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ONE DIMENSIONAL MAXIMUM PRINCIPLE IN GEOMETRIC SETTING

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Dedicated to Professor Ştefan Mititelu
on the occasion of his seventieth birthday

Abstract. Using a geometric setting of the one dimensional maximum principle, we introduce and study some classes of affine functions. Then we extend our study to differential inequalities which are of great interest in Mathematical Physics. Section 1 introduces the idea of maximum principle in geometrical language. Section 2 refreshes the one dimensional maximum principle theory. Section 3 gives the behavior of Γ-linear affine functions at the boundary. Section 4 analyzes important differential inequalities and Section 5 underlines new properties of special functions.

1. BASIC ROOTS OF THE MAXIMUM PRINCIPLE

Consider $(\mathbb{R}, \Gamma)$ as a manifold endowed with the connection Γ and $f\colon \mathbb{R} \to \mathbb{R}$ a C^2-class function. Then $\mathrm{Hess}_\Gamma f = f'' - \Gamma f'$, is the Hessian of f with respect to Γ, for details see [6]. The function f is called:

1) Γ-*linear affine* if $\mathrm{Hess}_\Gamma f = 0$;
2) Γ-*supralinear affine* or Γ-*convex* if $\mathrm{Hess}_\Gamma f \geq 0$;
3) Γ-*sublinear affine* or Γ-*concave* if $\mathrm{Hess}_\Gamma f \leq 0$.

Suppose that f satisfies the differential inequality $\mathrm{Hess}_\Gamma f > 0$, where Γ is a bounded function on the interval $I = (a, b)$. Then the function f cannot have a local maximum point on (a, b) since the relations $f'(x) = 0$, $f''(x) \leq 0$ are contradicted. Consequently, the maximum of f on $[a, b]$ is attained either at a or at b (or both).

This is the simplest version of maximum principle, [1].

Counterexample. The function $f(x) = \cos x$ is Γ-linear affine on the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ with respect to the connection $\Gamma(x) = \cot x$. However, the function f has a maximum at the point $x = 0$. The explanation comes from the fact that

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$\Gamma(x)$ is unbounded at $x = 0$. This explains how $f(x) = \cos x$ satisfies the ODE $\mathrm{Hess}_\Gamma f = f''(x) - \Gamma(x)f'(x) = 0$ and attains an interior maximum without being constant.

Definition 1.1. Let $(\mathbb{R}, g)$ be 1-dimensional Riemannian manifold. A function $f\colon \mathbb{R} \to \mathbb{R}$ is called *geodesically convex* iff

$$\mathrm{Hess}_g f = f'' - \frac{1}{2g} g' f' \geq 0.$$

Theorem 1.1. *If the function f is geodesically convex with respect to g, then it is geodesically convex with respect to the metric $\bar{g} = g + f'^2$.*

Proof. Indeed, we have $\mathrm{Hess}_{\bar{g}} f = \Big(f'' - \dfrac{1}{2g} g' f'\Big)\dfrac{g}{\bar{g}} \geq 0.$ □

2. ONE DIMENSIONAL MAXIMUM PRINCIPLE

Suppose that the function f is Γ-supralinear affine. Then each point of (a, b) is a maximum point of the constant function solution $f(x) = c$ of $\mathrm{Hess}_\Gamma f \geq 0$. It is our aim to prove that the constant function is the only solution which attains its maximum at an interior point of (a, b).

Theorem 2.1. *Suppose the function f is Γ-supralinear affine on (a, b), where $\Gamma(x)$ is a bounded function. If $f(x) \leq M$, for all $x \in (a, b)$, and if $f(x_0) = M$, $x_0 \in (a, b)$, then $f(x) = M$.*

Proof. Supposing that there exists a point $x_1 \in (a, b)$ with $f(x_1) < M$, we obtain a contradiction.

CASE $x_1 < x_0$. Let $h(x) = \exp(-\alpha(x - x_0)) - 1$, with positive parameter α. Remark that $h(x) > 0$, for all $x \in (a, x_0)$, $h(x) < 0$, for all $x \in (x_0, b)$ and $h(x_0) = 0$. On the other hand, $\mathrm{Hess}_\Gamma h = \alpha(\alpha + \Gamma(x)) \exp(-\alpha(x - x_0))$. Since $\Gamma(x)$ is bounded, we can select α as $\alpha > \sup\limits_{x\in(a,b)} \{-\Gamma(x)\}$. Then $\mathrm{Hess}_\Gamma h > 0$, for all $x \in (a, b)$.

