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# A UNIQUENESS RESULT ON MEROMORPHIC FUNCTIONS SHARING TWO SETS

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*Dedicated to the memory of Professor B. K. Lahiri*

Abstract. In the paper we employ the notion of weighted sharing of sets to deal with the well known question of Gross and obtain a uniqueness result on meromorphic functions sharing two sets which will improve an earlier result of Lahiri [14].

## 1. INTRODUCTION, DEFINITIONS AND MAIN RESULT

In this paper by meromorphic functions we will always mean meromorphic functions in the complex plane. We shall use the standard notations of value distribution theory:

$$T(r,f),\ \ m(r,f),\ \ N(r,\infty;f),\ \ \overline{N}(r,\infty;f),\ldots$$

(see [9]). It will be convenient to let $E$ denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For any non-constant meromorphic function $h(z)$ we denote by $S(r,h)$ any quantity satisfying

$$S(r,h)=o(T(r,h)) \qquad (r\longrightarrow\infty, r\notin E).$$

For any constant $a$, we define

$$\Theta(a;f)=1-\limsup_{r\longrightarrow\infty}\frac{\overline{N}(r,a;f)}{T(r,f)}.$$

If for some $a\in\mathbb{C}\cup\{\infty\}$, $f$ and $g$ have the same set of $a$-points with same multiplicities then we say that $f$ and $g$ share the value $a$ CM (counting multiplicities). If we do not take the multiplicities into account, $f$ and $g$ are said to share the value $a$ IM (ignoring multiplicities).

Let $S$ be a set of distinct elements of $\mathbb{C}\cup\{\infty\}$ and $E_f(S)=\bigcup_{a\in S}\{z:f(z)-a=0\}$, where each zero is counted according to its multiplicity. If we do not count the multiplicity, the set $E_f(S)$ is denoted by $\overline{E}_f(S)$. If $E_f(S)=E_g(S)$ we say that $f$ and $g$ share the set $S$ CM. On the other hand, if $\overline{E}_f(S)=\overline{E}_g(S)$, we say that $f$ and $g$ share the set $S$ IM.

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The following question is due to F. Gross [7] and plays an important role in the literature of meromorphic functions that share sets instead of values.

**Question A** *Can one find two finite sets* $S_j$ $(j = 1, 2)$ *such that any two non-constant entire functions* $f$ *and* $g$ *satisfying* $E_f(S_j) = E_g(S_j)$ *for* $j = 1, 2$ *must be identical?*

In [7] Gross wrote *If the answer of* Question A *is affirmative it would be interesting to know how large both sets would have to be?*

Corresponding to the Gross' question the following question [18] is a natural one.

**Question B** *Can one find two finite sets* $S_j$ $(j = 1, 2)$ *such that any two non-constant meromorphic functions* $f$ *and* $g$ *satisfying* $E_f(S_j) = E_g(S_j)$ *for* $j = 1, 2$ *must be identical?*

The shared set problems relative to a meromorphic function has been studied by many authors {see [1]-[6], [8], [10], [14], [16]-[17], [18]-[26]}.

In [5] Fang and Lahiri exhibited the following range set $S$ with smaller cardinalities than that obtained by the previous authors, imposing some restrictions on the poles of $f$ and $g$.

**Theorem A.** [5] *Let* $S = \{z : z^n + az^{n-1} + b = 0\}$, *where* $n(\geq 7)$ *be an integer and* $a$ *and* $b$ *be two nonzero constants such that* $z^n + az^{n-1} + b = 0$ *has no multiple root. If* $f$ *and* $g$ *be two non-constant meromorphic functions having no simple poles such that* $E_f(S) = E_g(S)$ *and* $E_f(\{\infty\}) = E_g(\{\infty\})$ *then* $f \equiv g$.

In 2001 an idea of gradation of sharing of values and sets known as weighted sharing has been introduced in {[12], [13]} which measure how close a shared value is to being shared CM or to being shared IM. Below we are giving the notion.

