

DEFLECTION D-TENSOR IDENTITIES IN THE RELATIVISTIC TIME DEPENDENT LAGRANGE GEOMETRY

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*Dedicated to Professor Ștefan Mititelu
on the occasion of his seventieth birthday*

ABSTRACT. The aim of this paper is to study the local components of the relativistic time dependent d-linear connections, d-torsions, d-curvatures and deflection d-tensors with respect to an adapted basis on the 1-jet space $J^1(\mathbb{R}, M)$. The Ricci identities, with their corresponding identities of deflection d-tensors, are also given.

1. SOME PHYSICAL AND GEOMETRICAL ASPECTS

According to Olver's opinion [11], we agree that the 1-jet fibre bundle is a basic object in the study of classical and quantum field theories. For such a reason, a lot of authors (Asanov [3], Saunders [12], Vondra [13] and many others) studied the differential geometry of the 1-jet spaces. Continuing the geometrical studies of Asanov [3] and using as a pattern the Lagrangian geometrical ideas developed by Anastasiei in the research work [2], Miron, Anastasiei and Bucătaru in the monographs [6] and [4], the first author of this paper has recently developed the *Riemann-Lagrange geometry of 1-jet spaces* [8]. This theory is very suitable for the geometrical study of the *relativistic non-autonomous (rheonomic) Lagrangians*, that is of the Lagrangians depending on an usual *relativistic time* [7], [9] or depending on a *relativistic multi-time* [8], [10].

It is important to note that a *classical non-autonomous (rheonomic) Lagrangian geometry* (i.e. a geometrization of the Lagrangians depending on an *absolute time*) was sketched by Miron and Anastasiei at the end of the book [6] and developed in the same way by Anastasiei and Kawaguchi [1] or Frigioiu [5].

In what follows we try to expose the main geometrical and physical aspects which differentiate the both geometrical theories: the *jet relativistic non-autonomous Lagrangian geometry* [9] and the *classical non-autonomous Lagrangian geometry* [6].

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In this direction, we point out that the *relativistic non-autonomous Lagrangian geometry* [9] has as natural habit the 1-jet space $J^1(\mathbb{R}, M)$, where \mathbb{R} is the manifold of real numbers having the coordinate t . This represents for us the usual *relativistic time*. We recall that the 1-jet space $J^1(\mathbb{R}, M)$ is regarded as a vector bundle over the product manifold $\mathbb{R} \times M$, having the fibre type \mathbb{R}^n , where n is the dimension of the *spatial* manifold M . In mechanical terms, if the manifold M has the spatial local coordinates $(x^i)_{i=\overline{1,n}}$, then the 1-jet vector bundle

$$J^1(\mathbb{R}, M) \rightarrow \mathbb{R} \times M$$

can be regarded as a *bundle of configurations* having the local coordinates (t, x^i, y_1^i) ; these transform by the rules [9]

$$\begin{cases} \tilde{t} = \tilde{t}(t) \\ \tilde{x}^i = \tilde{x}^i(x^j) \\ \tilde{y}_1^i = \frac{\partial \tilde{x}^i}{\partial x^j} \frac{dt}{d\tilde{t}} \cdot y_1^j. \end{cases} \quad (1.1)$$

Remark 1.1. The form of the jet transformation group (1.1) stands out by the *relativistic character of the time t* .

Comparatively, in the *classical non-autonomous Lagrangian geometry* [6] the *bundle of configurations* is the vector bundle

$$\mathbb{R} \times TM \rightarrow M,$$

whose local coordinates (t, x^i, y^i) transform by the rules

$$\begin{cases} \tilde{t} = t \\ \tilde{x}^i = \tilde{x}^i(x^j) \\ \tilde{y}^i = \frac{\partial \tilde{x}^i}{\partial x^j} \cdot y^j, \end{cases} \quad (1.2)$$

where TM is the tangent bundle of the spatial manifold M .

Remark 1.2. The form of the transformation group (1.2) stands out by the *absolute character of the time t* .

It is important to note that jet transformation group (1.1) from the *relativistic non-autonomous Lagrangian geometry* is more general and more natural than the transformation group (1.2) used in the *classical non-autonomous Lagrangian geometry*. This is because the last one ignores the temporal reparametrizations, emphasizing in this way the absolute character of the usual time coordinate t . Or, physically speaking, the relativity of time is a well-known fact.

From a geometrical point of view, we point out that the entire *classical rheonomic Lagrangian geometry* of Miron and Anastasiei [6] relies on the study of the *energy action functional*

$$\mathbb{E}_1(c) = \int_a^b L(t, x^i, y^i) dt,$$

where $L: \mathbb{R} \times TM \rightarrow \mathbb{R}$ is a Lagrangian function and $y^i = dx^i/dt$, whose Euler-Lagrange equations produce a semispray $G^i(t, x^k, y^k)$ and a corresponding nonlinear connection $N_j^i = \partial G^i / \partial y^j$. Therefore, the authors construct the adapted bases of vector and covector fields, with the adapted components of the N -linear connections and their corresponding d-torsions and d-curvatures. But, because $L(t, x^i, y^i)$ is a real function, we deduce that the previous geometrical theory has the following impediment: *-the energy action functional depends on the reparametrizations $t \longleftrightarrow \tilde{t}$ of the same curve c .* Thus, in order to avoid this inconvenience, the Finsler case imposes the 1-positive homogeneity condition

$$L(t, x^i, \lambda y^i) = \lambda L(t, x^i, y^i), \quad \forall \lambda > 0.$$

Alternatively, the *relativistic rheonomic Lagrangian geometry* from [9] uses the *relativistic energy action functional*

$$\mathbb{E}_2(c) = \int_a^b L(t, x^i, y_1^i) \sqrt{h_{11}(t)} dt,$$

where $L: J^1(\mathbb{R}, M) \rightarrow \mathbb{R}$ is a jet Lagrangian function and $h_{11}(t)$ is a Riemannian metric on the relativistic time manifold \mathbb{R} . This functional is now independent of the reparametrizations $t \longleftrightarrow \tilde{t}$ of the same curve c . The Euler-Lagrange equations of the Lagrangian $\mathcal{L} = L(t, x^i, y_1^i) \sqrt{h_{11}(t)}$ produce a relativistic time dependent semispray [9]

$$\mathcal{S} = \left(H_{(1)1}^{(i)}, G_{(1)1}^{(i)} \right),$$

which gives the jet nonlinear connection [7]

$$\Gamma_{\mathcal{S}} = \left(M_{(1)1}^{(j)} = 2H_{(1)1}^{(j)}, N_{(1)k}^{(j)} = \frac{\partial G_{(1)1}^{(j)}}{\partial y_1^k} \right).$$

With these geometrical tools we can construct in the relativistic rheonomic Lagrangian geometry the distinguished (d-) linear connections, with their d-torsions and d-curvatures, which naturally generalize the similar geometrical objects from the classical rheonomic Lagrangian geometry [6]. In this respect, the authors of this paper believe that the relativistic geometrical approach proposed in this paper has more geometrical and physical meanings than the theory proposed by Miron and Anastasiei in [6].

In conclusion, in order to remark the main similitudes and differences between these geometrical theories, we invite the reader to compare both *classical* and *relativistic non-autonomous Lagrangian geometries* exposed in works [6] and [9].

As a final remark, we point out that for a lot of mathematicians (such as Crampin, de Leon, Krupkova, Sarlet, Saunders and others) the non-autonomous Lagrangian geometry is constructed on the first jet bundle $J^1\pi$ of a fibered manifold $\pi: M^{n+1} \rightarrow \mathbb{R}$. In their papers, if (t, x^i) are the local coordinates on the $n + 1$ -dimensional manifold M such that t is a global coordinate for the fibers of

the submersion π and x^i are transverse coordinates of the induced foliation, then a change of coordinates on M is given by

$$\begin{cases} \tilde{t} = \tilde{t}(t), & \frac{d\tilde{t}}{dt} \neq 0 \\ \tilde{x}^i = \tilde{x}^i(x^j, t), & \text{rank} \left(\frac{\partial \tilde{x}^i}{\partial x^j} \right) = n. \end{cases} \quad (1.3)$$

Although the 1-jet extension of the transformation rules (1.3) is more general than the transformation group (1.1), the authors of this paper consider that the transformation group (1.1) is more appropriate for their final purpose, the development of a *relativistic rheonomic Lagrangian field theory*. For example, in the paper [9], starting with a non-degenerate Lagrangian function $L : J^1(\mathbb{R}, M) \rightarrow \mathbb{R}$ and an *a priori* given Riemannian metric $h_{11}(t)$ on the relativistic temporal manifold \mathbb{R} , one introduces a *relativistic time dependent electromagnetic field*

$$F = F_{(i)j}^{(1)} \delta y_1^i \wedge dx^j,$$

where

$$F_{(i)j}^{(1)} = \frac{1}{2} \left[D_{(i)j}^{(1)} - D_{(j)i}^{(1)} \right] \text{ and } \delta y_1^i = dy_1^i + M_{(1)1}^{(i)} dt + N_{(1)j}^{(i)} dx^j,$$

the *metrical deflection d-tensors* $D_{(i)j}^{(1)}$ being produced only by the jet Lagrangian $\mathcal{L} = L\sqrt{h_{11}(t)}$. In such a perspective, the relativistic time dependent electromagnetic field F has an *intrinsic geometrical character*. Moreover, the electromagnetic components $F_{(i)j}^{(1)}$ are governed by some natural *generalized Maxwell equations*. These equations are exposed in [9] and naturally generalize the already classical Maxwell equations from Miron and Anastasiei's theory [6].

