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RADIUS PROBLEMS OF CERTAIN STARLIKE FUNCTIONS

HIRO KOBASHI, HITOSHI SHIRAISHI AND SHIGEYOSHI OWA

Dedicated to Professor Ştefan Mititelu
on the occasion of his seventieth birthday

Abstract. For analytic functions $f(z)$ normalized by $f(0) = f'(0) - 1 = 0$ in the open unit disk $\mathbb{U}$, a class $\mathcal{P}(\beta_1, \beta_2; \lambda)$ of $f(z)$ defined by some conditions with some complex numbers β_1 and β_2 is introduced. The object of the present paper is to consider some radius problems of $\frac{1}{\delta}f(\delta z)$ for $f(z) \in \mathcal{S}_1^*(\alpha)$.

1. INTRODUCTION AND PRELIMINARIES

Let $\mathcal{A}$ be the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of all univalent functions $f(z)$ in $\mathbb{U}$. Also let $\mathcal{S}^*(\alpha)$ be defined by

$$\mathcal{S}^*(\alpha) = \left\{ f(z) \in \mathcal{A} : \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha, 0 \leqq \alpha < 1 \right\}.$$

A function $f(z) \in \mathcal{S}^*(\alpha)$ is said to be starlike of order α in $\mathbb{U}$ (cf. Robertson [3]).

For the class $\mathcal{S}^*(\alpha)$, we introduce the subclass $\mathcal{S}_1^*(\alpha)$ of $\mathcal{S}^*(\alpha)$ by

$$\mathcal{S}_1^*(\alpha) = \left\{ f(z) \in \mathcal{S}^*(\alpha) : \frac{z}{f(z)} = 1 + \sum_{n=1}^{\infty} b_n z^n, b_n = |b_n| e^{in\theta} \right\}.$$

For $f(z) \in \mathcal{A}$, we say that $f(z) \in \mathcal{P}(\beta_1, \beta_2; \lambda)$ if $f(z)$ satisfies $\frac{f(z)}{z} \neq 0$ $(z \in \mathbb{U})$ and

$$\left| \beta_1 \left(\frac{z}{f(z)}\right)'' + \beta_2 \left(\frac{z}{f(z)}\right)''' \right| \leqq \lambda \qquad (z \in \mathbb{U})$$

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for some complex numbers β_1 and β_2, and for some real $\lambda > 0$. Obradović and Ponnusamy [2] have studied the subclass $\mathcal{P}_2(\lambda)$ of $\mathcal{A}$ consisting of $f(z)$ satisfying $\dfrac{f(z)}{z} \neq 0 \quad (z \in \mathbb{U})$ and $\left|\left(\dfrac{z}{f(z)}\right)''\right| \leqq \lambda \quad (z \in \mathbb{U})$, for some real $\lambda > 0$.

Recently, Kobashi, Kuroki, Shiraishi and Owa [1] have considered the class $\mathcal{P}_4(\lambda)$ defined by

$$\left|\left(\frac{z}{f(z)}\right)''''\right| \leqq \lambda \quad (z \in \mathbb{U}).$$

Consider a function $f(z)$ given by $f(z) = \dfrac{z}{(1-z)^\rho} \qquad (\rho \geqq 0)$. We see that

$$\frac{f(z)}{z} = \frac{1}{(1-z)^\rho} \neq 0 \quad (z \in \mathbb{U}),)$$

$$\left|\left(\frac{z}{f(z)}\right)''\right| = \left|\rho(\rho-1)(1-z)^{\rho-2}\right| < \rho(\rho-1)2^{\rho-2} \qquad (\rho \geqq 2)$$

and

$$\left|\left(\frac{z}{f(z)}\right)'''\right| = \left|\rho(\rho-1)(\rho-2)(1-z)^{\rho-3}\right| < \rho(\rho-1)(\rho-2)2^{\rho-3} \qquad (\rho \geqq 3)\ .$$

Therefore, Koebe function $f(z) = \dfrac{z}{(1-z)^2}$ belongs to the classes $\mathcal{P}(1, 0; 2)$ and $\mathcal{P}(0, 1; \lambda)$ for any $\lambda > 0$.