**Definition 1.1.** [12, 13] *Let* $k$ *be a nonnegative integer or infinity. For* $a \in \mathbb{C} \cup \{\infty\}$ *we denote by* $E_k(a; f)$ *the set of all* $a$*-points of* $f$, *where an* $a$*-point of multiplicity* $m$ *is counted* $m$ *times if* $m \leq k$ *and* $k + 1$ *times if* $m > k$. *If* $E_k(a; f) = E_k(a; g)$, *we say that* $f, g$ *share the value* $a$ *with weight* $k$.

The definition implies that if $f$, $g$ share a value $a$ with weight $k$ then $z_0$ is an $a$-point of $f$ with multiplicity $m$ $(\leq k)$ if and only if it is an $a$-point of $g$ with multiplicity $m$ $(\leq k)$ and $z_0$ is an $a$-point of $f$ with multiplicity $m$ $(> k)$ if and only if it is an $a$-point of $g$ with multiplicity $n$ $(> k)$, where $m$ is not necessarily equal to $n$.

We write $f$, $g$ share $(a, k)$ to mean that $f$, $g$ share the value $a$ with weight $k$. Clearly if $f$, $g$ share $(a, k)$ then $f, g$ share $(a, p)$ for any integer $p$, $0 \leq p < k$. Also we note that $f$, $g$ share a value $a$ IM or CM if and only if $f$, $g$ share $(a, 0)$ or $(a, \infty)$ respectively.

**Definition 1.2.** [12] *Let* $S$ *be a set of distinct elements of* $\mathbb{C} \cup \{\infty\}$ *and* $k$ *be a nonnegative integer or* $\infty$. *We denote by* $E_f(S, k)$ *the set* $\bigcup_{a \in S} E_k(a; f)$.

With the notion of weighted sharing of sets improving *Theorem A*, Lahiri [14] proved the following theorem.

**Theorem B.** [14] *Let $S$ be defined as in* Theorem A *and $n(\geq 7)$ be an integer. If for two non-constant meromorphic functions $f$ and $g$, $\Theta(\infty;f)+\Theta(\infty;g)>1$, $E_f(S,2)=E_g(S,2)$ and $E_f(\{\infty\},\infty)=E_g(\{\infty\},\infty)$ then $f\equiv g$.*

Suppose that the polynomial (see [25]) $P(w)$ is defined by

$$P(w)=aw^n-n(n-1)w^2+2n(n-2)bw-(n-1)(n-2)b^2 \qquad (1.1)$$

where $n\geq 3$ is an integer and $a$ and $b$ are two nonzero complex numbers satisfying $ab^{n-2}\neq 2$. In fact we consider the following rational function

$$R(w)=\frac{aw^n}{n(n-1)(w-\alpha_1)(w-\alpha_2)}, \qquad (1.2)$$

where $\alpha_1$ and $\alpha_2$ are two distinct roots of

$$n(n-1)w^2-2n(n-2)bw+(n-1)(n-2)b^2=0.$$

We have from (1.2)

$$R^{'}(w)=\frac{(n-2)aw^{n-1}\,(w-b)^2}{n(n-1)\,(w-\alpha_1)^2\,(w-\alpha_2)^2}. \qquad (1.3)$$

From (1.3) we know that $w=0$ is a root with multiplicity $n$ of the equation $R(w)=0$ and $w=b$ is a root with multiplicity 3 of the equation $R(w)-c=0$, where $c=\dfrac{ab^{n-2}}{2}$. Then

$$R(w)-c=\frac{a(w-b)^3\,Q_{n-3}(w)}{n(n-1)(w-\alpha_1)(w-\alpha_2)}, \qquad (1.4)$$

where $Q_{n-3}(w)$ is a polynomial of degree $n-3$.
Moreover from (1.1) and (1.2) we have

$$R(w)-1=\frac{P(w)}{n(n-1)(w-\alpha_1)(w-\alpha_2)}. \qquad (1.5)$$

Noting that $c=\frac{ab^{n-2}}{2}\neq 1$, from (1.3) and (1.5) we have

$$P(w)=aw^n-n(n-1)w^2+2n(n-2)bw-(n-1)(n-2)b^2$$

has only simple zeros.