2. THE ADAPTED COMPONENTS OF THE JET Γ -LINEAR CONNECTIONS

Let us suppose that on the 1-jet space $E = J^1(\mathbb{R}, M)$ is fixed a nonlinear connection Γ given by the *temporal components* $M_{(1)1}^{(i)}$ and the *spatial components* $N_{(1)j}^{(i)}$. We recall that the transformation rules of the local components of the nonlinear connection $\Gamma = \left(M_{(1)1}^{(i)}, N_{(1)j}^{(i)} \right)$ are expressed by [7]

$$\tilde{M}_{(1)1}^{(k)} = M_{(1)1}^{(j)} \left(\frac{dt}{d\tilde{t}} \right)^2 \frac{\partial \tilde{x}^k}{\partial x^j} - \frac{dt}{d\tilde{t}} \frac{\partial \tilde{y}_1^k}{\partial t}, \quad \tilde{N}_{(1)l}^{(k)} = N_{(1)j}^{(i)} \frac{dt}{d\tilde{t}} \frac{\partial x^i}{\partial \tilde{x}^l} \frac{\partial \tilde{x}^k}{\partial x^j} - \frac{\partial x^i}{\partial \tilde{x}^l} \frac{\partial \tilde{y}_1^k}{\partial x^i}.$$

Example 2.1. Let $(\mathbb{R}, h_{11}(t))$ and $(M, \varphi_{ij}(x))$ be semi-Riemannian manifolds. Let us consider the Christoffel symbols

$$H_{11}^1 = \frac{h^{11}}{2} \frac{dh_{11}}{dt}$$

and

$$\gamma_{jk}^i = \frac{\varphi^{im}}{2} \left(\frac{\partial \varphi_{jm}}{\partial x^k} + \frac{\partial \varphi_{km}}{\partial x^j} - \frac{\partial \varphi_{jk}}{\partial x^m} \right),$$

where $h^{11} = 1/h_{11}$. Then, using the transformation rules

$$\tilde{H}_{11}^1 = H_{11}^1 \frac{dt}{d\tilde{t}} + \frac{d\tilde{t}}{dt} \frac{d^2t}{d\tilde{t}^2} \quad (2.1)$$

and

$$\tilde{\gamma}_{qr}^p = \gamma_{jk}^i \frac{\partial \tilde{x}^p}{\partial x^i} \frac{\partial x^j}{\partial \tilde{x}^q} \frac{\partial x^k}{\partial \tilde{x}^r} + \frac{\partial \tilde{x}^p}{\partial x^l} \frac{\partial^2 x^l}{\partial \tilde{x}^q \partial \tilde{x}^r}, \quad (2.2)$$

we deduce that the set of local functions

$$\mathring{\Gamma} = \left(\mathring{M}_{(1)1}^{(j)}, \mathring{N}_{(1)i}^{(j)} \right),$$

where

$$\mathring{M}_{(1)1}^{(j)} = -H_{11}^1 y_1^j \quad \text{and} \quad \mathring{N}_{(1)i}^{(j)} = \gamma_{im}^j y_1^m, \quad (2.3)$$

represents a nonlinear connection on the 1-jet space $E = J^1(\mathbb{R}, M)$. This jet nonlinear connection is called the *canonical nonlinear connection attached to the pair of metrics* $(h_{11}(t), \varphi_{ij}(x))$.

Let

$$\left\{ \frac{\delta}{\delta t}, \frac{\delta}{\delta x^i}, \frac{\partial}{\partial y_1^i} \right\} \subset \mathcal{X}(E)$$

and

$$\{dt, dx^i, \delta y_1^i\} \subset \mathcal{X}^*(E)$$

be the dual bases adapted to the nonlinear connection $\Gamma = \left(M_{(1)1}^{(i)}, N_{(1)j}^{(i)} \right)$, where

$$\begin{aligned} \frac{\delta}{\delta t} &= \frac{\partial}{\partial t} - M_{(1)1}^{(j)} \frac{\partial}{\partial y_1^j}, \\ \frac{\delta}{\delta x^i} &= \frac{\partial}{\partial x^i} - N_{(1)i}^{(j)} \frac{\partial}{\partial y_1^j}, \\ \delta y_1^i &= dy_1^i + M_{(1)1}^{(i)} dt + N_{(1)j}^{(i)} dx^j. \end{aligned}$$

Remark 2.1. The components of the above dual adapted bases transform under a change of coordinates (1.1) as classical tensors.

In order to develop a theory of the Γ -linear connections on the 1-jet space $E = J^1(\mathbb{R}, M)$, we need the following result:

Proposition 2.1. a) *The Lie algebra $\mathcal{X}(E)$ of the vector fields on E decomposes in the direct sum*

$$\mathcal{X}(E) = \mathcal{X}(\mathcal{H}_{\mathbb{R}}) \oplus \mathcal{X}(\mathcal{H}_M) \oplus \mathcal{X}(\mathcal{V}),$$

where

$$\mathcal{X}(\mathcal{H}_{\mathbb{R}}) = \text{Span} \left\{ \frac{\delta}{\delta t} \right\}, \quad \mathcal{X}(\mathcal{H}_M) = \text{Span} \left\{ \frac{\delta}{\delta x^i} \right\}, \quad \mathcal{X}(\mathcal{V}) = \text{Span} \left\{ \frac{\partial}{\partial y_1^i} \right\}.$$

b) The Lie algebra $\mathcal{X}^*(E)$ of the covector fields on E decomposes in the direct sum

$$\mathcal{X}^*(E) = \mathcal{X}^*(\mathcal{H}_{\mathbb{R}}) \oplus \mathcal{X}^*(\mathcal{H}_M) \oplus \mathcal{X}^*(\mathcal{V}),$$

where

$$\mathcal{X}^*(\mathcal{H}_{\mathbb{R}}) = \text{Span}\{dt\}, \quad \mathcal{X}^*(\mathcal{H}_M) = \text{Span}\{dx^i\}, \quad \mathcal{X}^*(\mathcal{V}) = \text{Span}\{\delta y_1^i\}.$$

Denoting by $h_{\mathbb{R}}$, h_M , respectively v , the \mathbb{R} -horizontal, M -horizontal, respectively vertical canonical projections associated to the above decompositions, we get

Corollary 2.1. a) Any vector field on E can be uniquely written in the form:

$$X = h_{\mathbb{R}}X + h_MX + vX, \quad \forall X \in \mathcal{X}(E).$$

b) Any 1-form on E can be uniquely written in the form:

$$\omega = h_{\mathbb{R}}\omega + h_M\omega + v\omega, \quad \forall \omega \in \mathcal{X}^*(E).$$

Definition 2.1. A linear connection $\nabla : \mathcal{X}(E) \times \mathcal{X}(E) \rightarrow \mathcal{X}(E)$, which verifies the Ehresman-Koszul axioms

$$\nabla h_{\mathbb{R}} = 0, \quad \nabla h_M = 0, \quad \nabla v = 0,$$

is called a Γ -linear connection on the 1-jet vector bundle $E = J^1(\mathbb{R}, M)$.