Now, we introduce the variation $\bar{f}(x) = f(x) + \varepsilon h(x)$, where ε is selected by $0 < \varepsilon < \dfrac{M - f(x_1)}{h(x_1)}$. This possibility appears because $f(x_1) < M$ and $h(x_1) > 0$.

Since $h(x)$ is negative for $x_0 < x < b$, we have $h(x) < M$, for all $x \in (x_0, b)$. On the other hand, $\bar{f}(x_1) = f(x_1) + \varepsilon h(x_1) < f(x_1) + M - f(x_1) = M$. Now $\bar{f}(x_0) = f(x_0) + \varepsilon h(x_0) = M$.

Since $\bar{f}(x) < M$, for all $x \in (x_0, b)$, $\bar{f}(x_1) < M$ and $\bar{f}(x_0) = M$, it follows that $\bar{f}(x)$ attains a maximum value greater than or equal to M on (x_1, b). On the other hand, $\mathrm{Hess}_\Gamma \bar{f} = \mathrm{Hess}_\Gamma f + \varepsilon \mathrm{Hess}_\Gamma h > 0$, and so $\bar{f}$ cannot attain its maximum at an interior point in (x_1, b). Contradiction. □

Corollary 2.1. *Suppose the function* f *is* Γ*-sublinear affine on* (a, b)*, with* $\Gamma(x)$ *bounded function. If* $f(x) \geq m$ *on* (a, b) *and if* $f(x_0) = m$*,* $x_0 \in (a, b)$*, then* $f(x) = m$*.*

3. BEHAVIOR OF Γ-LINEAR AFFINE FUNCTIONS AT THE BOUNDARY

The Γ-linear affine functions have important properties at the boundary.

Theorem 3.1. *Consider that* f *is a non-constant* Γ*-linear affine function and let* $f'(b)$ *be the left derivative of* f *at* b*. Suppose that the connection* $\Gamma(x)$ *is bounded on each closed subinterval* $[a', b']$ *of* (a, b)*. If the maximum of* f *is attained at* $x = b$ *and* $\Gamma(x)$ *is bounded above at* $x = b$*, then* $f'(b) > 0$*.*

Proof. Suppose ODE $\mathrm{Hess}_\Gamma f = 0$, $f(b) = M$, $f(x) \leq M$, for all $x \in (a, b)$, and exists $x_1 \in (a, b)$ such that $f(x_1) < M$. Define $h(x) = \exp(-\alpha(x - b))$, with $\alpha > 0$. Again, let us consider $\alpha > \sup\limits_{x \in (a,b)} \{-\Gamma(x)\}$. For $x \in (x_1, b)$, $\mathrm{Hess}_\Gamma h > 0$. We build the variation $\bar{f}(x) = f(x) + \varepsilon h(x)$, $0 < \varepsilon < \dfrac{M - f(x_1)}{g(x_1)}$. Since $\mathrm{Hess}_\Gamma h(x) > 0$, for all $x \in (x_1, b)$, the maximum value of h must be attained either at x_1 or b. But $\bar{f}(b) = M > f(x_1)$, and so b is a maximum point. Then the left derivative of $\bar{f}$ at b cannot be negative, since otherwise $\bar{f}(b - \delta) > \bar{f}(b)$. So, $\bar{f}'(b) = f'(b) + \varepsilon h'(b) \geq 0$. Since $h'(b) = -\alpha < 0$, it follows $f'(b) > 0$. ☐

Corollary 3.1. *A nonconstant* Γ*-linear affine function cannot have a horizontal point of inflection* $x_0 \in (a, b)$*.*

Proof. Suppose $\mathrm{Hess}_\Gamma f = 0$ hold true. If we accept that there exists a horizontal inflection point $x_0 \in (a, b)$, then either on (a', x_0) or on (x_0, b') we have $f(x_0) = M$, $f(x) \leq M$, for all $x \in (a', x_0)$ or for all $x \in (x_0, b')$. Also, $f'(x_0) = 0$. Contradiction to Theorem 3.1. ☐

4. EXTENDED ONE DIMENSIONAL MAXIMUM PRINCIPLE

Let us analyze some complementary differential inequalities which can be introduced in theoretical problems of Mathematical Physics, [5].