In the paper our prime concern is to improve Theorem B. In fact we will show that in our result, the cardinality of the range set can be lowered further at the expense of allowing the replacement of the set $S$ in Theorem B by a new range set. The following theorem is the main result of the paper.

**Theorem 1.1.** *Let $S=\{w\mid P(w)=0\}$, where $P(w)$ is given by* (1.1) *and $n\geq 6$. Suppose that $f$ and $g$ are two non-constant meromorphic functions satisfying $E_f(S,2)=E_g(S,2)$ and $E_f(\{\infty\},\infty)=E_g(\{\infty\},\infty)$ and $\Theta_f+\Theta_g+\min\{\Theta(b;f),\Theta(b;g)\}>8-n$, where $\Theta_f=2\,\Theta(0;f)+\,\Theta(\infty;f)+\,\Theta(b;f)$ and $\Theta_g$ is defined similarly. Then $f\equiv g$.*

It is assumed that the readers are familiar with the standard definitions and notations of value distribution theory as those are available in [9]. We are still going to explain the following two notations as these are used in the paper.

**Definition 1.3.** [11] *For* $a \in \mathbb{C}\cup\{\infty\}$ *we denote by* $N(r,a;f\mid=1)$ *the counting function of simple* $a$*-points of* $f$*. For a positive integer* $m$ *we denote by* $N(r,a;f\mid\leq m)$ $(N(r,a;f\mid\geq m))$ *the counting function of those* $a$*-points of* $f$ *whose multiplicities are not greater (not less) than* $m$ *where each* $a$*-point is counted according to its multiplicity.*

$\overline{N}(r,a;f\mid\leq m)$ $(\overline{N}(r,a;f\mid\geq m))$ *are defined similarly, where in counting the* $a$*-points of* $f$ *we ignore the multiplicities.*

*Also* $N(r,a;f\mid< m)$, $N(r,a;f\mid> m)$, $\overline{N}(r,a;f\mid< m)$ *and* $\overline{N}(r,a;f\mid> m)$ *are defined analogously.*

**Definition 1.4.** *Let* $f$ *and* $g$ *be two non-constant meromorphic functions such that* $f$ *and* $g$ *share* $(1,0)$*. Let* $z_0$ *be a* $1$*-point of* $f$ *with multiplicity* $p$*, a* $1$*-point of* $g$ *with multiplicity* $q$*. We denote by* $\overline{N}_L(r,1;f)$ *the reduced counting function of those* $1$*-points of* $f$ *and* $g$ *where* $p>q$*, by* $N^{1)}_E(r,1;f)$ *the counting function of those* $1$*-points of* $f$ *and* $g$*, where* $p=q=1$*, by* $\overline{N}^{(2}_E(r,1;f)$ *the reduced counting function of those* $1$*-points of* $f$ *and* $g$ *where* $p=q\geq 2$*. In the same way we can define* $\overline{N}_L(r,1;g)$, $N^{1)}_E(r,1;g)$, $\overline{N}^{(2}_E(r,1;g)$*. In a similar manner we can define* $\overline{N}_L(r,a;f)$ *and* $\overline{N}_L(r,a;g)$ *for* $a\in\mathbb{C}\cup\{\infty\}$*. When* $f$ *and* $g$ *share* $(1,m)$*,* $m\geq 1$ *then* $N^{1)}_E(r,1;f)=N(r,1;f\mid=1)$*.*

**Definition 1.5.** [12, 13] *Let* $f$*,* $g$ *share* $(a,0)$*. We denote by* $\overline{N}_*(r,a;f,g)$ *the reduced counting function of those* $a$*-points of* $f$ *whose multiplicities differ from the multiplicities of the corresponding* $a$*-points of* $g$*.*

Clearly $\overline{N}_*(r,a;f,g)=\overline{N}_*(r,a;g,f)$ and $\overline{N}_*(r,a;f,g)=\overline{N}_L(r,a;f)+\overline{N}_L(r,a;g)$.