Using the adapted basis of vector fields on $E = J^1(\mathbb{R}, M)$ and the definition of a Γ -linear connection, we prove without difficulties

Proposition 2.2. A Γ -linear connection ∇ on $E = J^1(\mathbb{R}, M)$ is determined by nine local adapted components

$$\nabla\Gamma = \left(\bar{G}_{11}^1, G_{i1}^k, G_{(1)(j)1}^{(i)(1)}, \bar{L}_{1j}^1, L_{ij}^k, L_{(1)(j)k}^{(i)(1)}, \bar{C}_{1(k)}^{1(1)}, C_{i(k)}^{j(1)}, C_{(1)(j)(k)}^{(i)(1)(1)} \right),$$

which are uniquely defined by the relations:

$$\begin{aligned} (h_{\mathbb{R}}) \quad \nabla \frac{\delta}{\delta t} \frac{\delta}{\delta t} &= \bar{G}_{11}^1 \frac{\delta}{\delta t}, \quad \nabla \frac{\delta}{\delta t} \frac{\delta}{\delta x^i} = G_{i1}^k \frac{\delta}{\delta x^k}, \quad \nabla \frac{\delta}{\delta t} \frac{\partial}{\partial y_1^i} = G_{(1)(i)1}^{(k)(1)} \frac{\partial}{\partial y_1^k}, \\ (h_M) \quad \nabla \frac{\delta}{\delta x^j} \frac{\delta}{\delta t} &= \bar{L}_{1j}^1 \frac{\delta}{\delta t}, \quad \nabla \frac{\delta}{\delta x^j} \frac{\delta}{\delta x^i} = L_{ij}^k \frac{\delta}{\delta x^k}, \quad \nabla \frac{\delta}{\delta x^j} \frac{\partial}{\partial y_1^i} = L_{(1)(i)j}^{(k)(1)} \frac{\partial}{\partial y_1^k}, \\ (v) \quad \nabla \frac{\partial}{\partial y_1^j} \frac{\delta}{\delta t} &= \bar{C}_{1(j)}^{1(1)} \frac{\delta}{\delta t}, \quad \nabla \frac{\partial}{\partial y_1^j} \frac{\delta}{\delta x^i} = C_{i(j)}^{k(1)} \frac{\delta}{\delta x^k}, \quad \nabla \frac{\partial}{\partial y_1^j} \frac{\partial}{\partial y_1^i} = C_{(1)(i)(j)}^{(k)(1)(1)} \frac{\partial}{\partial y_1^k}. \end{aligned}$$

Taking into account the tensorial transformation laws of the adapted basis of vector fields on $E = J^1(\mathbb{R}, M)$, by laborious local computations, we deduce

Theorem 2.1. a) Under a change of coordinates (1.1) on the 1-jet vector bundle $E = J^1(\mathbb{R}, M)$, the adapted coefficients of the Γ -linear connection ∇ modify by the rules

$$\begin{aligned}
 (h_{\mathbb{R}}) \quad & \left\{ \begin{aligned} \bar{G}_{11}^1 &= \tilde{G}_{11}^1 \frac{d\tilde{t}}{dt} + \frac{dt}{d\tilde{t}} \frac{d^2\tilde{t}}{dt^2} \\ G_{i1}^k &= \tilde{G}_{j1}^r \frac{\partial x^k}{\partial \tilde{x}^r} \frac{\partial \tilde{x}^j}{\partial x^i} \frac{d\tilde{t}}{dt} \\ G_{(1)(i)1}^{(k)(1)} &= \tilde{G}_{(1)(j)1}^{(p)(1)} \frac{\partial x^k}{\partial \tilde{x}^p} \frac{\partial \tilde{x}^j}{\partial x^i} \frac{d\tilde{t}}{dt} + \delta_i^k \left(\frac{d\tilde{t}}{dt} \right)^2 \frac{d^2\tilde{t}}{dt^2}, \end{aligned} \right. \\
 (h_M) \quad & \left\{ \begin{aligned} \bar{L}_{1j}^1 &= \tilde{L}_{1l}^1 \frac{\partial \tilde{x}^l}{\partial x^j} \\ L_{ij}^r &= \tilde{L}_{pq}^s \frac{\partial x^r}{\partial \tilde{x}^s} \frac{\partial \tilde{x}^p}{\partial x^i} \frac{\partial \tilde{x}^q}{\partial x^j} + \frac{\partial x^r}{\partial \tilde{x}^s} \frac{\partial^2 \tilde{x}^s}{\partial x^i \partial x^j} \\ L_{(1)(i)j}^{(k)(1)} &= \tilde{L}_{(1)(p)s}^{(r)(1)} \frac{\partial x^k}{\partial \tilde{x}^r} \frac{\partial \tilde{x}^p}{\partial x^i} \frac{\partial \tilde{x}^s}{\partial x^j} + \frac{\partial x^k}{\partial \tilde{x}^r} \frac{\partial^2 \tilde{x}^r}{\partial x^i \partial x^j}, \end{aligned} \right. \\
 (v) \quad & \left\{ \begin{aligned} \bar{C}_{1(i)}^{1(1)} &= \tilde{C}_{1(j)}^{1(1)} \frac{\partial \tilde{x}^j}{\partial x^i} \frac{d\tilde{t}}{dt} \\ C_{i(j)}^{k(1)} &= \tilde{C}_{p(r)}^{s(1)} \frac{\partial x^k}{\partial \tilde{x}^s} \frac{\partial \tilde{x}^p}{\partial x^i} \frac{\partial \tilde{x}^r}{\partial x^j} \frac{d\tilde{t}}{dt} \\ C_{(1)(i)(j)}^{(k)(1)(1)} &= \tilde{C}_{(1)(p)(a)}^{(r)(1)(1)} \frac{\partial x^k}{\partial \tilde{x}^r} \frac{\partial \tilde{x}^p}{\partial x^i} \frac{\partial \tilde{x}^a}{\partial x^j} \frac{d\tilde{t}}{dt}. \end{aligned} \right.
 \end{aligned}$$

b) Conversely, to give a Γ -linear connection ∇ on the 1-jet vector bundle $E = J^1(\mathbb{R}, M)$ is equivalent to give a set of nine adapted local components $\nabla\Gamma$, which transform by the rules described in a).

Example 2.2. Let $h_{11}(t)$ (respectively $\varphi_{ij}(x)$) be a semi-Riemannian metric on \mathbb{R} (respectively M). We denote by $H_{11}^1(t)$ (respectively $\gamma_{ij}^k(x)$) the Christoffel symbols of the metric $h_{11}(t)$ (respectively $\varphi_{ij}(x)$). Let us consider on the 1-jet space $E = J^1(\mathbb{R}, M)$ the canonical nonlinear connection $\mathring{\Gamma}$ associated to the pair of metrics $(h_{11}(t), \varphi_{ij}(x))$, which is defined by the local coefficients (2.3). In this context, using the transformation laws (2.1) and (2.2), we deduce that the set of adapted local coefficients

$$B\mathring{\Gamma} = \left(\bar{G}_{11}^1, 0, G_{(1)(i)1}^{(k)(1)}, 0, L_{ij}^k, L_{(1)(i)j}^{(k)(1)}, 0, 0, 0 \right),$$

where

$$\bar{G}_{11}^1 = H_{11}^1, \quad G_{(1)(i)1}^{(k)(1)} = -\delta_i^k H_{11}^1, \quad L_{ij}^k = \gamma_{ij}^k, \quad L_{(1)(i)j}^{(k)(1)} = \gamma_{ij}^k,$$

defines a $\mathring{\Gamma}$ -linear connection on the 1-jet space E , which is called the *Berwald connection attached to the semi-Riemannian metrics $h_{11}(t)$ and $\varphi_{ij}(x)$* .

Remark 2.2. In the particular case $(\mathbb{R}, h) = (\mathbb{R}, \delta)$ our Berwald linear connection naturally generalizes the canonical N -linear connection induced by the canonical

spray $2G^i = \gamma_{jk}^i y^j y^k$ from the classical theory of Finsler and Lagrange spaces. For more details, please consult [4], [6].

Now, let us consider that ∇ is a fixed Γ -linear connection on the 1-jet space $E = J^1(\mathbb{R}, M)$, which is defined by the adapted local coefficients

$$\nabla\Gamma = \left(\bar{G}_{11}^1, G_{i1}^k, G_{(1)(j)1}^{(i)(1)}, \bar{L}_{1j}^1, L_{ij}^k, L_{(1)(j)k}^{(i)(1)}, \bar{C}_{1(k)}^{1(1)}, C_{i(k)}^{j(1)}, C_{(1)(j)(k)}^{(i)(1)(1)} \right). \quad (2.4)$$

Definition 2.2. A geometrical object $D = \left(D_{1k(1)(l)\dots}^{1i(j)(1)\dots} \right)$ on the 1-jet vector bundle $E = J^1(\mathbb{R}, M)$, whose local components transform by the rules

$$D_{1k(1)(l)\dots}^{1i(j)(1)\dots} = \tilde{D}_{1r(1)(s)\dots}^{1p(m)(1)\dots} \frac{dt}{dt} \frac{\partial x^i}{\partial \tilde{x}^p} \left(\frac{\partial x^j}{\partial \tilde{x}^m} \frac{d\tilde{t}}{dt} \right) \frac{d\tilde{t}}{dt} \frac{\partial \tilde{x}^r}{\partial x^k} \left(\frac{\partial \tilde{x}^s}{\partial x^l} \frac{d\tilde{t}}{dt} \right) \dots,$$

is called a *d-tensor field*.

