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EINSTEIN-LIKE CONDITIONS AND COSYMPLECTIC GEOMETRY

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ABSTRACT. We prove that every Einstein compact almost $\mathcal{C}$-manifold M^{2n+s} whose Reeb vector fields are Killing is a $\mathcal{C}$-manifold. Then we extend this result considering some generalizations of the Einstein condition (η-Einstein, generalized quasi Einstein, etc.). Moreover, we find some topological properties of compact almost $\mathcal{C}$-manifolds under the assumption that the Ricci tensor is transversally positive definite and the Reeb vector fields are Killing, namely we prove that the first Betti number is s and the first fundamental group is isomorphic to $\mathbb{Z}^s$. Finally, a splitting theorem for cosymplectic manifolds is found.

1. INTRODUCTION

In [19] S. I. Goldberg conjectured that any compact Einstein almost Kählerian manifold is necessarily Kählerian, giving rise to a well-known problem which is still unsolved, the so-called "Goldberg conjecture". The best step toward proving the conjecture has been given in 1987 by K. Sekigawa, who confirmed the Goldberg conjecture under the assumption of non-negative scalar curvature (cf. [30]).

The odd dimensional counterparts of Kählerian manifolds are, from different points of view, Sasakian manifolds and cosymplectic manifolds, so one can ask for a Goldberg-like conjecture for these manifolds. In 2001 C. P. Boyer and K. Galicki (cf. [8]) have investigated in this direction for Sasakian geometry, proving that every compact Einstein K-contact manifold is Sasakian. More recently (cf. [2]) V. Apostolov, T. Draghici and A. Moroianu have given an alternative proof of the same result.

The corresponding result for cosymplectic manifolds is one of the motivations of this paper. Thus we prove that every Einstein compact almost cosymplectic manifold whose Reeb vector field is Killing is a cosymplectic manifold. More general, we prove such a result in the context of f-structures and under the assumption that the metric is η-Einstein. However the proof of the Goldberg-like conjecture in this last

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more general setting leads us to study some interesting topological properties of $\mathcal{C}$-manifolds. For instance, under the assumptions that the Ricci tensor is transversally positive definite and the Reeb vector fields are Killing, we prove that the first Betti number of a compact almost $\mathcal{C}$-manifold M^{2n+s} is s and $\pi_1(M^{2n+s})$ is isomorphic to $\mathbb{Z}^s$. As a consequence we deduce the existence of odd-dimensional smooth manifolds which admit a $\mathcal{C}$-structure but no Sasakian structures. Conversely, we show that any compact almost $\mathcal{C}$-manifold M^{2n+s}, whose first Betti number is s and such that each Reeb vector field is Killing, necessarily fibres over the flat torus $\mathbb{T}^s$.

Finally, in the last section of the paper we shall discuss some counterexamples and make some final remarks.

2. PRELIMINARIES

An *almost contact manifold* is a $(2n+1)$-dimensional manifold M^{2n+1} which carries a field ϕ of endomorphisms of the tangent spaces, a vector field ξ, called *Reeb vector field*, and a 1-form η satisfying $\phi^2 = -I + \eta \otimes \xi$, $\eta(\xi) = 1$. From the definition it follows that $\phi\xi = 0$, $\eta \circ \phi = 0$ and the $(1,1)$-tensor field ϕ has constant rank $2n$ (cf. [6]). An almost contact metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ is said to be *normal* if the tensor field $N = [\phi, \phi] + 2d\eta \otimes \xi$ vanishes identically. It is known that any almost contact manifold $(M^{2n+1}, \phi, \xi, \eta)$ admits a Riemannian metric g such that

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$$

for all $X, Y \in \Gamma(TM^{2n+1})$. This metric g is called a *compatible metric* and the manifold M^{2n+1} together with the structure (ϕ, ξ, η, g) is called an *almost contact metric manifold*. Setting $\mathcal{D} = \ker(\eta)$, one can see that the tangent bundle TM^{2n+1} splits as the orthogonal sum $TM^{2n+1} = \mathcal{D} \oplus \mathbb{R}\xi$. The 2-form Φ on M^{2n+1} defined by $\Phi(X, Y) = g(X, \phi Y)$ is called the *fundamental 2-form* of the almost contact metric manifold M^{2n+1}. Almost contact metric manifolds such that $d\eta = \Phi$ are called *contact metric manifolds* and almost contact metric manifolds such that both η and Φ are closed are called *almost cosymplectic manifolds*. Adding the normality condition one has Sasakian and cosymplectic manifolds, respectively. It is known (cf. [6]) that an almost contact metric manifold is cosymplectic if and only if $\nabla\phi = 0$, ∇ denoting the Levi-Civita connection. Another fundamental property of almost cosymplectic manifolds is that the distribution $\mathcal{D}$ is integrable. Thus $\mathcal{D}$ defines a $2n$-dimensional foliation of M^{2n+1}. Every leaf of this foliation is a minimal almost Kählerian manifold with the almost complex structure given by the restriction of ϕ to the leaf.

In any almost cosymplectic manifold $\nabla_\xi \xi = 0$ and ξ is ∇-parallel if and only if it is a Killing vector field. This occurs in particular in any cosymplectic manifold. In fact in dimension 3 also the converse holds: if the Reeb vector field of a 3-dimensional almost cosymplectic manifold M^{2n+1} is Killing then M^{2n+1} is cosymplectic (cf. [20]).

