extremum point.

These remarks allow us to introduce a more general type of extremum, in which the constraints on points and the constraints on velocities are not necessarily correlated, even if Kuhn-Tucker conditions hold.

7. EXTREMUM WITH POINT AND/OR VELOCITY CONSTRAINTS

Let $\omega(x)=\sum_{k=1}^{p}\omega_k(x)dx^k$ be a C^1-class Pfaff form with $\operatorname{rank}[\omega_k(x)]=1$. Let S and bS be two arbitrary disjoint sets. The points of S will be called *interior points* and the points of bS will be called *boundary points*. In this context, we say that the set $N=S\cup bS$ represents the *constraints of inequality type*. The pair (ω,N) determines an *inequality selector* of curves:

$$\hat{\Gamma}(x_0)=\begin{cases}\Gamma(x_0), & \text{if } x_0\in S\\ \left\{\alpha\in\Gamma(x_0)\mid \int_{t_0}^{t}\langle\omega(\alpha(u)),\alpha'(u)\rangle\geq 0,\ t\in[t_0,t_0+\epsilon)\right\}, & \text{if } x_0\in bS\\ \emptyset, & \text{if } x_0\in D\setminus N.\end{cases}$$

If $T\subseteq D$ is an arbitrary subset, then the pair (ω,T) defines an *equality selector*

$$\hat{\Gamma}_0(x_0)=\begin{cases}\left\{\alpha\in\Gamma(x_0)\mid \int_{t_0}^{t}\langle\omega(\alpha(u)),\alpha'(u)\rangle\geq 0,\ t\in[t_0,t_0+\epsilon)\right\}, & \text{if } x_0\in T\\ \emptyset, & \text{if } x_0\in D\setminus T.\end{cases}$$

In this sense, we say that the set T represents the *constraints of equality type*.

We underline that any equality selector can be expressed by inequality selectors. More accurately, for an arbitrary subset $T\subseteq D$, let us consider the inequality selectors $\hat{\Gamma}_+$ and $\hat{\Gamma}_-$ defined by $(\omega,\hat{S}\cup bS)$ respectively $(-\omega,\hat{S}\cup bS)$, where $S=\emptyset$ and $bS=T$. Then $\hat{\Gamma}_0(x)=\hat{\Gamma}_+(x)\cap\hat{\Gamma}_-(x),\forall x\in D$. Also, the inequality selector $\hat{\Gamma}_0$ can be deactivated considering $T=D$.

Having in mind the previous idea, we can introduce inequality selectors $\hat{\Gamma}^j$ defined by the pairs $(\omega^j;S_j\cap bS_j)$, $j=\overline{1,s}$, and equality selectors $\hat{\Gamma}_0^i$ defined by the pairs $\left(\eta^i, T_i\right), i = \overline{1,m}$. Then we build

$$N = \bigcap_{j=1}^{s} (S_j \cap bS_j), \quad bS = \{x \in N | \exists j = \overline{1,s},\ x \in bS_j\}$$

$$S = N \setminus bS, \quad T = \bigcap_{i=1}^{m} T_i,\ M = N \cap T.$$

In case of absence of equality constraints, we have $M = N$ and in case of absence of inequality constraint we have $M = T$.

If ω means the Pfaff form system (ω^j) and η means the Pfaff form system $\left(\eta^i\right)$, then the triple (ω, η, M) defines the curve selector

$$\hat{\Gamma}(x) = \Big(\bigcap_{j=1}^{s} \Gamma^j(x)\Big) \cap \Big(\bigcap_{i=1}^{m} \Gamma_0^i(x)\Big),\ \forall x \in D.$$

Definition 7.1. Let $f\colon D \to \mathbb{R}$ be a real function. We say that $a \in M$ is a *minimum point of f constrained by* (ω, η, M), if a is a minimum point constrained by the selector $\hat{\Gamma}$. We say that ω and η represent the *velocity constraints* and M represents the *point constraints*. The triple (ω, η, M) is called *system of point/velocity constraints* or *system of constraints.*

A convenient selection of the objects ω, η, M and a family $\Gamma(x)$ leads to all the types of extremum mentioned in the previous section.

- *Case of free extremum*: $S_j = D$, $bS_j = \emptyset$, $\omega^j =$ arbitrary (without equation constraints), $\Gamma(x) = \Gamma^m(x)$.
- *Case of classical equality constraints*: $T_i = \{x \in D | g^i(x) = 0\}$, $\eta^i = dg^i$ (without inequation constraints), $\Gamma(x) = \Gamma^m(x)$.
- *Case of classical inequality constraints*: $S_j = \{x \in D | g^j(x) > 0\}$, $bS_j = \{x \in D | g^j(x) = 0\}$, $\omega^j = dg^j$ (without equation constraints), $\Gamma(x) = \Gamma^m(x)$.
- *Case of Pfaff equality constraints*: $T_i = D$, $\eta^i = \omega^i$ (without inequation constraints), $\Gamma(x)$ being the family of all regular parametrized curves of C^2-class passing through x.
- *Case of Pfaff inequality constraints*: $S_j = \emptyset$, $bS_j = D$ (without equation constraints), $\Gamma(x)$ being the family of all regular parametrized curves of C^2-class passing through x.