## 2. LEMMAS

In this section we present some lemmas which will be needed in the sequel. Let $F$ and $G$ be two non-constant meromorphic functions defined in $\mathbb{C}$. Henceforth we shall denote by $H$ the following function.

$$H=\left(\frac{F''}{F'}-\frac{2F'}{F-1}\right)-\left(\frac{G''}{G'}-\frac{2G'}{G-1}\right).$$

Let $f$ and $g$ be two non-constant meromorphic functions and

$$F=R(f),\qquad G=R(g),\tag{2.1}$$

where $R(w)$ is given by (1.2). From (1.2) and (2.1) it is clear that

$$T(r,f)=\frac{1}{n}T(r,F)+S(r,f),\quad T(r,g)=\frac{1}{n}T(r,G)+S(r,g).\tag{2.2}$$

**Lemma 2.1** (Lemma 2.18 in [2])**.** *Let $F$, $G$ be given by* (2.1) *and $H \not\equiv 0$. If $F$, $G$ share $(1,m)$ and $f$, $g$ share $(\infty,k)$. Then*

$$\begin{aligned} N_E^{1)}(r,1;F) \le{}& \overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+\overline{N}(r,0;f)+\overline{N}(r,b;f)+\overline{N}_*(r,\infty;f,g)\\ &+\overline{N}(r,0;g)+\overline{N}(r.b;g)+\overline{N}_0(r,0;f')+\overline{N}_0(r,0;g'), \end{aligned}$$

*where $\overline{N}_0(r,0;f')$ denotes the reduced counting function corresponding to the zeros of $f'$ which are not the zeros of $f(f-b)$ and $F-1$. Of course $\overline{N}_0(r,0;g')$ is defined similarly.*

**Lemma 2.2.** [15] *If $N(r,0;f^{(k)} \mid f \neq 0)$ denotes the counting function of those zeros of $f^{(k)}$ which are not zeros of $f$, where a zero of $f^{(k)}$ is counted according to its multiplicity then*

$$N(r,0;f^{(k)} \mid f \neq 0) \le k\overline{N}(r,\infty;f)+N(r,0;f \mid< k)+k\overline{N}(r,0;f \mid\ge k)+S(r,f).$$

**Lemma 2.3.** [19] *Let $f$ be a non-constant meromorphic function and $P(f)=a_0+a_1f+a_2f^2+\cdots+a_nf^n$, where $a_0,a_1,a_2\ldots,a_n$ are constants and $a_n \neq 0$. Then $T(r,P(f))=nT(r,f)+O(1)$.*

**Lemma 2.4.** *Let $f$ and $g$ be two non-constant meromorphic functions sharing* $(\infty,0)$ *and suppose $\alpha_1$ and $\alpha_2$ are two distinct roots of the equation*

$$n(n-1)w^2-2n(n-2)bw+(n-1)(n-2)b^2=0.$$

*Then*

$$\frac{f^n}{(f-\alpha_1)(f-\alpha_2)}\,\frac{g^n}{(g-\alpha_1)(g-\alpha_2)} \not\equiv \frac{n^2(n-1)^2}{a^2},$$

*where $n$ $(\ge 3)$ is an integer.*

*Proof.* We omit the proof since the proof can be found out in the proof of Theorem 3 [8] (second half of page 26). □

**Lemma 2.5.** *Let $F$, $G$ be given by* (2.1)*, where $n \ge 6$ is an integer. If $F \equiv G$, then $f \equiv g$.*

*Proof.* We omit the proof since the proof can be found out in [8] (page 27). □

**Lemma 2.6.** *Let $F$, $G$ be given by* (2.1)*. Also let $S$ be given as in* Theorem 1.1*, where $n \ge 3$ is an integer. If $E_f(S,0)=E_g(S,0)$ then $S(r,f)=S(r,g)$.*

*Proof.* Since $E_f(S,0)=E_g(S,0)$, it follows that $F$ and $G$ share $(1,0)$. We denote the distinct elements of $S$ by $w_j$, $j=1,2,\ldots n$. Since $F$, $G$ share $(1,0)$ from the second fundamental theorem we have

$$\begin{aligned} (n-2)T(r,g) &\le \sum_{j=1}^{n}\overline{N}(r,w_j;g)+S(r,g)=\sum_{j=1}^{n}\overline{N}(r,w_j;f)+S(r,g)\\ &\le nT(r,f)+S(r,g). \end{aligned}$$