The main curvature properties of cosymplectic manifolds were studied by S. Goldberg and K. Yano in [20]. In particular, they proved that in any cosymplectic manifold $R(\phi X, \phi Y) = R(X, Y)$ for all $X, Y \in \Gamma(TM^{2n+1})$. From this result it easily follows that

$$\mathrm{Ric}(\phi X, \phi Y) = \mathrm{Ric}(X, Y) \tag{2.1}$$

for any $X, Y \in \Gamma(TM^{2n+1})$, where Ric denotes the Ricci tensor of (M^{2n+1}, g). In particular, we have that $\mathrm{Ric}(X, \xi) = \mathrm{Ric}(\phi X, \phi\xi) = 0$ and $\mathrm{Ric}(X, \phi Y) = \mathrm{Ric}(\phi X, -Y + \eta(Y)\xi) = -\mathrm{Ric}(\phi X, Y)$. This last property implies that the tensor

$$\rho(X, Y) = \mathrm{Ric}(X, \phi Y) \tag{2.2}$$

is in fact a 2-form. Note that, since ξ is Killing and $\mathcal{L}_\xi \phi = 0$, the metric g and the tensor field ϕ locally project along the leaves of the 1-dimensional foliation defined by ξ, respectively, onto a Riemannian metric g' and a tensor field J' such that (J', g') is a Kählerian structure on the space of leaves $M'^{2n} = M^{2n+1}/\xi$. Then, as easily follows from the O'Neill equations ([15]), also the Ricci tensor of the cosymplectic manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ projects onto the Ricci tensor Ric' of M'^{2n}. This implies, in turn, that the above defined 2-form ρ is a foliated object, i.e. constant along the leaves of the foliation defined by ξ, and locally it is just the pull-back of the Ricci-form ρ' of the Kahlerian structure (J', g'). Then, since ρ' is closed (cf. [23]), also ρ is a closed 2-form.

The notion of almost contact metric manifold can be generalized and extended in the context of f-manifolds. Let M^{2n+s} be a manifold of dimension $2n+s$, $s \geq 1$. We say that M^{2n+s} has a *metric f-structure with complemented frames* (cf. [21]) if M^{2n+s} admits a tensor field f of type $(1,1)$ and constant rank $2n$, s vector fields $\xi_1, \dots, \xi_s$ and 1-forms $\eta_1, \dots, \eta_s$ satisfying

$$f^2 = -I + \sum_{\alpha=1}^{s} \eta_\alpha \otimes \xi_\alpha, \quad \eta_\alpha(\xi_\beta) = \delta_{\alpha\beta}, \quad f\xi_\alpha = 0, \quad \eta_\alpha \circ f = 0,$$

and a Riemannian metric g such that

$$g(fX, fY) = g(X, Y) - \sum_{\alpha=1}^{s} \eta_\alpha(X)\, \eta_\alpha(Y) \tag{2.3}$$

for all vector fields X, Y on M^{2n+s}. Note that, as an immediate consequence of (2.3), we have $\eta_\alpha(X) = g(X, \xi_\alpha)$ for all $X \in \Gamma(TM^{2n+s})$ and each $\alpha \in \{1, \dots, s\}$. Let F be the 2-form defined by $F(X, Y) = g(X, fY)$. Then $(M^{2n+s}, f, \xi_\alpha, \eta_\alpha, g)$, or simply M^{2n+s}, is said to be an *almost* $\mathcal{C}$*-manifold* if the forms F and η_α are closed for all $\alpha \in \{1, \dots, s\}$. Note that $\eta_1 \wedge \dots \wedge \eta_s \wedge F^n$ never vanishes, so that the manifold is orientable. When an almost $\mathcal{C}$-manifold is *normal*, in the sense that the tensor field $N = [f, f] + 2\sum_{\alpha=1}^{s} d\eta_\alpha \otimes \xi_\alpha$ vanishes identically, we say that M^{2n+s} is a $\mathcal{C}$*-manifold* (cf. [5]). In terms of the Levi-Civita connection, a necessary and sufficient condition for a metric f-manifold with complemented frames to be a $\mathcal{C}$-manifold is that $\nabla f = 0$ ([5]). Note that for $s = 1$ we reobtain the definition of (almost) cosymplectic structure.

As proven in [24], in an almost $\mathcal{C}$-manifold we have $\nabla_{\xi_\alpha}\xi_\beta = 0$ for all $\alpha, \beta \in \{1, \dots, s\}$. In particular $[\xi_\alpha, \xi_\beta] = 0$, so that the distribution spanned by $\xi_1, \dots, \xi_s$ is integrable and defines a totally geodesic s-dimensional foliation of M^{2n+s} denoted by $\mathcal{F}$. On the other hand, since $d\eta_1 = \cdots = d\eta_s = 0$, the equations $\eta_1 = 0, \dots, \eta_s = 0$ define a $2n$-dimensional foliation $\mathcal{D}$ which is orthogonal to $\mathcal{F}$. The leaves of $\mathcal{D}$ are minimal almost Kählerian manifolds if M^{2n+s} is an almost $\mathcal{C}$-manifold, and totally geodesic Kählerian if M^{2n+s} is a $\mathcal{C}$-manifold. However there are also examples of non-normal almost $\mathcal{C}$-manifolds such that the leaves of $\mathcal{D}$ are Kählerian. These are called *almost $\mathcal{C}$-manifolds with Kählerian leaves*. With regard to this we have the following result.