Theorem 4.1. *Suppose that* f *satisfies the weak differential inequality*

$$\mathrm{Hess}_\Gamma f + h(x) f \geq 0 \tag{4.1}$$

and ℓ *satisfies the weak differential inequality*

$$\mathrm{Hess}_\Gamma \ell + h(x) \ell \leq 0, \tag{4.2}$$

with $h(x)$ *bounded and* $\Gamma(x)$ *bounded above.*

If f satisfies the differential inequality (4.1), *then for any sufficiently short interval* $[a,b]$, *there exists a strictly positive function* ℓ *on* $[a,b]$ *satisfying* (4.2) *on* (a,b).

If f is a function which satisfies (4.1) *in* (a,b), *then* $\frac{f}{\ell}$ *has the properties:*

1) *If* $\frac{f}{\ell}$ *has a nonnegative maximum value* M *at a interior point* $x_0 \in (a,b)$, *then* $\frac{f}{\ell}(x) = M$;

2) *The right derivative at the point* b, $\left(\frac{f}{\ell}\right)'(b)$, *exists and if* $\left(\frac{f}{\ell}\right)(b) = M > 0$, *then* $\left(\frac{f}{\ell}\right)'(b) > 0$;

3) *If* $f \geq 0$, *then* $\frac{f}{\ell}$ *is increasing.*

To give a proof of Theorem 4.1, we need two lemmas.

Lemma 4.1. *If f satisfies the differential inequality* (4.1) *on the interval* (a,b) *with* $h(x) \leq 0$, *if* Γ *and* h *are bounded on every closed subinterval, and if* f *assumes a non-negative maximum value* M *at an interior point* c, *then* $f(x) = M$.

Proof. Almost the same as the proof of Theorem 2.1. Here we choose α such that $\alpha^2 - \alpha|\Gamma(x)| + h(x) > 0$. □

Lemma 4.2. *Suppose that the function f is a non-constant solution of the differential inequality* (4.1) *whose left derivative at b exists, and that the conditions of Lemma* 4.1 *apply. If* $f(b) = M \geq 0$, $f(x) \leq M$, $x \in (a,b)$ *and if* $-\Gamma(x) - (b-x)h(x)$ *is bounded above at* $x = b$, *then* $f'(b) > 0$.

Proof. Almost the same as the proof of Theorem 3.1. Of course

$$\begin{aligned}(\mathrm{Hess}_\Gamma + h)(e^{-\alpha(x-b)} - 1) &= e^{-\alpha(x-b)}(\alpha^2 + \alpha\Gamma(x) + h(x)\big(1 - e^{\alpha(x-b)}\big)\\ &\leq e^{-\alpha(x-b)}\big(\alpha^2 + \alpha\Gamma(x) + \alpha(b-x)h(x)\big).\end{aligned}$$

This completes the proof. □

Now we are able to prove Theorem 4.1. Let $k = \frac{f}{\ell}$. Then

$$\begin{aligned}0 &\leq \mathrm{Hess}_\Gamma f + hf = \mathrm{Hess}_\Gamma(k\ell) + hk\ell\\ &= \ell k'' + (2\ell' - \Gamma(x)\ell)k' + (\ell'' - \Gamma(x)\ell' + h\ell)k.\end{aligned}$$

But $\ell > 0$. Then

$$k'' + \Big(2\frac{\ell'}{\ell} - \Gamma(x)\Big)k' + \frac{1}{\ell}\Big(\ell'' - \Gamma(x)\ell' + h\ell\Big)k \geq 0.$$

Taking $\ell_1(x) = 2\frac{\ell'}{\ell} - \Gamma(x)$ and $h_1(x) = \frac{1}{\ell}\Big(\ell'' - \Gamma(x)\ell' + h\ell\Big)$, the conditions in Lemma 4.1 and Lemma 4.2 are satisfied. To prove the existence of ℓ, we remark that $\ell(x) = 1 - \beta(x-a)^2$, β being a suitable determined constant, satisfies our requirements. We find

$$\mathrm{Hess}_\Gamma \ell + h\ell = -2\beta(1 - \Gamma(x)(x-a) + \frac{1}{2}h(x)(x-a)^2) + h(x).$$

Since $-\Gamma$ and h are bounded below, there exists $-\Gamma$ and H such that $\Gamma(x) \leq \Gamma$ and $h(x) \geq H$, for all $x \in (a,b)$. Suppose that a and b are close enough so that

$$\beta \geq \frac{1}{2}\frac{h(x)}{1-(x-a)\Gamma + \frac{1}{2}(x-a)^2 H}.$$

It follows $\mathrm{Hess}_\Gamma \ell + h\ell \leq 0$ on (a,b). If $b-a$ is also small enough so that $\beta(b-a)^2 < 1$, then $\ell(x) = 1 - \beta(x-a)^2 > 0$, on $[a,b]$.

