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DECOMPOSITIONS OF ω-CONTINUITY IN TOPOLOGICAL SPACES

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Abstract. The aim of this paper is to give decompositions of a weaker form of continuity, namely ω-continuity, by providing the concepts of ω_t-sets, ω_{α^*}-sets, ω_t-continuity and ω_{α^*}-continuity.

1. INTRODUCTION

Various interesting problems arise when one considers continuity and generalized continuity. In recent years, the decomposition of continuity is one of the main interest for general topologists. In 1961, Levine [5] obtained a decomposition of continuity which was later improved by Rose [13]. Tong [17] decomposed continuity into α-continuity and $\mathcal{A}$-continuity and showed that his decomposition is independent of Levine's. Recently, Ganster and Reilly [2] have improved Tong's decomposition result and provided a decomposition of $\mathcal{A}$-continuity. Przemski [12] obtained some decomposition of continuity. Hatir et al. [3] also obtained a decomposition of continuity. Recently, Dontchev and Prezemski [1] and Noiri et al. [11] obtained some more decompositions of continuity. In this paper, we obtain decompositions of ω-continuity in topological spaces using αgs-continuity [15], pgs-continuity [19], ω_t-continuity and ω_{α^*}-continuity.

2. PRELIMINARIES

Throughout this paper, (X,τ) and (Y,σ) (simply, X and Y) denote topological spaces on which no separation axioms are assumed. Let S be a subset of a space X. The closure of S and the interior of S are denoted by $\mathrm{Cl}(S)$ and $\mathrm{Int}(S)$, respectively.

We shall recall some definitions used in the sequel.

Definition 2.1. A subset S of a space (X,τ) is called:
(1) a *semi-open set* [6] if $S \subset \mathrm{Cl}(\mathrm{Int}(S))$,
(2) a *pre-open set* [8] if $S \subset \mathrm{Int}(\mathrm{Cl}(S))$,
(3) an *α-set* [10] if $S \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(S)))$,

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(4) a *t-set* [18] if $\mathrm{Int}(S) = \mathrm{Int}(\mathrm{Cl}(S))$,
(5) an *α^*-set* [3] if $\mathrm{Int}(S) = \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(S)))$,
(6) *slc^*-set* [16] if $S = U \cap F$, where U is semi-open and F is closed in (X, τ),
(7) an *ω-closed* [16] (resp. g-closed [7]) set if $\mathrm{Cl}(A) \subset U$, whenever $A \subset U$ and U is semi-open (resp. open) in (X, τ),
(8) an *αgs-closed* [14] (resp. pgs-closed [19]) set if $\alpha cl(A) \subset U$ (resp. $pcl(A) \subset U$), whenever $A \subset U$ and U is semi-open in (X, τ),
(9) an *η^*-set* [4] if $S = U \cap F$, where U is semi-open and F is α-closed in (X, τ),
(10) an *η^{**}-set* [4] if $S = U \cap F$, where U is αgs-open and F is a t-set in (X, τ).

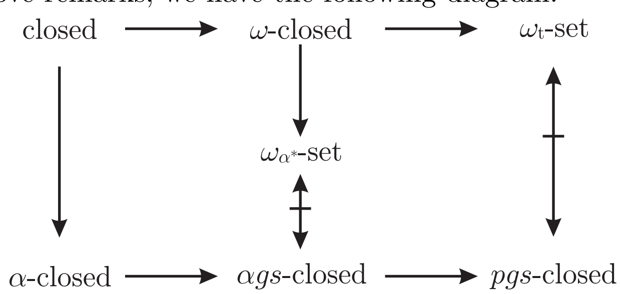
The complements of the above mentioned closed sets are their respective open sets.