Proposition 2.1 ([24]). *An almost $\mathcal{C}$-manifold with Kählerian leaves is a $\mathcal{C}$-manifold if and only if each ξ_α is a Killing vector field, that is $\mathcal{F}$ is a Riemannian foliation or, equivalently, $\mathcal{D}$ is totally geodesic.*

We remark that the condition that each ξ_α is Killing is equivalent to the requirement that $\nabla\xi_\alpha = 0$. We point out also that the foliation $\mathcal{F}$ defined by the vector fields $\xi_1, \dots, \xi_s$ is transversally orientable since the bundle $\Lambda^s(\mathcal{D})$ is trivial, $\eta_1 \wedge \ldots \wedge \eta_s$ being a nowhere vanishing section (cf. [32]).

The standard example of an almost $\mathcal{C}$-manifold is given by the product of an almost Kählerian manifold with an abelian Lie group. Note that in this case the almost Kählerian manifold is Kählerian if and only if the almost $\mathcal{C}$-structure is normal. Moreover, each vector field ξ_α is Killing. On the other hand it is well known that $\nabla\xi_\alpha = 0$ for all $\alpha \in \{1, \dots, s\}$ if and only if the almost $\mathcal{C}$-manifold M^{2n+s} in question is locally the Riemannian product of an almost Kählerian manifold and an s-dimensional flat manifold. However, there are also several examples of $\mathcal{C}$-manifolds which are not the global products of a Kählerian manifold with an abelian Lie group (see e.g. [12], [16], [17], [27]).

3. EINSTEIN CONDITIONS

In this section we focus on almost $\mathcal{C}$-manifolds such that the canonical foliation $\mathcal{F}$ is Riemannian, by stating and proving a Goldberg-like conjecture for these manifolds. Firstly we recall the following preliminary lemma.

Lemma 3.1 ([26]). *In any almost $\mathcal{C}$-manifold $(M^{2n+s}, f, \xi_\alpha, \eta_\alpha, g)$ the following relations hold, for any $\alpha \in \{1, \dots, s\}$,*

$$\mathrm{Ric}\,(\xi_\alpha, \xi_\alpha) + |\nabla\xi_\alpha|^2 = 0, \tag{3.1}$$

$$\tau - \tau^* - \sum_{\alpha=1}^{s} \mathrm{Ric}(\xi_\alpha, \xi_\alpha) + \frac{1}{2}|\nabla f|^2 = 0, \tag{3.2}$$

where τ denotes the scalar curvature and τ^* the $*$-scalar curvature defined as the trace of the Ricci-$*$ tensor.

Theorem 3.1. *Every compact Einstein almost $\mathcal{C}$-manifold $(M^{2n+s}, f, \xi_\alpha, \eta_\alpha, g)$ such that each ξ_α is a Killing vector field is a $\mathcal{C}$-manifold.*

Proof. Since ξ_α is Killing, we have $\nabla\xi_\alpha = 0$ and by (3.1) we get that M^{2n+s}, being Einstein, is Ricci-flat. Now let us consider the product $M' = M^{2n+s} \times \mathbb{T}^s$, where $\mathbb{T}^s$ denotes the s-dimensional flat torus. We define on M' an almost complex structure J' by putting $J'X = fX$ for all $X \in \Gamma(\mathcal{D})$ and $J'\xi_\alpha = u_\alpha$, $J'u_\alpha = -\xi_\alpha$ for each $\alpha \in \{1,\dots,s\}$, where u_α denotes a unit vector field on each factor of $\mathbb{T}^s = S^1 \times \cdots \times S^1$. Let us denote by g' the product metric on M'. Then one easily verifies that J' is compatible with respect to g'. Moreover, some straightforward computations show that the differential of the fundamental 2-form Ω' associated to J' and g' is given by

$$d\Omega'\left(X + \sum_{\alpha=1}^{s} h_\alpha u_\alpha, Y + \sum_{\beta=1}^{s} k_\beta u_\beta, Z + \sum_{\gamma=1}^{s} \ell_\gamma u_\gamma\right) =$$
$$= dF(X,Y,Z) - \frac{2}{3}\sum_{\alpha=1}^{s}\left(\ell_\alpha d\eta_\alpha(X,Y) + h_\alpha d\eta_\alpha(Y,Z) + k_\alpha d\eta_\alpha(Z,X)\right) = 0$$

for any vector fields X, Y, Z on M^{2n+s} and smooth functions h_α, k_α, ℓ_α on M', $\alpha \in \{1,\dots,s\}$. Hence (M', J', g') is an almost Kählerian manifold. Moreover the Riemannian metric g' is Ricci-flat, being the product of two Ricci-flat Riemannian metrics. In particular, M' is an Einstein manifold with zero scalar curvature and by Sekigawa theorem ([30]) J' is parallel. This last result, together with $\nabla\xi_\alpha = 0$, in turn implies that $\nabla f = 0$. Hence M^{2n+s} is a $\mathcal{C}$-manifold. □

Corollary 3.1. *Any compact Ricci-flat almost $\mathcal{C}$-manifold is a $\mathcal{C}$-manifold.*

Proof. In fact by virtue of (3.1) the Ricci flatness is equivalent to the requirement that the metric g is Einstein and the vector fields ξ_α are Killing. □

Corollary 3.2. *Every Einstein compact metric f-manifold $\left(M^{2n+s}, f, \xi_\alpha, \eta_\alpha, g\right)$ with each ξ_α Killing, satisfying* (3.1) *and*

$$\tau - \tau^* - \frac{s\tau}{2n+s} + \frac{1}{2}|\nabla f|^2 = 0.$$

is a $\mathcal{C}$-manifold.