To prove that $\frac{f}{\ell}$ is increasing, we write (4.1) and (4.2) as

$$f'' - \Gamma f' + hf \geq 0, \quad \ell'' - \Gamma\ell' + h\ell \leq 0,$$

multiply the first by ℓ and the second by f. Subtracting, we find

$$f''\ell - \ell'' f - \Gamma f'\ell + \Gamma\ell' f \geq 0.$$

Consequently, we get $\frac{d}{dx}(f'\ell - \ell' f) \geq (f'\ell - \ell' f)\Gamma$, or $f'\ell - \ell' f \geq \exp\left(\int_0^x \Gamma(\xi)d\xi\right)$, or

$$\frac{d}{dt}\Big(\frac{f}{\ell}\Big) \geq \frac{1}{\ell^2}\exp\left(\int_0^x \Gamma(\xi)d\xi\right).$$

Theorem 4.2. *If x_0 is a maximum point of a function f satisfying* (4.1)*, then $h(x_0)f(x_0) \geq 0$, while if x_0 is a minimum point of a function ℓ satisfying* (4.2)*, then $h(x_0)f(x_0) \leq 0$.*

5. PROPERTIES OF SPECIAL FUNCTIONS

The most important special functions are known as: Bessel functions, Hermite functions, Legendre functions, Laguerre functions, Chebyshev functions etc., [10]. The zeros of these functions are important in applied and theoretical sciences such as: approximation theory, electromagnetism, astronomy, fluid dynamics, control theory, learning control systems, neural networks. The origin of their study can be traced back to Bernoulli, Bessel, Laplace, Laguerre, Legendre and Chebyshev.

Theorem 5.1. *Suppose that $y(x)$ is a nonzero solution of the* ODE

$$y'' - \Gamma(x)y' + h(x)y = 0,$$

where $\Gamma(x)$ and $h(x)$ are bounded functions. If $y(a) = 0$ and if $y(x)$ has zeros at the right of a, then the first of these, denoted by a^*, is called the conjugate point of

a. In these conditions, there exists $\ell(x)$ such that Theorem 4.1 *holds on the interval $[a,b]$ if and only if $b < a^*$.*

Proof. We use $\ell(x) = y(x) + \varepsilon\,(2 - \exp(\alpha(x-a))$. Without loss of generality, we assume that $y(x)$ is strictly positive on the interval (a, a^*). Now assume that we have a function ℓ which is positive on the whole interval $[a, a^*]$. Then $\dfrac{y}{\ell}$ is zero both at a and a^*, hence has a maximum on (a, a^*). But then by Theorem 4.1, ℓ cannot satisfy $\mathrm{Hess}_\Gamma \ell + h\ell \leq 0$.

If $a < b < a^*$, since $y(x)$ is bounded below on any subinterval $[c,b]$ in (a,a^*), then for sufficiently small $\varepsilon > 0$, $\ell(x) = y(x) + \varepsilon\,(2 - \exp(\alpha(x-a))$ is positive on $[a,b]$. The value α is selected such that $(\mathrm{Hess}_\Gamma + h)\,(2 - \exp(\alpha(x-a))) \leq 0$ in (a,b), and ℓ is a function for which Theorem 4.1 holds. ☐

Definition 5.1. The function f is said to be Γ-*conex* along the solution of ODE $y'' - \Gamma y' + hy = 0$ if its composition with any solution of the ODE is a Γ-convex function.

Theorem 5.2. *The functionf is Γ-convex along the solution of* ODE $y''-\Gamma y'+hy = 0$ *if and only if $f''y'^2 - f'hy \geq 0$.*

- CASE STUDY. Consider the equation of Bessel

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0, \quad x \in (-\infty, 0) \cup (0, \infty),$$

ν being a real parameter.