For a subset S of a space X, the α-closure (resp. semi-closure, pre-closure) of S, denoted by $\alpha\mathrm{cl}(S)$ (resp. $\mathrm{scl}(S)$, $\mathrm{pcl}(S)$), is the intersection of all α-closed (resp. semi-closed, pre-closed) subsets of X containing S. Dually, the α-interior (resp. semi-interior, pre-interior) of S, denoted by $\alpha\mathrm{int}(S)$ (resp. $\mathrm{sint}(S)$, $\mathrm{pint}(S)$), is the union of all α-open (resp. semi-open, pre-open) subsets of X contained in S.

The following remarks hold:

(1) Every ω-closed set is αgs-closed, but not conversely [14].
(2) Every αgs-closed set is pgs-closed but, not conversely [19].
(3) The concepts of g-closed sets and αgs-closed sets are independent [14].
(4) The concepts of α-closed sets and ω-closed sets are independent [16].

From the above remarks, we have the following diagram:



None of the implications is reversible.

Remark 2.1. [3]

(1) Every t-set is an α^*-set, but not conversely.
(2) An open set need not be an α^*-set.
(3) The union of two α^*-sets need not be an α^*-set.
(4) Arbitrary intersection of α^*-sets is an α^*-set.

Definition 2.2. A function $f\colon (X, \tau) \to (Y, \sigma)$ is said to be

(1) *α-continuous* [9] if for each $V \in \sigma$, $f^{-1}(V)$ is an α-open set in (X, τ),
(2) *ω-continuous* [16] if for each $V \in \sigma$, $f^{-1}(V)$ is an ω-open set in (X, τ),
(3) *αgs-continuous* [15] (resp. pgs-continuous [19]) if for each $V \in \sigma$, $f^{-1}(V)$ is an αgs-open set (resp. a pgs-open set) in (X, τ).

Recently, the following decompositions have been established.

Theorem 2.1. [16] *A function* $f\colon (X,\tau)\to(Y,\sigma)$ *is continuous if and only if it is both* ω*-continuous and* slc^**-continuous.*

Theorem 2.2. [4] *A function* $f\colon (X,\tau)\to(Y,\sigma)$ *is* α*-continuous if and only if it is both* αgs*-continuous and* η^**-continuous.*

Theorem 2.3. [4] *A function* $f\colon (X,\tau)\to(Y,\sigma)$ *is* αgs*-continuous if and only if it is both* pgs*-continuous and* η^{**}*-continuous.*

3. ON ω_T-SETS AND ω_{α^*}-SETS

Definition 3.1. A subset S of a space (X,τ) is called

(1) an ω_t*-set* if $S=U\cap F$, where U is ω-open in X, and F is a t-set in X,
(2) an ω_{α^*}*-set* if $S=U\cap F$, where U is ω-open in X, and F is an α^*-set in X.

The family of all ω_t-sets (resp. ω_{α^*}-sets) in a space (X,τ) is denoted by $\omega_t(X,\tau)$ (resp. $\omega_{\alpha^*}(X,\tau)$).

Proposition 3.1. *Let* S *be a subset of* X*.*

(a) *If* S *is a* t*-set, then* $S\in\omega_t(X,\tau)$*.*
(b) *If* S *is an* α^**-set, then* $S\in\omega_{\alpha^*}(X,\tau)$*.*
(c) *If* S *is an* ω*-open set in* X*, then* $S\in\omega_t(X,\tau)$ *and* $S\in\omega_{\alpha^*}(X,\tau)$*.*

Proof. The proof is obvious. □

Proposition 3.2. *In a space* X*, every* ω_t*-set is an* ω_{α^*}*-set, but not conversely.*

Example 3.1. Let $X=\{a,b,c\}$ and $\tau=\{\varnothing,\{a,b\},X\}$. In (X,τ), the set $\{a,c\}$ is an ω_{α^*}-set but it is not an ω_t-set.

Remark 3.1. The following examples show that

(1) the converse of Proposition 3.1 need not be true.
(2) the concepts of ω_t-sets and pgs-open sets are independent.
(3) the concepts of ω_{α^*}-sets and αgs-open sets are independent.