Proof. Theorem 4.1 in [26] states that a metric f-manifold satisfying (3.1) and (3.2) is an almost $\mathcal{C}$-manifold. Thus, the result follows from Theorem 3.1 applying the relation $\sum_{\alpha=1}^{s} \mathrm{Ric}(\xi_\alpha, \xi_\alpha) = \dfrac{s\tau}{2n+s}$. □

Corollary 3.3. *Let $(M^{2n+s}, f, \xi_\alpha, \eta_\alpha, g)$ be an Einstein compact $\mathcal{C}$-manifold. Then any other almost $\mathcal{C}$-structure $(f', \xi'_\alpha, \eta'_\alpha, g)$ on M^{2n+s} is a $\mathcal{C}$-structure.*

Proof. Since the Riemannian metric g is Einstein and $(f, \xi_\alpha, \eta_\alpha, g)$ is a $\mathcal{C}$-structure, (M^{2n+s}, g) is Ricci-flat. Then, by applying the formula $\mathrm{Ric}\,(\xi'_\alpha, \xi'_\alpha) + |\nabla\xi'_\alpha|^2 = 0$, we get $\nabla\xi'_\alpha = 0$, hence each ξ'_α is Killing and we are under the assumptions of Theorem 3.1. Thus $(f', \xi'_\alpha, \eta'_\alpha, g)$ is a $\mathcal{C}$-structure. □

In particular, for $s=1$ Theorem 3.1 implies the following result.

Theorem 3.2. *Every compact Einstein almost cosymplectic manifold such that the Reeb vector field is Killing is cosymplectic.*

Theorem 3.2 may be interpreted as an odd dimensional analogue of the celebrated Goldberg conjecture stating that any compact Einstein almost Kähler manifold is necessarily Kähler-Einstein ([19]). An analogous result in the context of Sasakian manifolds was proven by Boyer-Galicki ([8]) in 2001 and Apostolov-Draghici-Moroianu ([2]) in 2006.

4. η-EINSTEIN CONDITIONS

A generalization of Theorem 3.2 involves the notion of η-Einstein metric. An almost cosymplectic manifold $(M^{2n+1},\phi,\xi,\eta,g)$ is said to be *η-Einstein* if the Ricci tensor satisfies

$$\mathrm{Ric} = ag + b\eta\otimes\eta,$$

for some smooth functions a and b on M^{2n+1}. In particular, any η-Einstein almost cosymplectic manifold is *quasi Einstein* in the sense of [11]. Examples of this type are given by cosymplectic manifolds of constant ϕ-sectional curvature c, whose Ricci tensor is given by $\mathrm{Ric} = \dfrac{c\,(n+1)}{2}\,(g-\eta\otimes\eta)$ (cf. [25]). Thus we will consider compact η-Einstein almost cosymplectic manifolds whose Reeb vector field is Killing. Note that, according to Theorem 2 of [10], if $b>0$ and $a+b<0$ then there exist no Killing vector fields in the manifold, but in our case we have that $a+b=\mathrm{Ric}(\xi,\xi)=-|\nabla\xi|^2\leq 0$, so that it is meaningful to assume ξ Killing.

A similar notion can also be defined for almost $\mathcal{C}$-manifolds (cf. [22]). Let M^{2n+s} be an almost $\mathcal{C}$-manifold. Then the leaves of $\mathcal{D}$ are almost Kählerian and the induced metric is an Einstein metric if and only if

$$\mathrm{Ric} = ag + \sum_{\alpha=1}^{s} b_\alpha \eta_\alpha\otimes\eta_\alpha \tag{4.1}$$

for some smooth functions $a, b_1,\ldots,b_s$. In this case we say that M^{2n+s} is an *η-Einstein* almost $\mathcal{C}$-manifold.

Now we discuss the case of η-Einstein compact almost $\mathcal{C}$-manifolds, which includes, for $s=1$, the case of η-Einstein compact almost cosymplectic manifolds. We assume that each ξ_α is a Killing vector field and that $a>0$, since $a=0$ gives the Ricci-flatness already discussed in the previous section. We may assume that $n\geq 2$, since any almost $\mathcal{C}$-manifold of dimension $2+s$ such that each ξ_α is Killing is necessarily a $\mathcal{C}$-manifold. Then $a, b_1,\ldots,b_s$ are necessarily constant. We recall that, as proven in [26], the 1-forms η_α and the 2-form F are harmonic.

Proposition 4.1. *Let M^{2n+s} be a compact $\mathcal{C}$-manifold with transversally positive definite Ricci tensor. Then the space $\Lambda^1_{H,\mathcal{F}}(M^{2n+s})$ of the harmonic 1-forms which annihilate all the ξ_α is trivial. Moreover one has* $\dim\Lambda^1_H(M^{2n+s}) = s = b_1(M^{2n+s})$.

Proof. Let β be a harmonic 1-form such that $\beta(\xi_\alpha) = 0$ for each $\alpha \in \{1, \dots, s\}$ and denote by B its dual vector field. Since $\eta_\alpha(B) = g(B, \xi_\alpha) = \beta(\xi_\alpha) = 0$, we obtain $B \in \mathcal{D}$. Then, the well-known formula (cf. [18])

$$\int_{M^{2n+s}} \mathrm{Ric}(B,B) + |\nabla\beta|^2 = 0$$

implies $B = 0$ and $\beta = 0$.