1) If $\nu \neq n$, $n \in \mathbb{N}$, then the general solution of the Bessel ODE is

$$y(x) = aJ_\nu(x) + bJ_{-\nu}(x).$$

2) The general solution of the Bessel ODE is

$$y(x) = aJ_\nu(x) + bY_{-\nu}(x),$$

where J_ν is the Bessel function of the first kind, while $Y_\nu(x)$ is the Bessel function of the second kind. This function is defined as

$$\begin{aligned} Y_\nu(x) &= \frac{1}{\sin \nu\pi}\left(J_\nu(x)\cos\nu\pi - J_{-\nu}(x)\right), \quad \nu \neq n \\ Y_n(x) &= \lim_{\nu\to\infty} Y_\nu(x). \end{aligned}$$

Corollary 5.1. *Let be given the reals a and b.*

1) *There exist c and d such that $\dfrac{aJ_\nu(x) + bJ_{-\nu}(x)}{cJ_\nu(x) + dJ_{-\nu}(x)}$ satisfies the conditions in Theorem* 4.1 *on a certain interval.*

2) *There exist c and d such that $\dfrac{aJ_\nu(x) + bY_\nu(x)}{cJ_\nu(x) + dJ_{-\nu}(x)}$ satisfies the conditions in Theorem* 4.1 *on a certain interval.*

Open problem 5.1. Which is the significance of the previous theory on Bessel differential inequalities, with parameter ν,

$$x^2y'' + xy' + (x^2 - \nu^2)y \geq 0, \quad x^2z'' + xz' + (x^2 - \nu^2)z \leq 0\,?$$

Here $\Gamma(x) = -\dfrac{1}{x}$ and $h(x) = 1 - \dfrac{\nu^2}{x^2}$.

Open problem 5.2. Analyze the following PDEs:

1) metric flow in the direction of a Hessian [4],

$$\frac{\partial g}{\partial t}(t,x) = \frac{d^2 f}{dx^2}(x) - \frac{1}{2g(t,x)}\frac{\partial g}{\partial x}(t,x)\frac{df}{dx}(x);$$

2) metric dynamics in the direction of a Hessian,

$$\frac{\partial^2 g}{\partial t^2}(t,x) = \frac{d^2 f}{dx^2}(x) - \frac{1}{2g(t,x)}\frac{\partial g}{\partial x}(t,x)\frac{df}{dx}(x);$$

3) minus gradient flow,

$$\dot{x}(t) + \frac{1}{g(x(t))}f'(x(t)) = 0.$$

CONCLUSIONS

This paper reinforces the theory of special functions via some geometrical concepts: metric flows, metric dynamics, and ordinary differential inequalities [2], [3], [6]÷[9]. The original results include geometric setting of one dimensional maximum principle [1], [5], [10], geodesic convex functions, complementary differential inequalities, zeros of special functions and convexity along solutions of differential equations. Important open problems are stated, proposing the study of PDE metric flow in the direction of Hessian, of PDE metric dynamics in the direction of Hessian and of minus gradient flow ODE. The results included in this paper completes some recent studies obtained by our research group and published in [2], [3], [7], [8], [9].

REFERENCES

[1] G. K. Ambler: *The maximum principle in eliptic equations*, Final Year Project, University of Bristol, October 5, 1998.

[2] G. Bercu, C. Corcodel and M. Postolache: *On a study of distinguished structures of Hessian type on pseudo-Riemannian manifolds*, J. Adv. Math. Stud., **2**(2009), No. 1, 1-16.

[3] C. Corcodel and C. Udrişte: *Evolution metrics and geometric vector fields*, Proc. 9th Int. Conf. Syst. Sci. Sim. Engng. (ICOSSSE-10), Iwate, Japan, October 4-6, 2010 (in printing).

[4] D. Glickenstein, *Introduction to Flows on Riemannian Metrics*, Math. 538, Springer, 2009.

[5] M. H. Protter and H. F. Weinberger: *Maximum Principles in Differential Equations*, Springer-Verlag, 1984.

[6] C. Udrişte: *Convex Functions and Optimization Methods on Riemannian Manifolds*, MAIA 297, Kluwer, 1994.

[7] C. Udrişte: *Riemannian convexity in programming (II)*, Balkan J. Geom. Appl., **1**(1996), No. 1, 99-109.

[8] C. Udrişte and G. Bercu: *Riemannian Hessian metrics*, Analele Universităţii Bucureşti, **55**(2005), No. 1, 189-204.

[9] C. Udrişte, G. Bercu and M. Postolache: 2*D Hessian Riemannian manifolds*, J. Adv. Math. Stud., **1**(2008), No. 1-2, 135-142.

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