Similarly we can deduce

$$(n-2)T(r,f) \leq nT(r,g)+S(r,f).$$

The last inequalities imply $T(r,f)=O\left(T(r,g)\right)$ and $T(r,g)=O\left(T(r,f)\right)$ and so we have $S(r,f)=S(r,g)$. □

## 3. PROOF OF THE THEOREM

*Proof of Theorem* 1.1. Let $F$, $G$ be given by (2.1). Since $E_f(S,2)=E_g(S,2)$ it follows that $F$, $G$ share $(1,2)$. Also since $E_f(\{\infty\},\infty)=E_g(\{\infty\},\infty)$ we see that $\overline{N}_*(r,\infty;f,g)\equiv 0$. We first show that $H\equiv 0$. Let us suppose that $H\not\equiv 0$.

Since $m\geq 2$, using Lemma 2.2 we note that

$$\begin{aligned} & \overline{N}_0(r,0;g^{'})+\overline{N}(r,1;G\mid\geq 2)+\overline{N}_*(r,1;F,G) & (3.1)\\ \leq\ & \overline{N}_0(r,0;g^{'})+\overline{N}(r,1;G\mid\geq 2)+\overline{N}(r,1;G\mid\geq 3) \\ \leq\ & \overline{N}_0(r,0;g^{'})+\sum_{j=1}^{n}\{\overline{N}(r,\omega_j;g\mid=2)+2\overline{N}(r,\omega_j;g\mid\geq 3)\} \\ \leq\ & N(r,0;g^{'}\mid g\neq 0)+S(r,g)\leq \overline{N}(r,0;g)+\overline{N}(r,\infty;g)+S(r,g). \end{aligned}$$

Hence using (3.1), Lemmas 2.1 and 2.3, we get from the second fundamental theorem for $\varepsilon>0$ that

$$\begin{aligned} & (n+1)\,T(r,f) & (3.2)\\ \leq\ & \overline{N}(r,0;f)+\overline{N}(r,b;f)+\overline{N}(r,\infty;f)+N(r,1;F\mid=1)+\overline{N}(r,1;F\mid\geq 2) \\ & -N_0(r,0;f^{'})+S(r,f) \\ \leq\ & 2\left\{\overline{N}(r,0;f)+\overline{N}(r,b;f)\right\}+\overline{N}(r,\infty;f)+\overline{N}(r,0;g)+\overline{N}(r,b;g) \\ & +\overline{N}(r,1;G\mid\geq 2)+\overline{N}_*(r,1;F,G)+\overline{N}_0(r,0;g^{'})+S(r,f)+S(r,g) \\ \leq\ & 2\left\{\overline{N}(r,0;f)+\overline{N}(r,b;f)+\overline{N}(r,0;g)\right\}+\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}(r,b;g) \\ & +S(r,f)+S(r,g) \\ \leq\ & (9-2\Theta(0;f)-2\Theta(0;g)-\Theta(\infty;f)-\Theta(\infty;g)-2\Theta(b;f)-\Theta(b;g)+\varepsilon)\,T(r)+S(r), \end{aligned}$$

where $T(r)=\max\left\{T(r,f),T(r,g)\right\}$. In a similar way we can obtain

$$\begin{aligned} & (n+1)\,T(r,g) & (3.3)\\ \leq\ & (9-2\Theta(0;f)-2\Theta(0;g)-\Theta(\infty;f)-\Theta(\infty;g)-\Theta(b;f)-2\Theta(b;g)+\varepsilon)\,T(r)+S(r). \end{aligned}$$

Combining (3.2) and (3.3) we see that

$$\begin{aligned} & (n-8+2\Theta(0;f)+2\Theta(0;g)+\Theta(\infty;f)+\Theta(\infty;g)+\Theta(b;f) & (3.4)\\ & +\Theta(b;g)+\min\{\Theta(b;f),\Theta(b;g)\}-\varepsilon)\,T(r)\leq S(r). \end{aligned}$$