Obviously, $\dim \Lambda^1_H = r \geq s$. Let $\{\eta_1, \dots, \eta_s, \beta_1, \dots, \beta_{r-s}\}$ be a basis of Λ^1_H. Since the ξ_α are Killing, we have $\beta_i(\xi_\alpha) = \text{const.}$ for any $i \in \{1, \dots, r-s\}$ and $\alpha \in \{1, \dots, s\}$. We put for any $j \in \{1, \dots, r-s\}$

$$\tilde{\beta}_j = \beta_j - \sum_{\alpha=1}^{s} \beta_j(\xi_\alpha)\eta_\alpha$$

and it is easy to verify that $\{\eta_1, \dots, \eta_s, \tilde{\beta}_1, \dots, \tilde{\beta}_{r-s}\}$ is again a basis of Λ^1_H. Furthermore $\tilde{\beta}_j(\xi_\alpha) = 0$ and this implies $\tilde{\beta}_j = 0$ and $r = s$. □

It should be remarked that the proof of Proposition 4.1 does not depend on the normality of the $\mathcal{C}$-structure, but just on the weaker condition of all the Reeb vector fields being Killing. Thus we have the following

Proposition 4.2. *Let M^{2n+s} be a compact almost $\mathcal{C}$-manifold with each ξ_α Killing and transversally positive definite Ricci tensor. Then*

$$\dim \Lambda^1_H(M^{2n+s}) = s = b_1(M^{2n+s}).$$

Corollary 4.1. *Let M^{2n+s} be a compact almost $\mathcal{C}$-manifold with each ξ_α Killing and transversally positive definite Ricci tensor and assume that s is odd. Then M^{2n+s} can not admit any Sasakian structure.*

Proof. The result follows from Proposition 4.2 and a well-known theorem of Tachibana stating that the first Betti number of a Sasakian manifold is even or zero ([31]). □

For instance, the product of a compact Einstein Kählerian manifold of strictly positive scalar curvature with the s-dimensional flat torus, s being an odd number, provides an example of a $\mathcal{C}$-manifold which satisfies the assumptions of Corollary 4.1. Thus in particular we have:

Corollary 4.2. *There exist odd-dimensional manifolds which admit a $\mathcal{C}$-structure but no Sasakian structures.*

We notice that Theorem 3.1 in [14] stated for compact cosymplectic manifolds holds also for compact almost cosymplectic manifolds whose Reeb vector field is Killing. Now, we extend such a result to compact almost $\mathcal{C}$-manifolds. Clearly, assuming $s = 1$ and the normality condition, we reobtain the quoted theorem.

Theorem 4.1. *Let M^{2n+s} be a compact almost $\mathcal{C}$-manifold such that $b_1(M^{2n+s}) = s$ and each ξ_α is Killing.*

(1) *If L is a leaf of the foliation $\mathcal{D}$, then the inclusion map induces a monomorphism $i\colon \pi_1(L) \to \pi_1(M^{2n+s})$ and the quotient group $\dfrac{\pi_1(M^{2n+s})}{\pi_1(L)}$ is isomorphic to $\mathbb{Z}^s$.*
(2) *There exists a fibration $\pi\colon M^{2n+s} \to \mathbb{T}^s$ of M^{2n+s} onto the flat torus $\mathbb{T}^s$ such that the leaves of $\mathcal{D}$ are the fibres of π.*
(3) *The leaves of $\mathcal{D}$ are compact.*

Proof. We know that, since the ξ_α are Killing, M^{2n+s} is locally a Riemannian product of an almost Kählerian manifold and a flat manifold of dimension s. Then, its almost $\mathcal{C}$-structure lifts to an almost $\mathcal{C}$-structure $(\tilde{\phi}, \tilde{\xi}_\alpha, \tilde{\eta}_\alpha, \tilde{g})$ on the universal covering space $\tilde{M}$. Moreover, $\tilde{M}$ is the product of a simply connected almost Kählerian manifold K with $\mathbb{R}^s$. The foliation $\mathcal{D}$ lifts to a foliation $\tilde{\mathcal{D}}$ of $\tilde{M}$, whose leaves are of the form $K \times \{x\}$, $x \in \mathbb{R}^s$, and they are the lift of the leaves L of $\mathcal{D}$.

As in the proof of Theorem 3.1 in [14], we get a homomorphism $\rho\colon \pi_1(M^{2n+s}) \to \mathrm{Diff}(\mathbb{R}^s)$ such that $\ker(\rho) = \pi_1(L)$ and $\mathrm{Im}(\rho)$ is an abelian group isomorphic to $\dfrac{\pi_1(M^{2n+s})}{\pi_1(L)}$, which is in turn isomorphic to $\mathbb{Z}^r$, with $r \geq 0$, being finitely generated owing to the compactness of M^{2n+s}. Now, $r = 0$ would imply $M^{2n+s} \simeq L \times \mathbb{R}^s$ against the compactness of M^{2n+s}. Assuming $r \geq 1$, since $\dfrac{\pi_1(M^{2n+s})}{\pi_1(L)} \simeq \mathbb{Z}^r$, there exists an epimorphism

$$\beta : \frac{\pi_1(M^{2n+s})}{[\pi_1(M^{2n+s}), \pi_1(M^{2n+s})]} \longrightarrow \frac{\pi_1(M^{2n+s})}{\pi_1(L)} \simeq \mathbb{Z}^r$$

and since $\dfrac{\pi_1(M^{2n+s})}{[\pi_1(M^{2n+s}), \pi_1(M^{2n+s})]}$ is isomorphic to the first integral homology group $H_1(M^{2n+s}, \mathbb{Z})$ whose rank is the first Betti number, we obtain $r \leq s$.