Example 2.3. The geometrical object $\mathbf{C} = \left(\mathbf{C}_{(1)}^{(i)} \right)$, where $\mathbf{C}_{(1)}^{(i)} = y_1^i$, represents a d-tensor field on the 1-jet space $E = J^1(\mathbb{R}, M)$. This is called the *canonical Liouville d-tensor field* of the 1-jet vector bundle E . Remark that the d-tensor field \mathbf{C} naturally generalizes the Liouville vector field [6]

$$\mathbf{C} = y^i \frac{\partial}{\partial y^i},$$

used in the Lagrangian geometry of the tangent bundle TM .

The Γ -linear connection ∇ naturally induces a linear connection on the set of the d-tensors of the 1-jet vector bundle E , in the following way: - starting with $X \in \mathcal{X}(E)$ a vector field and D a d-tensor field on E , locally expressed by

$$X = X^1 \frac{\delta}{\delta t} + X^r \frac{\delta}{\delta x^r} + X_{(1)}^{(r)} \frac{\partial}{\partial y_1^r},$$

$$D = D_{1k(1)(l)\dots}^{1i(j)(1)\dots} \frac{\delta}{\delta t} \otimes \frac{\delta}{\delta x^i} \otimes \frac{\partial}{\partial y_1^l} \otimes dt \otimes dx^k \otimes \delta y_1^l \dots,$$

we introduce the covariant derivative

$$\nabla_X D = X^1 \nabla_{\frac{\delta}{\delta t}} D + X^p \nabla_{\frac{\delta}{\delta x^p}} D + X_{(1)}^{(p)} \nabla_{\frac{\partial}{\partial y_1^p}} D = \left\{ X^1 D_{1k(1)(l)\dots/1}^{1i(j)(1)\dots} + X^p \cdot D_{1k(1)(l)\dots|p}^{1i(j)(1)\dots} + X_{(1)}^{(p)} D_{1k(1)(l)\dots|^{(1)(p)}}^{1i(j)(1)\dots} \right\} \frac{\delta}{\delta t} \otimes \frac{\delta}{\delta x^i} \otimes \frac{\partial}{\partial y_1^j} \otimes dt \otimes dx^k \otimes \delta y_1^l \dots,$$

where

$$(h_{\mathbb{R}}) \quad \left\{ \begin{array}{l} D_{1k(1)(l)\dots/1}^{1i(j)(1)\dots} = \frac{\delta D_{1k(1)(l)\dots}^{1i(j)(1)\dots}}{\delta t} + D_{1k(1)(l)\dots}^{1i(j)(1)\dots} \bar{G}_{11}^1 + \\ \quad + D_{1k(1)(l)\dots}^{1r(j)(1)\dots} G_{r1}^i + D_{1k(1)(l)\dots}^{1i(r)(1)\dots} G_{(1)(r)1}^{(j)(1)} + \dots - \\ \quad - D_{1k(1)(l)\dots}^{1i(j)(1)\dots} \bar{G}_{11}^1 - D_{1r(1)(l)\dots}^{1i(j)(1)\dots} G_{k1}^r - D_{1k(1)(r)\dots}^{1i(j)(1)\dots} G_{(1)(l)1}^{(r)(1)} - \dots, \end{array} \right.$$

$$\begin{aligned}
 (h_M) \quad & \left\{ \begin{aligned} D_{1k(1)(l)\dots|p}^{1i(j)(1)\dots} &= \frac{\delta D_{1k(1)(l)\dots}^{1i(j)(1)\dots}}{\delta x^p} + D_{1k(1)(l)\dots}^{1i(j)(1)\dots} \bar{L}_{1p}^1 + \\ &+ D_{1k(1)(l)\dots}^{1r(j)(1)\dots} L_{rp}^i + D_{1k(1)(l)\dots}^{1i(r)(1)\dots} L_{(1)(r)p}^{(j)(1)} + \dots - \\ &- D_{1k(1)(l)\dots}^{1i(j)(1)\dots} \bar{L}_{1p}^1 - D_{1r(1)(l)\dots}^{1i(j)(1)\dots} L_{kp}^r - D_{1k(1)(r)\dots}^{1i(j)(1)\dots} L_{(1)(l)p}^{(r)(1)} - \dots, \end{aligned} \right. \\
 (v) \quad & \left\{ \begin{aligned} D_{1k(1)(l)\dots|^{(1)}(p)}^{1i(j)(1)\dots} &= \frac{\partial D_{1k(1)(l)\dots}^{1i(j)(1)\dots}}{\partial y_1^p} + D_{1k(1)(l)\dots}^{1i(j)(1)\dots} \bar{C}_{1(p)}^{1(1)} + \\ &+ D_{1k(1)(l)\dots}^{1r(j)(1)\dots} C_{r(p)}^{i(1)} + D_{1k(1)(l)\dots}^{1i(r)(1)\dots} C_{(1)(r)(p)}^{(j)(1)(1)} + \dots - \\ &- D_{1k(1)(l)\dots}^{1i(j)(1)\dots} \bar{C}_{1(p)}^{1(1)} - D_{1r(1)(l)\dots}^{1i(j)(1)\dots} C_{k(p)}^{r(1)} - D_{1k(1)(r)\dots}^{1i(j)(1)\dots} C_{(1)(l)(p)}^{(r)(1)(1)} - \dots. \end{aligned} \right.
 \end{aligned}$$

Definition 2.3. The local derivative operators “ $_{/1}$ ”, “ $_{|p}$ ” and “ $_{|^{(1)}(p)}$ ” are called the \mathbb{R} -horizontal covariant derivative, the M -horizontal covariant derivative and the vertical covariant derivative associated to the Γ -linear connection $\nabla\Gamma$. These apply to the local components of an arbitrary d-tensor field on the 1-jet space $E = J^1(\mathbb{R}, M)$.

Remark 2.3. a) In the particular case of a function $f(t, x^k, y_1^k)$ on the 1-jet space $E = J^1(\mathbb{R}, M)$ the above covariant derivatives reduce to

$$f_{/1} = \frac{\delta f}{\delta t} = \frac{\partial f}{\partial t} - M_{(1)1}^{(k)} \frac{\partial f}{\partial y_1^k}, \quad f_{|p} = \frac{\delta f}{\delta x^p} = \frac{\partial f}{\partial x^p} - N_{(1)p}^{(k)} \frac{\partial f}{\partial y_1^k}, \quad f_{|^{(1)}(p)} = \frac{\partial f}{\partial y_1^p}.$$

b) Starting with a d-vector field $D = Y$ on the 1-jet space $E = J^1(\mathbb{R}, M)$, locally expressed by

$$Y = Y^1 \frac{\delta}{\delta t} + Y^i \frac{\delta}{\delta x^i} + Y_{(1)}^{(i)} \frac{\partial}{\partial y_1^i},$$

the following expressions of the local covariant derivatives hold good:

$$\begin{aligned}
 (h_{\mathbb{R}}) \quad & \left\{ \begin{aligned} Y_{/1}^1 &= \frac{\delta Y^1}{\delta t} + Y^1 \bar{G}_{11}^1 \\ Y_{/1}^i &= \frac{\delta Y^i}{\delta t} + Y^r G_{r1}^i \\ Y_{(1)/1}^{(i)} &= \frac{\delta Y_{(1)}^{(i)}}{\delta t} + Y_{(1)}^{(r)} G_{(1)(r)1}^{(i)(1)}, \end{aligned} \right. \\
 (h_M) \quad & \left\{ \begin{aligned} Y_{|p}^1 &= \frac{\delta Y^1}{\delta x^p} + Y^1 \bar{L}_{1p}^1 \\ Y_{|p}^i &= \frac{\delta Y^i}{\delta x^p} + Y^r L_{rp}^i \\ Y_{(1)|p}^{(i)} &= \frac{\delta Y_{(1)}^{(i)}}{\delta x^p} + Y_{(1)}^{(r)} L_{(1)(r)p}^{(i)(1)}, \end{aligned} \right.
 \end{aligned}$$

$$(v) \quad \begin{cases} Y^1|_{(p)}^{(1)} = \frac{\partial Y^1}{\partial y_1^p} + Y^1 \bar{C}_{1(p)}^{1(1)} \\ Y^i|_{(p)}^{(1)} = \frac{\partial Y^i}{\partial y_1^p} + Y^r C_{r(p)}^{i(1)} \\ Y_{(1)}^{(i)}|_{(p)}^{(1)} = \frac{\partial Y_{(1)}^{(i)}}{\partial y_1^p} + Y_{(1)}^{(r)} C_{(1)(r)(p)}^{(i)(1)(1)}. \end{cases}$$

Denoting generically by “: A ” one of the local covariant derivatives “/ $_1$ ”, “/ $_p$ ” or “/ $_{(p)}^{(1)}$ ”, we obtain the following properties of the covariant derivative operators:

Proposition 2.3. *If T_{\dots} and S_{\dots} are two arbitrary d -tensors on $E = J^1(\mathbb{R}, M)$, then the following statements hold good:*

i) *The local coefficients $T_{\dots A}$ represent the components of a new d -tensor field on the 1-jet space $E = J^1(\mathbb{R}, M)$.*

$$\text{ii) } (T_{\dots} + S_{\dots})_{:A} = T_{\dots A} + S_{\dots A}.$$

$$\text{iii) } (T_{\dots} \otimes S_{\dots})_{:A} = T_{\dots A} \otimes S_{\dots} + T_{\dots} \otimes S_{\dots A}.$$

3. TORSION AND CURVATURE D-TENSORS

In the sequel, we shall study the torsion tensor $\mathbf{T}: \mathcal{X}(E) \times \mathcal{X}(E) \rightarrow \mathcal{X}(E)$ associated to the Γ -linear connection ∇ , which is given by the formula

$$\mathbf{T}(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y], \quad \forall X, Y \in \mathcal{X}(E).$$

In order to obtain an adapted local characterization of the torsion tensor \mathbf{T} of the Γ -linear connection ∇ , we firstly deduce, by direct computations, the following important result:

Proposition 3.1. *The following identities of the Poisson brackets are true:*

$$\begin{aligned} \left[\frac{\delta}{\delta t}, \frac{\delta}{\delta t} \right] &= 0, & \left[\frac{\delta}{\delta t}, \frac{\delta}{\delta x^j} \right] &= R_{(1)1j}^{(r)} \frac{\partial}{\partial y_1^r}, \\ \left[\frac{\delta}{\delta t}, \frac{\partial}{\partial y_1^j} \right] &= \frac{\partial M_{(1)1}^{(r)}}{\partial y_1^j} \frac{\partial}{\partial y_1^r}, & \left[\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j} \right] &= R_{(1)ij}^{(r)} \frac{\partial}{\partial y_1^r}, \\ \left[\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y_1^j} \right] &= \frac{\partial N_{(1)i}^{(r)}}{\partial y_1^j} \frac{\partial}{\partial y_1^r}, & \left[\frac{\partial}{\partial y_1^i}, \frac{\partial}{\partial y_1^j} \right] &= 0, \end{aligned}$$

where $M_{(1)1}^{(r)}$ and $N_{(1)i}^{(r)}$ are the local coefficients of the nonlinear connection Γ , while

the components $R_{(1)1j}^{(r)}$ and $R_{(1)ij}^{(r)}$ are d -tensors given by the formulas

$$\begin{aligned} R_{(1)1j}^{(r)} &= \frac{\delta M_{(1)1}^{(r)}}{\delta x^j} - \frac{\delta N_{(1)j}^{(r)}}{\delta t}, \\ R_{(1)ij}^{(r)} &= \frac{\delta N_{(1)i}^{(r)}}{\delta x^j} - \frac{\delta N_{(1)j}^{(r)}}{\delta x^i}. \end{aligned} \quad (3.1)$$

In these conditions, working with a basis of vector fields, adapted to the non-linear connection

$$\Gamma = \left(M_{(1)1}^{(i)}, N_{(1)j}^{(i)} \right)$$

on the 1-jet space $E = J^1(\mathbb{R}, M)$, by local computations, we obtain

Theorem 3.1. *The torsion tensor \mathbf{T} of the Γ -linear connection (2.4) is determined by the following adapted torsion d -tensors:*

$$\begin{aligned} h_{\mathbb{R}}\mathbf{T} \left(\frac{\delta}{\delta t}, \frac{\delta}{\delta t} \right) &= 0, \quad h_M\mathbf{T} \left(\frac{\delta}{\delta t}, \frac{\delta}{\delta t} \right) = 0, \quad v\mathbf{T} \left(\frac{\delta}{\delta t}, \frac{\delta}{\delta t} \right) = 0, \\ h_{\mathbb{R}}\mathbf{T} \left(\frac{\delta}{\delta x^j}, \frac{\delta}{\delta t} \right) &= \bar{T}_{1j}^1 \frac{\delta}{\delta t}, \quad h_M\mathbf{T} \left(\frac{\delta}{\delta x^j}, \frac{\delta}{\delta t} \right) = T_{1j}^r \frac{\delta}{\delta x^r}, \\ v\mathbf{T} \left(\frac{\delta}{\delta x^j}, \frac{\delta}{\delta t} \right) &= R_{(1)1j}^{(r)} \frac{\partial}{\partial y_1^r}, \\ h_{\mathbb{R}}\mathbf{T} \left(\frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^i} \right) &= 0, \quad h_M\mathbf{T} \left(\frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^i} \right) = T_{ij}^r \frac{\delta}{\delta x^r}, \\ v\mathbf{T} \left(\frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^i} \right) &= R_{(1)ij}^{(r)} \frac{\partial}{\partial y_1^r}, \\ h_{\mathbb{R}}\mathbf{T} \left(\frac{\partial}{\partial y_1^j}, \frac{\delta}{\delta t} \right) &= \bar{P}_{1(j)}^{1(1)} \frac{\delta}{\delta t}, \quad h_M\mathbf{T} \left(\frac{\partial}{\partial y_1^j}, \frac{\delta}{\delta t} \right) = 0, \\ v\mathbf{T} \left(\frac{\partial}{\partial y_1^j}, \frac{\delta}{\delta t} \right) &= P_{(1)1(j)}^{(r)} \frac{\partial}{\partial y_1^r}, \\ h_{\mathbb{R}}\mathbf{T} \left(\frac{\partial}{\partial y_1^j}, \frac{\delta}{\delta x^i} \right) &= 0, \quad h_M\mathbf{T} \left(\frac{\partial}{\partial y_1^j}, \frac{\delta}{\delta x^i} \right) = P_{i(j)}^{r(1)} \frac{\delta}{\delta x^r}, \\ v\mathbf{T} \left(\frac{\partial}{\partial y_1^j}, \frac{\delta}{\delta x^i} \right) &= P_{(1)i(j)}^{(r)} \frac{\partial}{\partial y_1^r}, \\ h_{\mathbb{R}}\mathbf{T} \left(\frac{\partial}{\partial y_1^j}, \frac{\partial}{\partial y_1^i} \right) &= 0, \quad h_M\mathbf{T} \left(\frac{\partial}{\partial y_1^j}, \frac{\partial}{\partial y_1^i} \right) = 0, \\ v\mathbf{T} \left(\frac{\partial}{\partial y_1^j}, \frac{\partial}{\partial y_1^i} \right) &= S_{(1)(i)(j)}^{(r)(1)(1)} \frac{\partial}{\partial y_1^r}, \end{aligned}$$

where

$$\bar{T}_{1j}^1 = \bar{L}_{1j}^1, \quad T_{1j}^r = -G_{j1}^r, \quad T_{ij}^r = L_{ij}^r - L_{ji}^r, \quad \bar{P}_{1(j)}^{1(1)} = \bar{C}_{1(j)}^{1(1)},$$

$$P_{i(j)}^{r(1)} = C_{i(j)}^{r(1)}, \quad S_{(1)(i)(j)}^{(r)(1)(1)} = C_{(1)(i)(j)}^{(r)(1)(1)} - C_{(1)(j)(i)}^{(r)(1)(1)},$$

$$P_{(1)1(j)}^{(r)(1)} = \frac{\partial M_{(1)1}^{(r)}}{\partial y_1^j} - G_{(1)(j)1}^{(r)}, \quad P_{(1)i(j)}^{(r)(1)} = \frac{\partial N_{(1)i}^{(r)}}{\partial y_1^j} - L_{(1)(j)i}^{(r)},$$

and the d -tensors $R_{(1)1j}^{(r)}$ and $R_{(1)ij}^{(r)}$ are given by (3.1).

Corollary 3.1. *The torsion tensor \mathbf{T} of an arbitrary Γ -linear connection ∇ on the 1-jet space $E = J^1(\mathbb{R}, M)$ is determined by ten effective adapted local torsion d -tensors, which we arrange in the following table:*

	$h_{\mathbb{R}}$	h_M	v
$h_{\mathbb{R}}h_{\mathbb{R}}$	0	0	0
$h_Mh_{\mathbb{R}}$	\bar{T}_{1j}^1	T_{1j}^r	$R_{(1)1j}^{(r)}$
h_Mh_M	0	T_{ij}^r	$R_{(1)ij}^{(r)}$
$vh_{\mathbb{R}}$	$\bar{P}_{1(j)}^{1(1)}$	0	$P_{(1)1(j)}^{(r)(1)}$
vh_M	0	$P_{i(j)}^{r(1)}$	$P_{(1)i(j)}^{(r)(1)}$
vv	0	0	$S_{(1)(i)(j)}^{(r)(1)(1)}$

Example 3.1. In the particular case of the Berwald $\overset{\circ}{\Gamma}$ -linear connection $B\overset{\circ}{\Gamma}$, associated to the semi-Riemannian metrics $h_{11}(t)$ and $\varphi_{ij}(x)$, all torsion d -tensors vanish, except

$$R_{(1)ij}^{(k)} = \mathfrak{A}_{mij}^k y_1^m,$$

where $\mathfrak{A}_{mij}^k(x)$ are classical local curvature tensors of the spatial semi-Riemannian metric $\varphi_{ij}(x)$.