Example 3.2. Let $X=\{a,b,c\}$ and $\tau=\{\varnothing,\{a\},\{a,b\},X\}$. In (X,τ), the set $\{a\}$ is an ω_t-set but not a t-set and the set $\{a,b\}$ is an ω_{α^*}-set, but not an α^*-set.

Example 3.3. Let $X=\{a,b,c\}$ and $\tau=\{\varnothing,\{b\},\{a,b\},X\}$. In (X,τ), the set $\{a,c\}$ is both an ω_t-set and an ω_{α^*}-set, but it is not an ω-open set.

Example 3.4. In Example 3.1, the set $\{c\}$ is an ω_t-set, but not a pgs-open set, and the set $\{b,c\}$ is a pgs-open set but not an ω_t-set.

Example 3.5. Let $X=\{a,b,c\}$ and $\tau=\{\varnothing,\{a\},\{a,c\},X\}$. In (X,τ), the set $\{b,c\}$ is an ω_{α^*}-set but not an αgs-open set and the set $\{a,b\}$ is an αgs-open set but not an ω_{α^*}-set.

Remark 3.2. (1) The union of two ω_t-sets need not be an ω_t-set.

(2) The union of two ω_{α^*}-sets need not be an ω_{α^*}-set.

In Example 3.3, $\{b\}$ and $\{c\}$ are ω_t-sets, but $\{b\} \cup \{c\} = \{b, c\}$ is not an ω_t-set. It is known [[3], Remark 3.2] that the intersection of any numbers of ω_t-sets belongs to $\omega_t(X, \tau)$.

In Example 3.2, $\{a\}$ and $\{c\}$ are ω_{α^*}-sets, but $\{a\} \cup \{c\} = \{a, c\}$ is not an ω_{α^*}-set. It is known [[18], Proposition 3] that the intersection of any numbers of ω_{α^*}-sets belongs to $\omega_{\alpha^*}(X, \tau)$.

Lemma 3.1. (a) *A subset S of (X, τ) is ω-open* [16] *if and only if $F \subset \mathrm{Int}(S)$, whenever $F \subset S$ and F is semi-closed in X.*

(b) *A subset S of (X, τ) is αgs-open* [19] *if and only if $F \subset \alpha\mathrm{Int}(S)$ whenever $F \subset S$ and F is semi-closed in X.*

(c) *A subset S of a space (X, τ) is pgs-open* [19] *if and only if $F \subset \mathrm{pint}(S)$, whenever $F \subset S$ and F is semi-closed in X.*

Theorem 3.1. *A subset S is ω-open in (X, τ) if and only if it is both αgs-open and an ω_{α^*}-set in (X, τ).*

Proof. Necessity. The proof is obvious.

Sufficiency. Let S be an αgs-open set and an ω_{α^*}-set. Since S is an ω_{α^*}-set, $S = A \cap B$, where A is ω-open and B is an α^*-set. Assume that $F \subset S$, where F is semi-closed in X. Since A is ω-open, by Lemma 3.1(a), $F \subset \mathrm{Int}(A)$. Since S is αgs-open in X, by Lemma 3.1(b),

$$\begin{aligned} F \subset \alpha\mathrm{Int}(S) = S \cap \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(S))) &= (A \cap B) \cap \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A \cap B))) \\ &\subset A \cap B \cap \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A))) \cap \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(B))) \\ &= A \cap B \cap \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A))) \cap \mathrm{Int}(B) \subset \mathrm{Int}(B). \end{aligned}$$

Therefore, we obtain $F \subset \mathrm{Int}(B)$ and hence $F \subset \mathrm{Int}(A) \cap \mathrm{Int}(B) = \mathrm{Int}(S)$. Hence S is ω-open. □

Theorem 3.2. *A subset S is ω-open in (X, τ) if and only if it is both pgs-open and an ω_t-set in (X, τ).*

Proof. Similar to Theorem 3.1. □

4. DECOMPOSITIONS OF ω-CONTINUITY

Definition 4.1. A function $f\colon (X, \tau) \to (Y, \sigma)$ is said to be

(1) *ω_t-continuous* if for each V in σ, $f^{-1}(V) \in \omega_t(X, \tau)$.
(2) *ω_{α^*}-continuous* if for each V in σ, $f^{-1}(V) \in \omega_{\alpha^*}(X, \tau)$.