Denote by $\gamma\colon \mathbb{Z}^r \to \mathrm{Im}(\rho)$ an isomorphism, put $\gamma(\varepsilon_i) = \zeta_i$, $i \in \{1, \ldots, r\}$, $\varepsilon_i = (0, \ldots, 1, \ldots, 0)$ being generators of $\mathbb{Z}^r$. Then $\mathrm{Im}(\rho)$ is a normal subgroup of the translation group of $\mathbb{R}^s$, isomorphic to the subgroup $\left\{ \sum_{i=1}^{r} m_i a_i \mid m_i \in \mathbb{Z} \right\}$ of $\mathbb{R}^s$, $a_i \in (\mathbb{R}^s)^*$ denoting the displacement vector of ζ_i. Hence, we obtain a fibration

$$p\colon M^{2n+s} \longrightarrow \mathbb{R}^s/\mathbb{Z}^r$$

such that the diagram

$$\begin{array}{ccc} K \times \mathbb{R}^s & \xrightarrow{\ \pi\ } & M^{2n+s} \\ {\scriptstyle pr_2}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ \mathbb{R}^s & \xrightarrow[\ \pi'\ ]{} & \mathbb{R}^s/\mathbb{Z}^r \end{array}$$

commutes. Since M^{2n+s} is compact we have $r = s$ and $\mathbb{R}^s/\mathbb{Z}^r = \mathbb{T}^s$ (cf. [23]). Finally, the leaves of $\mathcal{D}$ are the fibres of p so they are compact. □

Then, we have:

Theorem 4.2. *Every compact η-Einstein almost $\mathcal{C}$-manifold M^{2n+s} such that $a > 0$ and each ξ_α is a Killing vector field is a $\mathcal{C}$-manifold. Furthermore, M^{2n+s} is locally a product of a simply connected compact Einstein Kählerian manifold and a flat torus $\mathbb{T}^s$ and $\pi_1(M^{2n+s})$ is isomorphic to $\mathbb{Z}^s$.*

Proof. By (4.1) and (3.1), since each vector field ξ_α is Killing, we have $b_1 = \cdots = b_s = -a$, the Ricci tensor is transversally positive definite and Proposition 4.2 gives $b_1(M^{2n+s}) = s$. From Theorem 3.2.1 of [18] we know that the first Betti number of a compact orientable Riemannian manifold of positive definite Ricci curvature is zero. Hence, Theorem 4.1 implies that any leaf of $\mathcal{D}$ is a compact simply connected Einstein almost Kählerian manifold with positive scalar curvature and then it is Kählerian, by the quoted Sekigawa's result. Thus M^{2n+s} is an almost $\mathcal{C}$-manifold with Kählerian leaves and in view of Proposition 2.1, since each ξ_α is Killing, it is a $\mathcal{C}$-manifold. Finally, the last statement follows again from Theorem 4.1. □

Remark 4.1. One could try to weaken the hypotheses, requiring that the foliation $\mathcal{F}$ is a conformal foliation, that is $(\mathcal{L}_V g)(X,Y) = \lambda(V)g(X,Y)$ for all $X, Y \in \Gamma(\mathcal{D})$ and $V \in \Gamma(\mathcal{D}^\perp)$, λ being a function depending on V. However this forces the ξ_α to be Killing and we return to the discussed cases. Indeed, assuming that $\mathcal{F}$ is conformal, the distribution $\mathcal{D}$ is totally umbilical and this implies that its leaves, being minimal, are totally geodesic. Therefore $\mathcal{F}$ is a Riemannian foliation and each ξ_α is Killing.

Note that Theorem 4.2 also holds in the more general setting of generalized quasi Einstein almost cosymplectic manifolds. Recall that a non-flat Riemannian manifold (M^n, g) $(n \geq 3)$ is called a *generalized quasi Einstein manifold* if its Ricci tensor is not identically zero and satisfies the condition

$$\mathrm{Ric}(X,Y) = ag(X,Y) + bA(X)A(Y) + c(A(X)B(Y) + A(Y)B(X)) \tag{4.2}$$

where $a, c \in \mathbb{R}$, $b \in \mathbb{R}^*$ and A, B are non-zero 1-forms such that

$$A = g(\cdot, U), \quad B = g(\cdot, V), \tag{4.3}$$

U, V being two unit, mutually orthogonal vector fields on the manifold ([11]). Thus let $(M^{2n+1}, \phi, \xi, \eta, g)$ be a compact generalized quasi Einstein almost cosymplectic manifold such that $\xi = U$ is Killing. Let us assume that $a > 0$. Then by (4.2) we get that the Ricci tensor is transversally positive definite, so that the proof of Theorem 4.2 still works in this situation and M^{2n+1} is cosymplectic. Moreover, M^{2n+1} is locally a product of a simply connected Kähler-Einstein manifold and S^1, and the first fundamental group is isomorphic to $\mathbb{Z}$.

Another possible generalization comes from the notion of mixed generalized quasi Einstein metrics [4]. Recall that a non-flat Riemannian manifold is called *mixed generalized quasi Einstein manifold* if the Ricci tensor is non-zero and satisfies the condition

$$\mathrm{Ric}(X,Y) = ag(X,Y) + bA(X)A(Y) + cB(X)B(Y) + d(A(X)B(Y) + A(Y)B(X)) \tag{4.4}$$

where a, b, c, d are non-zero scalars and A, B are 1-forms satisfying (4.3) as above. The previous arguments works also in this setting, so we conclude that any compact mixed generalized quasi Einstein almost cosymplectic manifold, such that the Reeb vector field $\xi = U$ is Killing and $a > 0$, is cosymplectic. Moreover, one can try to adapt this last generalization of the Einstein condition to the context of $\mathcal{C}$-manifolds. For instance we can consider a $(2n+2)$-dimensional almost $\mathcal{C}$-manifold whose Ricci tensor satisfies (4.4) with $A = \eta_1$ and $B = \eta_2$. However, as soon as we suppose that the Reeb vector fields ξ_1, ξ_2 are Killing, we have $\mathrm{Ric}(\xi_1, \xi_1) = \mathrm{Ric}(\xi_1, \xi_2) = \mathrm{Ric}(\xi_2, \xi_2) = 0$, which yields $d = 0$ and $b = c = -a$, so that we come back to the already studied η-Einstein condition.