In order to study the curvature of the Γ -linear connection ∇ , we recall that the curvature tensor \mathbf{R} of ∇ is given by the formula

$$\mathbf{R}(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z, \quad \forall X, Y, Z \in \mathfrak{X}(E).$$

Using again a basis of vector fields adapted to the nonlinear connection Γ , with the properties of the Γ -linear connection ∇ , by direct computations, we obtain

Theorem 3.2. *The curvature tensor \mathbf{R} associated to the Γ -linear connection (2.4) is determined by fifteen effective adapted local curvature d -tensors*

$$\mathbf{R}\left(\frac{\delta}{\delta t}, \frac{\delta}{\delta t}\right) \frac{\delta}{\delta t} = 0, \quad \mathbf{R}\left(\frac{\delta}{\delta t}, \frac{\delta}{\delta t}\right) \frac{\delta}{\delta x^i} = 0, \quad \mathbf{R}\left(\frac{\delta}{\delta t}, \frac{\delta}{\delta t}\right) \frac{\partial}{\partial y_1^i} = 0,$$

$$\begin{aligned}
 \mathbf{R}\left(\frac{\delta}{\delta x^k}, \frac{\delta}{\delta t}\right) \frac{\delta}{\delta t} &= \bar{R}_{11k}^1 \frac{\delta}{\delta t}, & \mathbf{R}\left(\frac{\delta}{\delta x^k}, \frac{\delta}{\delta t}\right) \frac{\delta}{\delta x^i} &= R_{i1k}^l \frac{\delta}{\delta x^l}, \\
 \mathbf{R}\left(\frac{\delta}{\delta x^k}, \frac{\delta}{\delta t}\right) \frac{\partial}{\partial y_1^i} &= R_{(1)(i)1k}^{(l)(1)} \frac{\partial}{\partial y_1^l}, \\
 \mathbf{R}\left(\frac{\delta}{\delta x^k}, \frac{\delta}{\delta x^j}\right) \frac{\delta}{\delta t} &= \bar{R}_{1jk}^1 \frac{\delta}{\delta t}, & \mathbf{R}\left(\frac{\delta}{\delta x^k}, \frac{\delta}{\delta x^j}\right) \frac{\delta}{\delta x^i} &= R_{ijk}^l \frac{\delta}{\delta x^l}, \\
 \mathbf{R}\left(\frac{\delta}{\delta x^k}, \frac{\delta}{\delta x^j}\right) \frac{\partial}{\partial y_1^i} &= R_{(1)(i)jk}^{(l)(1)} \frac{\partial}{\partial y_1^l}, \\
 \mathbf{R}\left(\frac{\partial}{\partial y_1^k}, \frac{\delta}{\delta t}\right) \frac{\delta}{\delta t} &= \bar{P}_{11(k)}^{1(1)} \frac{\delta}{\delta t}, & \mathbf{R}\left(\frac{\partial}{\partial y_1^k}, \frac{\delta}{\delta t}\right) \frac{\delta}{\delta x^i} &= P_{i1(k)}^{l(1)} \frac{\delta}{\delta x^l}, \\
 \mathbf{R}\left(\frac{\partial}{\partial y_1^k}, \frac{\delta}{\delta t}\right) \frac{\partial}{\partial y_1^i} &= P_{(1)(i)1(k)}^{(l)(1)(1)} \frac{\partial}{\partial y_1^l}, \\
 \mathbf{R}\left(\frac{\partial}{\partial y_1^k}, \frac{\delta}{\delta x^j}\right) \frac{\delta}{\delta t} &= \bar{P}_{1j(k)}^{1(1)} \frac{\delta}{\delta t}, & \mathbf{R}\left(\frac{\partial}{\partial y_1^k}, \frac{\delta}{\delta x^j}\right) \frac{\delta}{\delta x^i} &= P_{ij(k)}^{l(1)} \frac{\delta}{\delta x^l}, \\
 \mathbf{R}\left(\frac{\partial}{\partial y_1^k}, \frac{\delta}{\delta x^j}\right) \frac{\partial}{\partial y_1^i} &= P_{(1)(i)j(k)}^{(l)(1)(1)} \frac{\partial}{\partial y_1^l}, \\
 \mathbf{R}\left(\frac{\partial}{\partial y_1^k}, \frac{\partial}{\partial y_1^j}\right) \frac{\delta}{\delta t} &= \bar{S}_{1(j)(k)}^{1(1)(1)} \frac{\delta}{\delta t}, & \mathbf{R}\left(\frac{\partial}{\partial y_1^k}, \frac{\partial}{\partial y_1^j}\right) \frac{\delta}{\delta x^i} &= S_{i(j)(k)}^{l(1)(1)} \frac{\delta}{\delta x^l}, \\
 \mathbf{R}\left(\frac{\partial}{\partial y_1^k}, \frac{\partial}{\partial y_1^j}\right) \frac{\partial}{\partial y_1^i} &= S_{(1)(i)(j)(k)}^{(l)(1)(1)(1)} \frac{\partial}{\partial y_1^l},
 \end{aligned}$$

whose local components we arrange in the following table:

	$h_{\mathbb{R}}$	h_M	v
$h_{\mathbb{R}}h_{\mathbb{R}}$	0	0	0
$h_Mh_{\mathbb{R}}$	\bar{R}_{11k}^1	R_{i1k}^l	$R_{(1)(i)1k}^{(l)(1)}$
h_Mh_M	\bar{R}_{1jk}^1	R_{ijk}^l	$R_{(1)(i)jk}^{(l)(1)}$
$vh_{\mathbb{R}}$	$\bar{P}_{11(k)}^{1(1)}$	$P_{i1(k)}^{l(1)}$	$P_{(1)(i)1(k)}^{(l)(1)(1)}$
vh_M	$\bar{P}_{1j(k)}^{1(1)}$	$P_{ij(k)}^{l(1)}$	$P_{(1)(i)j(k)}^{(l)(1)(1)}$
vv	$\bar{S}_{1(j)(k)}^{1(1)(1)}$	$S_{i(j)(k)}^{l(1)(1)}$	$S_{(1)(i)(j)(k)}^{(l)(1)(1)(1)}$

Moreover, by a laborious local computation, we deduce the following result:

Theorem 3.3. *The expressions of the preceding local curvature d -tensors are given by:*

- $h_{\mathbb{R}}$ -components

1. $\bar{R}_{11k}^1 = \frac{\delta \bar{G}_{11}^1}{\delta x^k} - \frac{\delta \bar{L}_{1k}^1}{\delta t} + \bar{C}_{1(r)}^{1(1)} R_{(1)1k}^{(r)}$
2. $\bar{R}_{1jk}^1 = \frac{\delta \bar{L}_{1j}^1}{\delta x^k} - \frac{\delta \bar{L}_{1k}^1}{\delta x^j} + \bar{C}_{1(r)}^{1(1)} R_{(1)jk}^{(r)}$
3. $\bar{P}_{11(k)}^{1(1)} = \frac{\partial \bar{G}_{11}^1}{\partial y_1^k} - \bar{C}_{1(k)/1}^{1(1)} + \bar{C}_{1(r)}^{1(1)} P_{(1)1(k)}^{(r)(1)}$
4. $\bar{P}_{1j(k)}^{1(1)} = \frac{\partial \bar{L}_{1j}^1}{\partial y_1^k} - \bar{C}_{1(k)|j}^{1(1)} + \bar{C}_{1(r)}^{1(1)} P_{(1)j(k)}^{(r)(1)}$
5. $\bar{S}_{1(j)(k)}^{1(1)} = \frac{\partial \bar{C}_{1(j)}^{1(1)}}{\partial y_1^k} - \frac{\partial \bar{C}_{1(k)}^{1(1)}}{\partial y_1^j}$