If we consider the function $f(z)$ by $f(z) = \dfrac{z}{\sum_{k=0}^{n} z^k}$, then

$$\begin{aligned}\left|\beta_1\left(\frac{z}{f(z)}\right)'' + \beta_2\left(\frac{z}{f(z)}\right)'''\right| &< |\beta_1|\sum_{k=2}^{n} k(k-1) + |\beta_2|\sum_{k=3}^{n} k(k-1)(k-2)\\ &= \frac{n(n+1)(n-1)(4|\beta_1| + 3(n-2)|\beta_2|)}{12}\ .\end{aligned}$$

This means that $f(z) \in \mathcal{P}(\beta_1, \beta_2; \lambda)$, with

$$\lambda = \frac{n(n+1)(n-1)(4|\beta_1| + 3(n-2)|\beta_2|)}{12}\ .$$

2. MAIN RESULT

To consider our problems for the class $\mathcal{P}(\beta_1, \beta_2; \lambda)$, we need the following lemma.

Lemma 2.1. *If* $f(z) \in \mathcal{S}_1^*(\alpha)$ *and*

$$\frac{z}{f(z)} = 1 + \sum_{n=1}^{\infty} b_n z^n,$$

then we have

$$\sum_{n=1}^{\infty}(n-1+\alpha)\big|b_n\big| \leqq 1-\alpha.$$

Proof. Let $F(z)$ be defined by

$$F(z)=\frac{z}{f(z)}=1+\sum_{n=1}^{\infty}b_n z^n\ .$$

Noting that $b_n=|b_n|e^{in\theta}$ $(n=1,2,3,\ldots)$, we have that

$$\begin{aligned}\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right)&=\mathrm{Re}\left(1-\frac{zF'(z)}{F(z)}\right)\\&=\mathrm{Re}\left(\frac{1-\sum_{n=1}^{\infty}(n-1)b_n z^n}{1+\sum_{n=1}^{\infty}b_n z^n}\right)\\&=\mathrm{Re}\left(\frac{1-\sum_{n=1}^{\infty}(n-1)|b_n|e^{in\theta}z^n}{1+\sum_{n=1}^{\infty}|b_n|e^{in\theta}z^n}\right)>\alpha\quad(z\in\mathbb{U}).\end{aligned}$$

If we consider a point $z=|z|e^{-i\theta}$, then we have

$$\frac{1+\sum_{n=1}^{\infty}(n-1)|b_n||z|^n}{1+\sum_{n=1}^{\infty}|b_n||z|^n}>\alpha.$$

Therefore, letting $|z|\to 1^{-1}$, we obtain that $\sum_{n=1}^{\infty}(n-1+\alpha)|b_n|\leqq 1-\alpha$, and this completes the proof. □

Remark 2.1. If $f(z)\in\mathcal{S}_1^*(\alpha)$, then the inequality

$$\sum_{n=1}^{\infty}(n-1+\alpha)|b_n|\leqq 1-\alpha$$

implies that

$$\sum_{n=1}^{\infty}(n-1)|b_n|^2\leqq 1-\alpha.$$

Further, we need the following lemma.

Lemma 2.2. *Let* $f(z) \in \mathcal{A}$ *and* $\dfrac{z}{f(z)} = 1 + \sum\limits_{n=1}^{\infty} b_n z^n \neq 0 \quad (z \in \mathbb{U})$*. If* $f(z)$ *satisfies*

$$\sum_{n=2}^{\infty} n(n-1)\Big(|\beta_1| + (n-2)|\beta_2|\Big)|b_n| \leqq \lambda, \tag{2.1}$$

for some complex numbers β_1 *and* β_2*, then* $f(z) \in \mathcal{P}(\beta_1, \beta_2; \lambda)$*.*

Proof. We note that

$$\begin{aligned}\left|\beta_1\left(\frac{z}{f(z)}\right)'' + \beta_2\left(\frac{z}{f(z)}\right)'''\right| &= \Big|2(\beta_1 b_2 + 3\beta_2 b_3) + 6(\beta_1 b_3 + 4\beta_2 b_4)z + \cdots \\ &\qquad \cdots + n(n-1)\left(\beta_1 b_n + (n-2)\beta_2 b_{n+1}\right) z^n + \cdots\Big| \\ &< 2|\beta_1||b_2| + \sum_{n=3} n(n-1)\left(|\beta_1| + (n-2)|\beta_2|\right)|b_n| \\ &= \sum_{n=2}^{\infty} n(n-1)\Big(|\beta_1| + (n-2)|\beta_2|\Big)|b_n|.\end{aligned}$$

Thus, if $f(z)$ satisfies inequality (2.1), then $f(z) \in \mathcal{P}(\beta_1, \beta_2; \lambda)$. □