Note that (4.4) reduces to (4.2) for $c = 0$. In turn (4.2) putting $c = 0$ reduces to

$$\mathrm{Ric}(X, Y) = ag(X, Y) + bA(X)A(Y) \tag{4.5}$$

which is clearly an extension of the η-Einstein condition to every Riemannian manifold. Any non-flat Riemannian manifold satisfying (4.5) is called *quasi Einstein manifold* ([11]). Thus, in particular, by Theorem 4 of [11] we have that any η-Einstein almost cosymplectic manifold such that ξ is Killing is conformally conservative, i.e. the divergence of the conformal curvature is zero.

The geometric meaning of the scalars a and b in (4.5) is that $a+b$ and a are the (only and distinct) eigenvalues of the Ricci operator, of which the former is simple and the latter is of multiplicity $\dim(M^n) - 1$ ([11]). Now we just discuss the general case of a cosymplectic manifold whose Ricci operator admits two distinct constant eigenvalues.

Theorem 4.3. *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be a cosymplectic manifold. Let us assume that the Ricci operator Q of (M^{2n+1}, g) admits constant distinct eigenvalues λ and μ such that $TM^{2n+1} = E_\lambda \oplus E_\mu \oplus \mathbb{R}\xi$, where E_λ and E_μ denote the eigenspace distributions corresponding to λ and μ, respectively. Then M^{2n+1} admits a further almost cosymplectic structure (ϕ', ξ, η, g) such that $\phi\phi' = \phi'\phi$. Moreover, (ϕ', ξ, η, g) is cosymplectic if and only if (M^{2n+1}, g) is locally product of an η-Einstein cosymplectic manifold and a Kähler-Einstein manifold.*

Proof. Let λ and μ the two distinct eigenvalues of the Ricci operator Q (defined, as usual, by $g(QX, Y) = \mathrm{Ric}(X, Y)$), stated by the theorem. Note that, since $\mathrm{Ric}(\xi, \xi) = 0$, ξ is an eigenvector of Q associated to the eigenvalue 0. Thus we may distinguish the two cases (i) $\lambda \neq 0$, $\mu \neq 0$ and (ii) $\lambda = 0$, $\mu \neq 0$. We consider the first case. By (2.1) we have that E_λ, E_μ, $\mathbb{R}\xi$ are invariant under the action of ϕ and this in turn implies that E_λ and E_μ have even dimension. Now we define a tensor field ϕ' by setting $\phi'|_{E_\lambda} = \phi|_{E_\lambda}$, $\phi'|_{E_\mu} = -\phi|_{E_\mu}$, $\phi'\xi = \phi\xi = 0$. Note that the definition of ϕ' is well-posed just because $\phi E_i \subset E_i$ for each $i \in \{\lambda, \mu\}$. Easily one verifies that (ϕ', ξ, η, g) is an almost contact metric structure such that ϕ' commutes with ϕ. Let α and β the 2-forms on M^{2n+1} defined by $\alpha(X, Y) = \Phi(X_\lambda, Y_\lambda)$, $\beta(X, Y) = \Phi(X_\mu, Y_\mu)$, for all $X, Y \in \Gamma(TM^{2n+1})$, where X_λ and X_μ denote the projections of X onto the subbundles E_λ and E_μ of TM^{2n+1}, respectively. Then the tensor ρ defined in (2.2) is given by

$$\begin{aligned}\rho(X,Y) &= \mathrm{Ric}(X_\lambda + X_\mu + \eta(X)\xi, \phi Y_\lambda + \phi Y_\mu + \phi(\eta(Y)\xi))\\ &= \mathrm{Ric}(X_\lambda, \phi Y_\lambda) + \mathrm{Ric}(X_\mu, \phi Y_\mu)\\ &= g(QX_\lambda, \phi Y_\lambda) + g(QX_\mu, \phi Y_\mu)\\ &= \lambda g(X_\lambda, \phi Y_\lambda) + \mu g(X_\mu, \phi Y_\mu)\\ &= \lambda\alpha(X,Y) + \mu\beta(X,Y),\end{aligned}$$

thus $\rho = \lambda\alpha + \mu\beta$. On the other hand $\Phi = \alpha + \beta$, so that, as both ρ and Φ are closed, also α and β are closed 2-forms. Now let us compute the fundamental 2-form of the almost contact metric structure (ϕ', ξ, η, g). We have

$$\begin{aligned}\Phi'(X,Y) &= g(X_\lambda + X_\mu + \eta(X)\xi, \phi' Y_\lambda + \phi' Y_\mu + \phi'(\eta(Y)\xi))\\ &= g(X_\lambda + X_\mu + \eta(X)\xi, \phi Y_\lambda - \phi Y_\mu)\\ &= \Phi(X_\lambda, Y_\lambda) - \Phi(X_\mu, Y_\mu)\\ &= \alpha(X,Y) - \beta(X,Y).\end{aligned}$$