- h_M -components

6. $R_{i1k}^l = \frac{\delta G_{i1}^l}{\delta x^k} - \frac{\delta L_{ik}^l}{\delta t} + G_{i1}^r L_{rk}^l - L_{ik}^r G_{r1}^l + C_{i(r)}^{l(1)} R_{(1)1k}^{(r)}$
7. $R_{ijk}^l = \frac{\delta L_{ij}^l}{\delta x^k} - \frac{\delta L_{ik}^l}{\delta x^j} + L_{ij}^r L_{rk}^l - L_{ik}^r L_{rj}^l + C_{i(r)}^{l(1)} R_{(1)jk}^{(r)}$
8. $P_{i1(k)}^{l(1)} = \frac{\partial G_{i1}^l}{\partial y_1^k} - C_{i(k)/1}^{l(1)} + C_{i(r)}^{l(1)} P_{(1)1(k)}^{(r)(1)}$
9. $P_{ij(k)}^{l(1)} = \frac{\partial L_{ij}^l}{\partial y_1^k} - C_{i(k)|j}^{l(1)} + C_{i(r)}^{l(1)} P_{(1)j(k)}^{(r)(1)}$
10. $S_{i(j)(k)}^{l(1)} = \frac{\partial C_{i(j)}^{l(1)}}{\partial y_1^k} - \frac{\partial C_{i(k)}^{l(1)}}{\partial y_1^j} + C_{i(j)}^{r(1)} C_{r(k)}^{l(1)} - C_{i(k)}^{r(1)} C_{r(j)}^{l(1)}$

- v -components

11. $R_{(1)(i)1k}^{(l)(1)} = \frac{\delta G_{(1)(i)1}^{(l)(1)}}{\delta x^k} - \frac{\delta L_{(1)(i)k}^{(l)(1)}}{\delta t} + G_{(1)(i)1}^{(r)(1)} L_{(1)(r)k}^{(l)(1)} - L_{(1)(i)k}^{(r)(1)} G_{(1)(r)1}^{(l)(1)} + C_{(1)(i)(r)}^{(l)(1)(1)} R_{(1)1k}^{(r)}$
12. $R_{(1)(i)jk}^{(l)(1)} = \frac{\delta L_{(1)(i)j}^{(l)(1)}}{\delta x^k} - \frac{\delta L_{(1)(i)k}^{(l)(1)}}{\delta x^j} + L_{(1)(i)j}^{(r)(1)} L_{(1)(r)k}^{(l)(1)} - L_{(1)(i)k}^{(r)(1)} L_{(1)(r)j}^{(l)(1)} + C_{(1)(i)(r)}^{(l)(1)(1)} R_{(1)jk}^{(r)}$
13. $P_{(1)(i)1(k)}^{(l)(1)(1)} = \frac{\partial G_{(1)(i)1}^{(l)(1)}}{\partial y_1^k} - C_{(1)(i)(k)/1}^{(l)(1)(1)} + C_{(1)(i)(r)}^{(l)(1)(1)} P_{(1)1(k)}^{(r)(1)}$

$$\begin{aligned}
 \text{14. } P_{(1)(i)j(k)}^{(l)(1)(1)} &= \frac{\partial L_{(1)(i)j}^{(l)(1)}}{\partial y_1^k} - C_{(1)(i)(k)|j}^{(l)(1)(1)} + C_{(1)(i)(r)}^{(l)(1)(1)} P_{(1)j(k)}^{(r)(1)} \\
 \text{15. } S_{(1)(i)(j)(k)}^{(l)(1)(1)(1)} &= \frac{\partial C_{(1)(i)(j)}^{(l)(1)(1)}}{\partial y_1^k} - \frac{\partial C_{(1)(i)(k)}^{(l)(1)(1)}}{\partial y_1^j} + C_{(1)(i)(j)}^{(r)(1)(1)} C_{(1)(r)(k)}^{(l)(1)(1)} - \\
 &\quad - C_{(1)(i)(k)}^{(r)(1)(1)} C_{(1)(r)(j)}^{(l)(1)(1)}.
 \end{aligned}$$

Example 3.2. In the case of the Berwald $\overset{\circ}{\Gamma}$ -linear connection $B\overset{\circ}{\Gamma}$, associated to the pair of semi-Riemannian metrics $(h_{ab}(t), \varphi_{ij}(x))$, all local curvature d -tensors vanish, except

$$R_{ijk}^l = \mathfrak{R}_{ijk}^l, \quad R_{(1)(i)jk}^{(l)(1)} = \mathfrak{R}_{ijk}^l,$$

where $\mathfrak{R}_{ijk}^l(x)$ are classical local curvature tensors of the spatial semi-Riemannian metric $\varphi_{ij}(x)$.

4. RICCI IDENTITIES AND DEFLECTION D-TENSORS

Using the properties of a Γ -linear connection ∇ given by (2.4), and the definitions of its torsion tensor \mathbf{T} and its curvature tensor \mathbf{R} , we can prove the following important result which is used in the Lagrangian geometrical theory of the relativistic time dependent electromagnetism, in order to describe its generalized Maxwell equations. For more details, see [9].

Theorem 4.1. *If X is an arbitrary d -vector field on the 1-jet vector bundle $E = J^1(\mathbb{R}, M)$, locally expressed by*

$$X = X^1 \frac{\delta}{\delta t} + X^i \frac{\delta}{\delta x^i} + X_{(1)}^{(i)} \frac{\partial}{\partial y_1^i},$$

then the following Ricci identities of the Γ -linear connection ∇ are true:

$$(h_{\mathbb{R}}) \left\{ \begin{array}{l}
 X_{/1|k}^1 - X_{|k/1}^1 = X^1 \bar{R}_{11k}^1 - X_{/1}^1 \bar{T}_{1k}^1 - X_{|r}^1 T_{1k}^r - X^1 |_{(r)}^{(1)} R_{(1)1k}^{(r)} \\
 X_{|j|k}^1 - X_{|k|j}^1 = X^1 \bar{R}_{1jk}^1 - X_{|r}^1 T_{jk}^r - X^1 |_{(r)}^{(1)} R_{(1)jk}^{(r)} \\
 X_{/1|}^{(1)} - X_{|}^{(1)}/1 = X^1 \bar{P}_{11(k)}^{(1)} - X_{/1}^1 \bar{C}_{1(k)}^{(1)} - X^1 |_{(r)}^{(1)} P_{(1)1(k)}^{(r)(1)} \\
 X_{|j|}^{(1)} - X_{|}^{(1)}/j = X^1 \bar{P}_{1j(k)}^{(1)} - X_{|r}^1 C_{j(k)}^{r(1)} - X^1 |_{(r)}^{(1)} P_{(1)j(k)}^{(r)(1)} \\
 X^1 |_{(j)}^{(1)} - X^1 |_{(k)}^{(1)} = X^1 \bar{S}_{1(j)(k)}^{(1)(1)} - X^1 |_{(r)}^{(1)} S_{(1)(j)(k)}^{(r)(1)(1)}
 \end{array} \right.$$

$$\begin{aligned}
(h_M) \quad & \left\{ \begin{aligned}
X^i_{/1|k} - X^i_{|k/1} &= X^r R^i_{r1k} - X^i \bar{T}^1_{1k} - X^i_{|r} T^r_{1k} - X^i_{(r)} R^{(r)}_{(1)1k} \\
X^i_{|j|k} - X^i_{|k|j} &= X^r R^i_{rjk} - X^i_{|r} T^r_{jk} - X^i_{(r)} R^{(r)}_{(1)jk} \\
X^i_{/1|k} - X^i_{|k/1} &= X^r P^i_{r1(k)} - X^i \bar{C}^1_{1(k)} - X^i_{(r)} P^{(r)}_{(1)1(k)} \\
X^i_{|j|k} - X^i_{|k|j} &= X^r P^i_{rj(k)} - X^i_{|r} C^r_{j(k)} - X^i_{(r)} P^{(r)}_{(1)j(k)} \\
X^i_{|j|k} - X^i_{|k|j} &= X^r S^i_{r(j)(k)} - X^i_{(r)} S^{(r)}_{(1)(j)(k)}
\end{aligned} \right. \\
(v) \quad & \left\{ \begin{aligned}
X^{(i)}_{(1)/1k} - X^{(i)}_{(1)|k/1} &= X^{(r)}_{(1)} R^{(i)(1)}_{(1)(r)1k} - X^{(i)}_{(1)/1} \bar{T}^1_{1k} - X^{(i)}_{(1)|r} T^r_{1k} - \\
&\quad - X^{(i)}_{(1)|r} R^{(r)}_{(1)1k} \\
X^{(i)}_{(1)/j|k} - X^{(i)}_{(1)|j|k} &= X^{(r)}_{(1)} R^{(i)(1)}_{(1)(r)jk} - X^{(i)}_{(1)/r} T^r_{jk} - X^{(i)}_{(1)|r} R^{(r)}_{(1)jk} \\
X^{(i)}_{(1)/1|k} - X^{(i)}_{(1)|k/1} &= X^{(r)}_{(1)} P^{(i)(1)}_{(1)(r)1(k)} - X^{(i)}_{(1)/1} \bar{C}^1_{1(k)} - \\
&\quad - X^{(i)}_{(1)|r} P^{(r)}_{(1)1(k)} \\
X^{(i)}_{(1)/j|k} - X^{(i)}_{(1)|k|j} &= X^{(r)}_{(1)} P^{(i)(1)}_{(1)(r)j(k)} - X^{(i)}_{(1)/r} C^r_{j(k)} - \\
&\quad - X^{(i)}_{(1)|r} P^{(r)}_{(1)j(k)} \\
X^{(i)}_{(1)/j|k} - X^{(i)}_{(1)|k|j} &= X^{(r)}_{(1)} S^{(i)(1)(1)}_{(1)(r)(j)(k)} - X^{(i)}_{(1)/r} S^{(r)(1)(1)}_{(1)(j)(k)}.
\end{aligned} \right.
\end{aligned}$$