As we know, in the classical constrained extrema theory, the conditions of Karush-Kuhn-Tucker type are necessary conditions *only if* a particular proviso is satisfied. That proviso, called the *constraint qualification* or *regularity condition*, imposes a certain restriction on the constraint functions of a nonlinear programming problem.

In the following, we introduce two sets of constraint qualification conditions which can ensure theorems of Karush-Kuhn-Tucker type.

Definition 7.2. Let (ω, η, M) be a system of point/velocity constraints. Let $a \in M$ and $B(a) = \{j | a \in bS_j\} \subseteq \{1, \ldots, s\}$. The system (ω, η) is called *regular* at a if $\operatorname{rank}\left(\omega^j(a), \eta^i(a)\right) = m + \operatorname{card} B(a)$, where $j \in B(a)$, $i = \overline{1, m}$.

A more general constraint qualification condition is given in

Definition 7.3. We say that (ω, η, M) satisfies *the Kuhn-Tucker regularity condition at* $a \in M$ if from $a \in bS \cap T$ it follows that for any vector $v \neq 0$ with $\left\langle \omega^j(a), v \right\rangle \geq 0$, $\forall j \in B(a) = \{j | a \in bS_j\}$ and $\left\langle \eta^i(a), v \right\rangle = 0$, for all $i = \overline{1, m}$, it exists a parametrized curve $\alpha \in \hat{\Gamma}(a)$, $\alpha(t_0) = a$, such that $\alpha'(t_0) = v$.

In what follows [15], suppose $\Gamma(x)$ is the family of all regular parametrized curves of C^2-class passing through the point x.

Theorem 7.1 (Karush-Kuhn-Tucker necessary conditions)**.** *Let $f\colon D \to \mathbb{R}$ be a C^1-class function. Suppose the constraints triple (ω, η, M) satisfies one of the two constraint qualification conditions at $a \in M$. If a is a minimum point of f constrained by (ω, η, M), then there exist $\lambda_j \geq 0$, $j = \overline{1, s}$ and $\mu_i \in \mathbb{R}$ such that*

$$df(a) = \sum_{j=1}^{s} \lambda_j \omega^j(a) + \sum_{i=1}^{m} \mu_i \eta^i(a).$$

Moreover, if $\lambda_j > 0$, then $a \in bS_j$.

Remark that in the case of the first constraint qualification condition, which is more powerful, the multipliers λ and μ from the above theorem are unique.

Theorem 7.2 (KKT second-order sufficient conditions)**.** *Let us consider the constraints (ω, η, M). Let $f\colon D \to \mathbb{R}$ be a C^2-class function and $a \in D$. Suppose that*

i) KKT conditions hold, that is there exist $\lambda_j \geq 0$, $j = \overline{1, s}$, $\mu_i \in \mathbb{R}$ such that

$$df(a) = \sum_{j=1}^{s} \lambda_j \omega^j(a) + \sum_{i=1}^{m} \mu_i \eta^i(a),$$

and, if $\lambda_j > 0$, then $a \in bS_j$;

ii) the restriction of the quadratic form

$$d^2 f(a) - \frac{1}{2} \sum_{k=1}^{s} \lambda_k \sum_{i,j=1}^{p} \left(\frac{\partial \omega_i^k}{\partial x^j} + \frac{\partial \omega_j^k}{\partial x^i} \right)(a)\, dx^i dx^j$$
$$-\frac{1}{2} \sum_{k=1}^{m} \mu_k \sum_{i,j=1}^{p} \left(\frac{\partial \omega_i^k}{\partial x^j} + \frac{\partial \omega_j^k}{\partial x^i} \right)(a)\, dx^i dx^j$$

to the velocity subspace

$$\begin{cases} \sum_{i=1}^{p} \omega_i^j(a)\,dx^i = 0,\ j \in J^1(a) = \{j \in B(a)\,|\lambda_j > 0\} \\ \sum_{k=1}^{p} \eta_k^i(a)\,dx^i = 0,\ i = \overline{1,m} \end{cases} \tag{7.1}$$

is positive definite.

Then, a is a minimum point of f constrained by (ω, η, M).