Since $\varepsilon > 0$, (3.4) leads to a contradiction to the conditions of the theorem. Hence $H \equiv 0$. Then from $\frac{F''}{F'} - \frac{2F'}{F-1} = \frac{G''}{G'} - \frac{2G'}{G-1}$ we obtain by integration

$$F \equiv \frac{AG+B}{CG+D}, \tag{3.5}$$

where $A$, $B$, $C$, $D$ are constants such that $AD - BC \neq 0$. Also

$$T(r,F) = T(r,G) + O(1),$$

and hence from Lemma 2.3 we have

$$T(r,f) = T(r,g) + O(1). \tag{3.6}$$

From (1.4) we note that

$$\overline{N}(r,c;F) \leq \overline{N}(r,b;f) + (n-3)T(r,f) \leq (n-2)T(r,f) + S(r,f).$$

Similarly $\overline{N}(r,c;G) \leq (n-2)T(r,g) + S(r,g)$. From (3.5) and the condition that $f$ and $g$ share $(\infty,0)$ it follows that $\infty$ is a Picard exceptional value of $f$ and $g$ when $C \neq 0$. So in view of (1.2) and (2.1) we observe that when $C \neq 0$, $\overline{N}(r,\infty;F) = \overline{N}(r,\alpha_1;f) + \overline{N}(r,\alpha_2;f)$ and $\overline{N}(r,\infty;G) = \overline{N}(r,\alpha_1;g) + \overline{N}(r,\alpha_2;g)$. We now consider the following cases.

**Case I.** Let $AC \neq 0$. Suppose $B \neq 0$. From (3.5) we get

$$\overline{N}\left(r,-\frac{B}{A};G\right) = \overline{N}(r,0;F). \tag{3.7}$$

In view of (3.6), (3.7), Lemma 2.3 and the second fundamental theorem we get

$$\begin{aligned} nT(r,g) &\leq \overline{N}(r,0;G) + \overline{N}(r,\infty;G) + \overline{N}\left(r,-\frac{B}{A};G\right) + S(r,G) \\ &\leq \overline{N}(r,0;g) + \overline{N}(r,\alpha_1;g) + \overline{N}(r,\alpha_2;g) + \overline{N}(r,0;f) + S(r,g) \\ &\leq 3T(r,g) + T(r,f) + S(r,g) \leq 4T(r,g) + S(r,g), \end{aligned}$$

which is a contradiction for $n \geq 6$.
So we must have $B = 0$ and in this case (3.5) changes to

$$F \equiv \frac{\frac{A}{C}G}{G + \frac{D}{C}}. \tag{3.8}$$

From (3.8) we see that

$$\overline{N}(r,\infty;F) = \overline{N}\left(r,-\frac{D}{C};G\right). \tag{3.9}$$

Now in view of (3.9), Lemma 2.3 and the second fundamental theorem we obtain

$$\begin{aligned} nT(r,g) &\leq \overline{N}(r,0;G)+\overline{N}(r,\infty;G)+\overline{N}\left(r,-\frac{D}{C};G\right)+S(r,G) \\ &\leq \overline{N}(r,0;g)+2T(r,g)+2T(r,f)+S(r,g)\leq 5T(r,g)+S(r,g), \end{aligned}$$

which implies a contradiction for $n\geq 6$.

**Case II.** Let $A\neq 0$ and $C=0$. Then $F=\alpha G+\beta$, where $\alpha=\dfrac{A}{D}$ and $\beta=\dfrac{B}{D}$. If $F$ has no 1-point, by the second fundamental theorem and Lemma 2.3, we get

$$nT(r,f)\leq \overline{N}(r,0;F)+\overline{N}(r,\infty;F)+S(r,f)\leq 4T(r,f)+S(r,f),$$

which implies a contradiction for $n\geq 6$.