Now, we derive

Theorem 2.1. *Let* $f(z) \in \mathcal{S}_1^*(\alpha)$ *and* $\delta \in \mathbb{C}$ $(|\delta| < 1)$*. Then the function* $\dfrac{1}{\delta} f(\delta z)$ *belongs to the class* $\mathcal{P}(\beta_1, \beta_2; \lambda)$ *for* $0 < |\delta| \leqq |\delta_0(\lambda)|$*, where* $|\delta_0| = |\delta_0(\lambda)|$ *is the smallest root of the equation*

$$|\beta_1| \frac{|\delta|^2\sqrt{2\big(|\delta|^2+2\big)}}{\big(1-|\delta|^2\big)^2}\Big(1-\alpha\Big)^{\frac{1}{2}} + |\beta_2| \frac{|\delta|^3\sqrt{6\big(3|\delta|^4+14|\delta|^2+3\big)}}{\big(1-|\delta|^2\big)^3}\Big(1-\alpha-|b_2|^2\Big)^{\frac{1}{2}} = \lambda \tag{2.2}$$

in $0 < |\delta| < 1$.

Proof. Since $\dfrac{z}{f(z)} \neq 0 \quad (z \in \mathbb{U})$ for $f(z) \in \mathcal{S}_1^*(\alpha)$, if we write

$$\frac{z}{f(z)} = 1 + \sum_{n=1}^{\infty} b_n z^n,$$

then

$$\frac{z}{\frac{1}{\delta}f(\delta z)} = 1 + \sum_{n=1}^{\infty} \delta^n b_n z^n$$

for $0 < |\delta| < 1$.

To show that $\frac{1}{\delta}f(\delta z)\in\mathcal{P}(\beta_1,\beta_2;\lambda)$, we have to prove that

$$|\beta_1|\sum_{n=2}^{\infty}\frac{n!}{(n-2)!}\big|\delta^n b_n\big|+|\beta_2|\sum_{n=3}^{\infty}\frac{n!}{(n-3)!}\big|\delta^n b_n\big|\leqq\lambda \tag{2.3}$$

which is equivalent to (2.1) by means of Lemma 2.2. Indeed, applying the Cauchy-Schwarz inequality for the left hand of (2.3), we obtain that

$$\begin{aligned}
&|\beta_1|\sum_{n=2}^{\infty}\frac{n!}{(n-2)!}\big|\delta^n b_n\big|+|\beta_2|\sum_{n=3}^{\infty}\frac{n!}{(n-3)!}\big|\delta^n b_n\big|\\
&=|\beta_1|\sum_{n=2}^{\infty}\Big(n^2(n-1)|\delta|^{2n}\Big)^{\frac{1}{2}}\Big((n-1)|b_n|^2\Big)^{\frac{1}{2}}\\
&\quad+|\beta_2|\sum_{n=3}^{\infty}\Big(n^2(n-1)(n-2)^2|\delta|^{2n}\Big)^{\frac{1}{2}}\Big((n-1)|b_n|^2\Big)^{\frac{1}{2}}\\
&\leqq|\beta_1|\Big(\sum_{n=2}^{\infty}n^2(n-1)|\delta|^{2n}\Big)^{\frac{1}{2}}\Big(\sum_{n=2}^{\infty}(n-1)|b_n|^2\Big)^{\frac{1}{2}}\\
&\quad+|\beta_2|\Big(\sum_{n=3}^{\infty}n^2(n-1)(n-2)^2|\delta|^{2n}\Big)^{\frac{1}{2}}\Big(\sum_{n=3}^{\infty}(n-1)|b_n|^2\Big)^{\frac{1}{2}}\\
&\leqq|\beta_1|\Big(\sum_{n=2}^{\infty}n^2(n-1)|\delta|^{2n}\Big)^{\frac{1}{2}}\Big(1-\alpha\Big)^{\frac{1}{2}}\\
&\quad+|\beta_2|\Big(\sum_{n=3}^{\infty}n^2(n-1)(n-2)^2|\delta|^{2n}\Big)^{\frac{1}{2}}\Big(1-\alpha-|b_2|^2\Big)^{\frac{1}{2}}\\
&=|\beta_1|\frac{|\delta|^2\sqrt{2\big(|\delta|^2+2\big)}}{\big(1-|\delta|^2\big)^2}\Big(1-\alpha\Big)^{\frac{1}{2}}+|\beta_2|\frac{|\delta|^3\sqrt{6\big(3|\delta|^4+14|\delta|^2+3\big)}}{\big(1-|\delta|^2\big)^3}\Big(1-\alpha-|b_2|^2\Big)^{\frac{1}{2}}.
\end{aligned}$$