Theorem 7.3 (KKT SECOND-ORDER NECESSARY CONDITIONS)**.** *Let $f\colon D \to \mathbb{R}$ be a C^2-class function and (ω, η, M) a system of restrictions on D. Let $a \in M$ be a point at which is satisfied one of constraint qualification conditions in Definitions* 7.2 *and* 7.3*. Suppose a is a minimum point of f constrained by (ω, η, M). Then the restriction of the quadratic form in the previous Theorem to subspace* (7.1) *is positive semidefinite.*

8. SADDLE POINT THEORY IN NOHOLONOMIC OPTIMIZATION

Let D be an open set in $\mathbb{R}^n$ and $\omega(x) = \sum_{j=1}^{n} \omega_j(x)dx^j$ be a C^0-class Pfaff form. Let $\Gamma^1(D)$ the family of all the piecewise C^1-class parametrized curves in D. Let $\Gamma(D) \subseteq \Gamma^1(D)$. Each parametrized curve $\alpha \in \Gamma(D)$ generates a family $\{g_\alpha\}$ of functions,

$$g_\alpha\colon \operatorname{dom}\alpha \to \mathbb{R}, \quad g'_\alpha(t) = \left\langle \omega(\alpha(t)), \alpha'(t) \right\rangle,\ t \in \operatorname{dom}\alpha,$$

called *primitives of ω along α*. On the other hand, each curve α defines an equivalence class $\tilde{a} = \{\beta = \alpha \circ \varphi | \varphi\colon J \to I\}$, where φ is a diffeomorphism.

Definition 8.1. Let g be a function which associates to each parameterized curve $\alpha \in \Gamma(D)$ a function g_α from the family $\{g_\alpha\}$. If $g_\beta = g_\alpha \circ \varphi$, for any equivalent curves α and β, then the function g is called *system of ω-primitives* on $\Gamma(D)$). In other words, the function $g\colon \Gamma(D) \to \Gamma^1(\mathbb{R})$ is a system of ω-primitives (on $\Gamma(D)$) if g is a Γ-function in the sense of Definition 3.1, which satisfies Axiom (A_1) (see §3).

Example 8.1. Suppose the Pfaff form ω is exact, that is $\omega = dG$. The function $g\colon \Gamma(D) \to \Gamma^1(\mathbb{R})$ with $g_\alpha = G \circ \alpha$ is a system of ω-primitives.

Example 8.2. Let $\Gamma_s^1(D)$ be the family of all the simple parametrized curves $\alpha \in \Gamma^1(D)$. For each curve $\tilde{a}$ with $\alpha \in \Gamma_s^1(D)$, we choose a point $x_0 = \alpha(t_0) \in \operatorname{Im}(\alpha)$ and for any $\beta \in \tilde{a}$ we consider the function

$$g_\beta(t) = \int_{t_0}^{t} \left\langle \omega(\beta(u)), \beta'(u) \right\rangle du + c_{\tilde{a}}, \quad c_{\tilde{a}} \in \mathbb{R}.$$

We obtain a Γ-function $g\colon \Gamma_s^1(D) \to \Gamma^1(\mathbb{R})$, $\alpha \to g_\alpha$, that is a system of ω-primitives (on $\Gamma_s^1(D)$).

Theorem 8.1. *The Pfaff form* ω *is exact iff the* Γ*-function* $g\colon \Gamma_s^1(D) \to \Gamma^1(\mathbb{R})$ *given above can be prolonged to* $\Gamma^1(D)$ *satisfying Axiom* (A_2)*, see* §3.

Proof. Suppose g can be prolonged to the Γ-function $\overline{g}\colon \Gamma^1(D) \to \Gamma^1(\mathbb{R})$ which satisfies (A_2). Applying Theorem 3.3 it follows that there exists a continuous function $G\colon D \to \mathbb{R}$ having the first order partial derivatives, such that $G \circ a = g_\alpha$ for any $\alpha \in \Gamma^1(D)$. It results $\dfrac{\partial G}{\partial x^j} = \omega^j$, $j = \overline{1,p}$; thus G is a C^1-class function and $dG = \omega$. The converse is obvious. □

From Theorem 3.4 it follows

Corollary 8.1. *A prolongation* $\overline{g}\colon \Gamma^1(D) \to \Gamma^1(\mathbb{R})$ *of the* Γ*-function* g *satisfies* (A_2) *if and only if satisfies* (A_3).

For a Pfaff form ω and for a system of ω-primitives we can associate the set $M = S \cup bS$, where

$$bS = \{x_0 \in D | \exists \alpha \in \Gamma^1(x_0),\ \alpha(t_0) = x_0,\ g_\alpha(t_0) = 0\}$$
$$S = \{x_0 \in D \setminus bS | \exists \alpha \in \Gamma^1(x_0),\ \alpha(t_0) = x_0,\ g_\alpha(t_0) > 0\}.$$

The pair (ω, M) induces a selector $\hat{\Gamma}$ of curves.