If $F$ and $G$ have some 1-points then $\alpha+\beta=1$ and so

$$F\equiv \alpha G+1-\alpha. \tag{3.10}$$

Suppose $\alpha\neq 1$. If $1-\alpha\neq c$ then in view of (3.6), Lemma 2.3 and the second fundamental theorem we get

$$\begin{aligned} 2nT(r,f) &\leq \overline{N}(r,0;F)+\overline{N}(r,c;F)+\overline{N}(r,1-\alpha;F)+\overline{N}(r,\infty;F)+S(r,F) \\ &\leq (n+2)T(r,f)+\overline{N}(r,0;G)+S(r,f)\leq (n+3)T(r,f)+S(r,f), \end{aligned}$$

which implies a contradiction for $n\geq 6$. If $1-\alpha=c$, then we have from (3.10)

$$F\equiv (1-c)G+c.$$

Since $c\neq 1$, by the second fundamental theorem we can obtain using (3.6) and Lemma 2.3 that

$$\begin{aligned} 2nT(r,g) &\leq \overline{N}(r,0;G)+\overline{N}(r,c;G)+\overline{N}\left(r,\frac{c}{c-1};G\right)+\overline{N}(r,\infty;G)+S(r,G) \\ &\leq (n+2)T(r,g)+\overline{N}(r,0;F)+S(r,g)\leq (n+3)T(r,g)+S(r,g), \end{aligned}$$

which implies a contradiction since $n\geq 6$.

So $\alpha=1$ and hence $F\equiv G$. So by Lemma 2.5, we get $f\equiv g$.

**Case III.** Let $A=0$ and $C\neq 0$. Then $F\equiv\dfrac{1}{\gamma G+\delta}$, where $\gamma=\dfrac{C}{B}$ and $\delta=\dfrac{D}{B}$. If $F$ has no 1-point then as in *Case II* we can deduce a contradiction.

If $F$ and $G$ have some 1-points then $\gamma+\delta=1$ and so

$$F\equiv\frac{1}{\gamma G+1-\gamma}. \tag{3.11}$$

Suppose $\gamma \neq 1$. If $\dfrac{1}{1-\gamma} \neq c$, then by the second fundamental theorem and Lemma 2.3 we get

$$\begin{aligned}2nT(r,f) &\leq \overline{N}(r,0;F) + \overline{N}(r,\frac{1}{1-\gamma};F) + \overline{N}(r,c;F) + \overline{N}(r,\infty;F) + S(r,f)\\ &\leq (n+3)T(r,f) + \overline{N}(r,0;G) + S(r,f) \leq (n+4)T(r,f) + S(r,f),\end{aligned}$$

which gives a contradiction for $n \geq 6$. If $\dfrac{1}{1-\gamma} = c$, from (3.11) we have

$$F \equiv \frac{c}{(c-1)\,G+1}. \tag{3.12}$$

If $c \neq \dfrac{1}{1-c}$, the second fundamental theorem with the help of (3.6), (3.12) and Lemma 2.3 yields

$$\begin{aligned}2nT(r,g) &\leq \overline{N}(r,0;G) + \overline{N}(r,c;G) + \overline{N}\left(r,\frac{1}{1-c};G\right) + \overline{N}(r,\infty;G) + S(r,G)\\ &\leq (n+1)T(r,g) + \overline{N}(r,\infty;F) + S(r,g) \leq (n+3)T(r,g) + S(r,g),\end{aligned}$$

which implies a contradiction since $n \geq 6$. On the other hand, if $c = \dfrac{1}{1-c}$ then from (3.12) we have

$$G \equiv \frac{c(F-c)}{F}.$$

So from the second fundamental theorem it follows that

$$\begin{aligned}nT(r,f) &\leq \overline{N}(r,0;F) + \overline{N}(r,c;F) + \overline{N}(r,\infty;F) + S(r,F)\\ &\leq 3T(r,f) + \overline{N}(r,0;G) + S(r,f) \leq 4T(r,f) + S(r,f),\end{aligned}$$

which implies a contradiction since $n \geq 6$. So we must have $\gamma = 1$ then $FG \equiv 1$, which is impossible by Lemma 2.4. This completes the proof of the theorem. ☐

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