Now, we consider the complex number δ $\big(0<|\delta|<1\big)$ such that

$$|\beta_1|\frac{|\delta|^2\sqrt{2\big(|\delta|^2+2\big)}}{\big(1-|\delta|^2\big)^2}\Big(1-\alpha\Big)^{\frac{1}{2}}+|\beta_2|\frac{|\delta|^3\sqrt{6\big(3|\delta|^4+14|\delta|^2+3\big)}}{\big(1-|\delta|^2\big)^3}\Big(1-\alpha-|b_2|^2\Big)^{\frac{1}{2}}=\lambda.$$

Let us define the function $h(|\delta|)$ by

$$\begin{aligned}
h(|\delta|)=&-\lambda|\delta|^6+\Big(3\lambda+|\beta_1|\sqrt{2(|\delta|^2+2)(1-\alpha)}\Big)|\delta|^4\\
&-|\beta_2|\sqrt{6(3|\delta|^4+14|\delta|^2+3)(1-\alpha-|b_2|^2)}|\delta|^3\\
&-\Big(3\lambda+|\beta_1|\sqrt{2(|\delta|^2+2)(1-\alpha)}\Big)|\delta|^2+\lambda=0.
\end{aligned}$$

Noting that $h(0) = \lambda > 0$ and $h(1) = -2\sqrt{30(1-\alpha-|b_2|^2)}|\beta_2| < 0$, we see that $h(|\delta|) = 0$ has a root $|\delta_0| = |\delta_0(\lambda)|$ in $0 < |\delta| < 1$. This completes the proof of the theorem. □

Remark 2.2. In the proof of Theorem 2.1, we calculate

$$\left(\sum_{n=2}^{\infty} n^2(n-1)|\delta|^{2n}\right)^{\frac{1}{2}} = \frac{|\delta|^2\sqrt{2\left(2+|\delta|^2\right)}}{\left(1-|\delta|^2\right)^2}$$

as follows. Note that

$$\begin{aligned}\sum_{n=2}^{\infty} n^2(n-1)t^n &= t^2\left(\sum_{n=2}^{\infty} nt^n\right)'' = t^2\left(\frac{-t^3+2t^2}{(1-t)^2}\right)'' \\ &= t^2\left(\frac{t^3-3t^2+4t}{(1-t)^3}\right)' = \frac{2t^2(t+2)}{(1-t)^4}.\end{aligned}$$

Letting $t = |\delta|^2$, we have

$$\left(\sum_{n=2}^{\infty} n^2(n-1)|\delta|^{2n}\right)^{\frac{1}{2}} = \frac{|\delta|^2\sqrt{2\left(2+|\delta|^2\right)}}{\left(1-|\delta|^2\right)^2} .$$

Further, we prove

$$\left(\sum_{n=2}^{\infty} n^2(n-1)(n-2)^2|\delta|^{2n}\right)^{\frac{1}{2}} = \frac{|\delta|^3\sqrt{6\left(3|\delta|^4+14|\delta|^2+3\right)}}{\left(1-|\delta|^2\right)^3}$$

as follows. Note that

$$\begin{aligned}\sum_{n=2}^{\infty} n^2(n-1)(n-2)^2t^n &= t^3\left(\sum_{n=3}^{\infty} n(n-2)t^n\right)''' = t^3\left(\frac{-t^4+3t^3}{(1-t)^3}\right)''' \\ &= t^3\left(\frac{t^4-4t^3+9t^2}{(1-t)^4}\right)'' = t^3\left(\frac{6t^2+18t}{(1-t)^5}\right)' \\ &= 6t^3\frac{3t^2+14t+3}{(1-t)^6}.\end{aligned}$$

Letting $t = |\delta|^2$, we have

$$\left(\sum_{n=2}^{\infty} n^2(n-1)(n-2)^2|\delta|^{2n}\right)^{\frac{1}{2}} = \frac{|\delta|^3\sqrt{6\left(3|\delta|^4+14|\delta|^2+3\right)}}{\left(1-|\delta|^2\right)^3}.$$

Remark 2.3. If we take $\delta = \frac{1}{2}e^{i\theta}$ in (2.2), then we have

$$\lambda = \frac{2\sqrt{2}}{3}|\beta_1|\sqrt{1-\alpha} + \frac{2\sqrt{642}}{27}|\beta_2|\sqrt{1-\alpha-|b_2|^2}.$$

If we put $\lambda = |\beta_1| = |\beta_2| = 1$, $|b_2| = \dfrac{1}{2}$ and $\alpha = 0$ in (2.1), then we have

$$(1-|\delta|)^2|\delta|^2\sqrt{2(|\delta|^2+2)} + |\delta|^3\sqrt{6(3|\delta|^4+14|\delta|^2+3)}\frac{\sqrt{3}}{2} - \left(1-|\delta|^2\right)^3 = 0.$$

It is easy to see that the above equation has a root $|\delta_0|$ such that $0.3999 < |\delta_0| < 0.4002$.