Proof. Let (Y_A) and (ω^A) be the dual bases adapted to the nonlinear connection Γ , where $A \in \left\{1, i, \binom{1}{i}\right\}$, and let $X = X^F Y_F$ be a distinguished vector field on the 1-jet space $E = J^1(\mathbb{R}, M)$. In this context, using the equalities

1. $\nabla_{Y_C} Y_B = \Gamma^F_{BC} Y_F$, 2. $[Y_B, Y_C] = R^F_{BC} Y_F$,
3. $\mathbf{T}(Y_C, Y_B) = \mathbf{T}^F_{BC} Y_F = \{\Gamma^F_{BC} - \Gamma^F_{CB} - R^F_{CB}\} Y_F$,
4. $\mathbf{R}(Y_C, Y_B) Y_A = \mathbf{R}^F_{ABC} Y_F$, 5. $\nabla_{Y_C} \omega^B = -\Gamma^B_{FC} \omega^F$,
6. $[\mathbf{R}(Y_C, Y_B) X] \cdot \omega^B \cdot \omega^C = \{\nabla_{Y_C} \nabla_{Y_B} X - \nabla_{Y_B} \nabla_{Y_C} X - \nabla_{[Y_C, Y_B]} X\} \cdot \omega^B \cdot \omega^C$,

where “ \cdot ” represents the tensorial product “ \otimes ”, we deduce by a direct calculation that

$$X^A_{:B:C} - X^A_{:C:B} = X^F \mathbf{R}^A_{FBC} - X^A_{:F} \mathbf{T}^F_{BC}, \quad (4.1)$$

where “ $:_D$ ” represents one of the local covariant derivatives “ $_{/1}$ ”, “ $_{|j}$ ” or “ $_{(j)}^{(1)}$ ” of the Γ -linear connection ∇ .

Taking into account that indices A, B, C, \dots belong to the set $\left\{1, i, \binom{1}{i}\right\}$, by complicated local computations, identities (4.1) imply the required Ricci identities. \square

Now, let us consider the *canonical Liouville d -tensor field*

$$\mathbf{C} = \mathbf{C}_{(1)}^{(i)} \frac{\partial}{\partial y_1^i} = y_1^i \frac{\partial}{\partial y_1^i}.$$

Definition 4.1. The distinguished tensors defined by the local components

$$\bar{D}_{(1)1}^{(i)} = \mathbf{C}_{(1)/1}^{(i)}, \quad D_{(1)j}^{(i)} = \mathbf{C}_{(1)|j}^{(i)}, \quad d_{(1)(j)}^{(i)(1)} = \mathbf{C}_{(1)|j}^{(i)(1)}$$

are called the *deflection d -tensors attached to the Γ -linear connection ∇ on the 1-jet space $E = J^1(\mathbb{R}, M)$.*

Taking into account the expressions of the local covariant derivatives of the Γ -linear connection ∇ given by (2.4), by a direct calculation, we find

Proposition 4.1. *The deflection d -tensors of the Γ -linear connection ∇ have the expressions:*

$$\begin{aligned} \bar{D}_{(1)1}^{(i)} &= -M_{(1)1}^{(i)} + G_{(1)(r)1}^{(i)(1)} y_1^r, & D_{(1)j}^{(i)} &= -N_{(1)j}^{(i)} + L_{(1)(r)j}^{(i)(1)} y_1^r, \\ d_{(1)(j)}^{(i)(1)} &= \delta_j^i + C_{(1)(r)(j)}^{(i)(1)(1)} y_1^r. \end{aligned}$$

In the sequel, applying the set (v) of the Ricci identities to the components of the canonical Liouville d -tensor field \mathbf{C} , we get

Theorem 4.2. *The deflection d -tensors attached to the Γ -linear connection ∇ on the 1-jet space $E = J^1(\mathbb{R}, M)$ verify the following identities:*

$$\left\{ \begin{array}{l} \bar{D}_{(1)1|k}^{(i)} - D_{(1)k/1}^{(i)} = y_1^r R_{(1)(r)1k}^{(i)(1)} - \bar{D}_{(1)1}^{(i)} \bar{T}_{1k}^1 - D_{(1)r}^{(i)} T_{1k}^r - d_{(1)(r)}^{(i)(1)} R_{(1)1k}^{(r)} \\ D_{(1)j|k}^{(i)} - D_{(1)k|j}^{(i)} = y_1^r R_{(1)(r)jk}^{(i)(1)} - D_{(1)r}^{(i)} T_{jk}^r - d_{(1)(r)}^{(i)(1)} R_{(1)jk}^{(r)} \\ \bar{D}_{(1)1|k}^{(i)(1)} - d_{(1)(k)/1}^{(i)(1)} = y_1^r P_{(1)(r)1(k)}^{(i)(1)(1)} - \bar{D}_{(1)1}^{(i)} \bar{C}_{1(k)}^{1(1)} - d_{(1)(r)}^{(i)(1)} P_{(1)1(k)}^{(r)(1)} \\ D_{(1)j|k}^{(i)(1)} - d_{(1)(k)|j}^{(i)(1)} = y_1^r P_{(1)(r)j(k)}^{(i)(1)(1)} - D_{(1)r}^{(i)} C_{j(k)}^{r(1)} - d_{(1)(r)}^{(i)(1)} P_{(1)j(k)}^{(r)(1)} \\ d_{(1)(j)|k}^{(i)(1)(1)} - d_{(1)(k)|j}^{(i)(1)(1)} = y_1^r S_{(1)(r)(j)(k)}^{(i)(1)(1)(1)} - d_{(1)(r)}^{(i)(1)} S_{(1)(j)(k)}^{(r)(1)(1)}. \end{array} \right.$$

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REFERENCES

- [1] M. Anastasiei and H. Kawaguchi: *A geometrical theory of time dependent Lagrangians, I. Nonlinear connections, II. M-Connections*, Tensor N. S., **48**(1989), 273-282, 283-293.
- [2] M. Anastasiei: *The geometry of time-dependent Lagrangians*, Math. Comput. Modelling, **20**(1994), No. 4-5, 67-81.
- [3] G. S. Asanov: *Jet extension of Finslerian gauge approach*, Fortschritte der Physik, **38**(1990), No. 8, 571-610.

- [4] I. Bucătaru and R. Miron: *Finsler-Lagrange Geometry. Applications to Dynamical Systems*, Editura Academiei Române, Bucharest, 2007.
- [5] C. Frigioiu: Noether's laws of conservation in time dependent Lagrange spaces, in *Lagrange and Hamilton Geometries and Their Applications* (Radu Miron (Ed.)), Fair Partners Publishers, Bucharest, 2004, 125-131.
- [6] R. Miron and M. Anastasiei: *The Geometry of Lagrange Spaces: Theory and Applications*, Kluwer Academic Publishers, 1994.
- [7] M. Neagu: *Jet geometrical objects depending on a relativistic time*, An. Ştiinţ. Univ. "Al. I. Cuza", Iaşi. Secţiunea Matematică (2009), 1-19, in press.
- [8] M. Neagu: *Riemann-Lagrange Geometry on 1-Jet Spaces*, Matrix Rom, Bucharest, 2005.
- [9] M. Neagu: *The geometry of relativistic rheonomic Lagrange spaces*, in Proceedings of Workshop on Global Analysis, Differ. Geom. and Lie Algebras (Grigorios Tsagas (Ed.)), Geometry Balkan Press, Bucharest, 2001, No. 5, 142-168.
- [10] M. Neagu and C. Udrişte: *Torsion, curvature and deflection d-tensors on $J^1(T, M)$* , Balkan J. Geom. Appl., **6**(2001), No. 1, 29-44.
- [11] P. J. Olver: *Applications of Lie Groups to Differential Equations*, Springer-Verlag, 1986.
- [12] D. Saunders: *The Geometry of Jet Bundles*, Cambridge University Press, New York, London, 1989.
- [13] A. Vondra: *Symmetries of connections on fibered manifolds*, Arch. Math. Brno, **30**(1994), No. 2, 97-115.

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