Proposition 4.1. *For a function $f\colon (X, \tau) \to (Y, \sigma)$, the following implications hold:*

(a) *ω-continuity $\Rightarrow$ ω_t-continuity;*
(b) *ω-continuity $\Rightarrow$ ω_{α^*}-continuity;*
(c) *ω-continuity $\Rightarrow$ αgs-continuity $\Rightarrow$ pgs-continuity.*

The reverse implications in Proposition 4.1 are not true as shown in the following examples.

Example 4.1. Let us consider $X = Y = \{a, b, c\}$, $\tau = \{\varnothing, \{a, b\}, X\}$ and $\sigma = \{\varnothing, \{a\}, \{c\}, \{a, c\}, Y\}$. Let $f\colon (X, \tau) \to (Y, \sigma)$ be the identity map. Then f is ω_t-continuous. However, f is neither ω-continuous nor pgs-continuous.

Example 4.2. Let be given $X = Y = \{a, b, c\}, \tau = \{\varnothing, \{a\}, \{b\}, \{a, b\}, X\}$ and $\sigma = \{\varnothing, \{a\}, \{c\}, \{a, c\}, Y\}$. Let $f\colon (X, \tau) \to (Y, \sigma)$ be the identity map. Then f is ω_{α^*}-continuous. However, f is neither ω-continuous nor αgs-continuous.

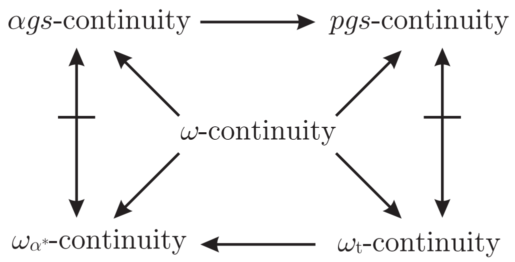
The following example and Example 4.2 show that the concept of ω_{α^*}-continuity and αgs-continuity are independent.

Example 4.3. Let us consider $X = Y = \{a, b, c\}$, $\tau = \{\varnothing, \{a\}, \{b, c\}, X\}$ and $\sigma = \{\varnothing, \{a\}, \{a, b\}, \{a, c\}, Y\}$. Define $f\colon (X, \tau) \to (Y, \sigma)$ as follows: $f(a) = a, f(b) = c$ and $f(c) = b$. Then f is αgs-continuous, but it is not ω_{α^*}-continuous.

Examples 4.1 and 4.4 in the following show that ω_t-continuity and pgs-continuity are independent.

Example 4.4. With the sets $X = Y = \{a, b, c\}$, consider $\tau = \{\varnothing, \{a, b\}, X\}$ and $\sigma = \{\varnothing, \{a\}, \{b\}, \{a, b\}, \{a, c\}, Y\}$. Let $f\colon (X, \tau) \to (Y, \sigma)$ be the identity map. Then f is pgs-continuous, but it is not ω_t-continuous.

From the above results, we have the following diagram.



None of the implications is reversible.

Theorem 4.1. *A function $f\colon (X, \tau) \to (Y, \sigma)$ is ω-continuous if and only if it is both αgs-continuous and ω_{α^*}-continuous.*

Proof. The proof follows immediately from Theorem 3.1. ☐

Theorem 4.2. *A function $f\colon (X, \tau) \to (Y, \sigma)$ is ω-continuous if and only if it is both pgs-continuous and ω_t-continuous.*

Proof. From Theorem 3.2, the proof is immediate. ☐

Corollary 4.1. *A function $f\colon (X, \tau) \to (Y, \sigma)$ is ω-continuous if and only if it is pgs-continuous, η^{**}-continuous and ω_{α^*}-continuous.*

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