Finally, we derive

Theorem 2.2. *Let* $f(z) \in \mathcal{P}(\beta_1, \beta_2; \lambda)$ *with* $0 \leqq \beta_1 \leqq \beta_2$*, and let*

$$\frac{z}{f(z)} = 1 + \sum_{n=1}^{\infty} b_n z^n \neq 0 \quad (z \in \mathbb{U}),$$

where $b_n = |b_n|e^{in\pi}$*. Then we have*

$$\sum_{n=3}^{\infty} n(n-1)\left((n-2)\beta_2 - \beta_1\right)|b_n| \leqq \lambda - 2|b_2|\beta_1.$$

Proof. For $f(z) \in \mathcal{P}(\beta_1, \beta_2; \lambda)$, we see that

$$\begin{aligned}
&\left|\beta_1\left(\frac{z}{f(z)}\right)'' + \beta_2\left(\frac{z}{f(z)}\right)'''\right| \\
&= \left|\beta_1\left(\sum_{n=2}^{\infty} n(n-1)|b_n|e^{in\pi}z^{n-2}\right) + \beta_2\left(\sum_{n=3}^{\infty} n(n-1)(n-2)|b_n|e^{in\pi}z^{n-3}\right)\right| \\
&= \left|2\beta_1|b_2| + \sum_{n=3}^{\infty} n(n-1)\left((n-2)\beta_2 + \beta_1 z\right)|b_n|e^{in\pi}z^{n-2}\right| \leqq \lambda
\end{aligned}$$

for all $z \in \mathbb{U}$. If we take a point z such that $z = |z|e^{-i\pi}$, we obtain that

$$2\beta_1|b_2| + \sum_{n=3}^{\infty} n(n-1)\left((n-2)\beta_2 - \beta_1|z|\right)|b_n||z|^{n-2} \leqq \lambda.$$

Therefore, letting $|z| \to 1^{-1}$, we have that

$$\sum_{n=3}^{\infty} n(n-1)\left((n-2)\beta_2 - \beta_1\right)|b_n| \leqq \lambda - 2\beta_1|b_2|,$$

which completes the proof. □

Corollary 2.1. *Let* $f(z) \in \mathcal{P}(\beta_1, \beta_2; \lambda)$ *with* $0 \leqq \beta_1 \leqq \beta_2$*, and let*

$$\frac{z}{f(z)} = 1 + \sum_{n=1}^{\infty} b_n z^n \neq 0 \quad (z \in \mathbb{U}),$$

where $b_n = |b_n|e^{in\pi}$. *Then we have*

$$|b_n| \leqq \frac{\lambda - 2\beta_1|b_2|}{n(n-1)((n-2)\beta_2 - \beta_1)} \qquad (n = 3, 4, 5, \ldots)$$

Example 2.1. Let us consider the function $f(z)$ given by

$$\frac{z}{f(z)} = 1 + \sum_{n=1}^{\infty} b_n z^n$$

with

$$b_n = \frac{\lambda - 2\beta_1|b_2|}{(n-2)n(n-1)^2((n-2)\beta_2 - \beta_1)} e^{in\pi} \qquad (n = 3, 4, 5, \ldots).$$

Then we have that

$$\begin{aligned}\sum_{n=3}^{\infty} n(n-1)((n-2)\beta_2 - \beta_1)|b_n| &= \sum_{n=3}^{\infty} (n-2)(n-1)(\lambda - 2\beta_1|b_2|) \\ &= (\lambda - 2\beta_1|b_2|) \sum_{n=3}^{\infty} \frac{1}{(n-2)(n-1)} \\ &= (\lambda - 2\beta_1|b_2|) \sum_{n=3}^{\infty} \left(\frac{1}{n-2} - \frac{1}{n-1} \right) = \lambda - 2\beta_1|b_2|.\end{aligned}$$

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Kinki University
Departmemt of Mathematics
Higashi-Osaka, Osaka 577-8502, *Japan*
E-mail address: `hero_of_earth_oo1@hotmail.com`

Kinki University
Departmemt of Mathematics
Higashi-Osaka, Osaka 577-8502, *Japan*
E-mail address: `step-625@hotmail.com`

Kinki University
Departmemt of Mathematics
Higashi-Osaka, Osaka 577-8502, *Japan*
E-mail address: `owa@math.kindai.ac.jp`