By analogy, for each index $i = \overline{1,s}$, we consider the pair (ω^i, M_i), where $\omega^i(x) = \sum_{j=1}^{p} \omega_j^i(x) dx^j$ are C^0-class Pfaff forms, and $M_i = S_i \cup bS_i$ are defined using the system of ω^i-primitives g^i. Let $M = \bigcap_{i=1}^{s} M_i$, $bS = \{x_0 \in D | \exists i \in \overline{1,s},\ x_0 \in S_i\}$ and $S = M \setminus bS$. The pair (ω, M) with $\omega = (\omega^i)$ induces a selector of curves via the system of primitives $g = (g^i)$.

Let $f\colon D \to \mathbb{R}$ be a C^0-class function. Using these ingredients, we define the *Lagrange function*

$$L_\alpha(t, \lambda) = f(\alpha(t)) - \sum_{i=1}^{s} \lambda_i g_\alpha^i(t),\ \forall t \in I,\ \lambda = (\lambda_i),\ \lambda_i \geq 0.$$

This function is defined along each curve $\alpha\colon I \to D$, using the restriction of function f to α and the primitives of the Pfaff forms ω^i along α. In this manner, we obtain a family of *Lagrange functions*, which will satisfy conditions of saddle point type.

Definition 8.2. Let $x_0 \in D$, and $\lambda^0 = (\lambda_i^0)$ with $\lambda_i^0 \geq 0$, $i = \overline{1,s}$. The point (x_0, λ^0) is called *saddle point* for the family of all Lagrange functions L_α if

a) $L_\alpha(t_0, \lambda^0) \leq L_\alpha(t, \lambda^0)$, $\forall \alpha \in \Gamma^1(x_0)$, $\alpha(t_0) = x_0$, $\forall t \in [t_0, t_0 + \epsilon)$;

b) there exists $\alpha \in \Gamma^1(x_0)$ with $\alpha(t_0) = x_0$ such that

$$L_\alpha(t_0, \lambda^0) \geq L_\alpha(t_0, \lambda),\ \forall \lambda = (\lambda_i) \geq 0.$$

Theorem 8.2 ([17]). *Let $x_0 \in D$. If there exists $\lambda^0 = (\lambda_i^0) \geq 0$, $i = \overline{1,s}$ such that (x_0, λ^0) is a saddle point for the family for all Lagrange functions L_α, then x_0 is a minimum point of the function f constrained by (ω, M).*

Suppose now that each Pfaff form ω^i is exact, that is $\omega^i = dG^i$ with $G^i : D \to \mathbb{R}$ a C^1-class function. In this case, each G^i induces the system of ω^i-primitives g^i, where $g^i_\alpha = G^i \circ \alpha$, for all $\alpha \in \Gamma^1(D)$. It is clear that $g^i_\alpha(t_0) = g^i_\beta(u_0)$, for all $\alpha, \beta \in \Gamma^1(D)$ with $\alpha(t_0) = \beta(t_0)$. In this manner, $S_i = \{x_0 | G^i(x_0) > 0\}$, $bS_i = \{x_0 | G^i(x_0) = 0\}$, and $M = \{x | G^i(x) \geq 0\}$, for all $i = \overline{1,s}$. We can define the *Lagrange function* associated to the function f

$$L(x, \lambda) = f(x) - \sum_{i=1}^{s} \lambda_i G^i(x), \ \lambda_i \geq 0.$$

It follows that $L_\alpha(t, \lambda) = L(\alpha(t), \lambda)$, for all $\alpha \in \Gamma^1(D)$ and $t \in \operatorname{dom} \alpha$.

Then $x_0 \in D$ is a saddle point for the family of all Lagrange functions L_α if and only if x_0 is a saddle point for the Lagrange function $L(x, \lambda)$, that is

a) $L(x_0, \lambda^0) \leq L(x, \lambda^0)$, $\forall x \in M$;

b) $L(x_0, \lambda^0) \geq L(x_0, \lambda)$, $\forall \lambda = (\lambda_i) \geq 0$.

Finally, Theorem 8.2 can be stated in the classical form.

Theorem 8.3 ([17]). *Let $x_0 \in D$. If there exists $\lambda^0 = (\lambda_i^0) \geq 0$, $i = \overline{1,s}$, such that (x_0, λ^0) is a saddle point for the Lagrange functions L, then x_0 is a minimum point of the function f constrained by $G^i \geq 0$, $i = \overline{1,s}$.*

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University "Politehnica" of Bucharest
Faculty of Applied Sciences
Splaiul Independenţei, No. 313*,* 060042 *Bucharest, Romania*
E-mail address: `oltin.horia@yahoo.com`

University "Politehnica" of Bucharest
Faculty of Applied Sciences
Splaiul Independenţei, No. 313*,* 060042 *Bucharest, Romania*
E-mail address: `mihai@mathem.pub.ro`