Therefore $\Phi' = \alpha - \beta$ and this implies that $d\Phi' = d\alpha - d\beta = 0$ since we have just proven that α and β are closed. Thus (ϕ', ξ, η, g) is almost cosymplectic. It is cosymplectic if and only if $\nabla\phi' = 0$. But note that

$$\begin{aligned}(\nabla_X\phi')Y &= (\nabla_X\phi')Y_\lambda + (\nabla_X\phi')Y_\mu + (\nabla_X\phi')\eta(Y)\xi\\ &= \nabla_X\phi Y_\lambda - \phi'(\nabla_X Y_\lambda) - \nabla_X\phi Y_\mu - \phi'(\nabla_X Y_\mu) - X(\eta(Y))\phi\xi\\ &\quad - \eta(Y)\phi'\nabla_X\xi\\ &= 2\phi(\nabla_X Y_\lambda)_\mu - 2\phi(\nabla_X Y_\mu)_\lambda.\end{aligned}$$

Hence $(\nabla_X\phi')Y = 0$ if and only if $\phi(\nabla_X Y_\lambda)_\mu = 0$ and $\phi(\nabla_X Y_\mu)_\lambda = 0$, i.e. if and only if $(\nabla_X Y_\lambda)_\mu = 0$ and $(\nabla_X Y_\mu)_\lambda = 0$. Note that the last conditions, together with $\nabla\xi = 0$, imply also the integrability of the distributions $E_\lambda \oplus \mathbb{R}\xi$ and $E_\mu \oplus \mathbb{R}\xi$. Then one can define by restriction a cosymplectic structure, whose fundamental 2-form is the restriction of α, on a manifold M_1 integral to the distribution $E_\lambda \oplus \mathbb{R}\xi$ and a Kähler structure, whose fundamental 2-form is the restriction of β, on a manifold M_2 integral to E_μ. It is clear that the Ricci tensor on M_1 is given by $\mathrm{Ric}_1 = \lambda g - \lambda\eta\otimes\eta$, whereas the Ricci tensor on M_2 is given by $\mathrm{Ric}_2 = \mu g$. This proves the theorem in the case (i). The case (ii) runs more or less in the same way, the only difference being that $\rho = \mu\beta$ (since $\lambda = 0$), so that we can conclude immediately that β is closed and hence also the fundamental 2-form Φ' of the new almost contact metric structure (ϕ', ξ, η, g) (defined in the same way as previously) is closed, and M_1 is Ricci-flat. Then the result comes as in the case (i). □

5. FINAL REMARKS

Without the assumption of compactness, Theorem 3.2 does not hold. For constructing counterexamples it is sufficient to consider the standard example of almost cosymplectic manifolds by means of product manifolds. Let (N^{2n}, J, ω, G) be any Einstein non-compact almost Kählerian manifold which is non-Kählerian (examples of such manifolds are given, for instance, in [1] and [3]). Then clearly $M^{2n+1} = N^{2n} \times S^1$ with its canonical almost cosymplectic structure (cf. [28]) is not cosymplectic. The same considerations can be done for almost $\mathcal{C}$-manifolds.

More delicate is the discussion about the necessity of assuming ξ a Killing vector field. However a class of counterexamples for this case can be given by compact almost cosymplectic manifolds with the Reeb vector field ξ belonging to the *k-nullity distribution*, i.e. satisfying $R(X,Y)\xi = k(\eta(Y) - \eta(X)Y)$ for any $X, Y \in \Gamma(TM^{2n+1})$. Indeed in this case the Ricci tensor is given by $\mathrm{Ric} = 2nk\eta \otimes \eta$ (cf. [13]), so that any such almost cosymplectic manifold is η-Einstein. Moreover, since it has been shown that ξ in never Killing unless $k = 0$, the almost cosymplectic structure can not be normal.

This last observation can been extended to the attempt of finding counterexamples to the Goldberg conjecture for contact metric manifolds. In [2] it was arisen the question whether every compact Einstein contact metric manifold is Sasakian-Einstein. As it is known, the answer is negative in general because the flat torus $\mathbb{T}^3$ with the 1-form $\eta = \cos(t)\,dx + \sin(t)\,dy$ provides an example of compact contact metric manifold which is not K-contact (cf. [6]). However, as remarked by the authors, this is the only known counterexample. In order to show a class of negative examples in higher dimensions, we recall the notion of *contact metric (κ, μ)-manifold*. Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be a contact metric manifold and $\kappa, \mu \in \mathbb{R}$. One says that ξ belongs to the (κ, μ)-nullity distribution or, simply, that M^{2n+1} is a contact metric (κ, μ)-manifold, if the curvature of the Levi-Civita connection satisfies

$$R(X,Y)\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY) \tag{5.1}$$

for all $X, Y \in \Gamma(TM^{2n+1})$, where $h = \frac{1}{2}\mathcal{L}_\xi\phi$. This notion was introduced by Blair, Koufogiorgos and Papantoniou ([7]), who proved also that $\kappa \leq 1$, being $\kappa = 1$ if and only if M^{2n+1} is Sasakian, and that for $\mu = 2(1-n)$ M^{2n+1} is η-Einstein. Moreover they proved also that the unit tangent sphere bundle of a space of constant sectional curvature c satisfies the condition (5.1) for $\kappa = c(2-c)$ and $\mu = -2c$ (if $c \neq 1$ the corresponding sphere bundle is not Sasakian). Thus any compact contact metric (κ, μ)-manifold with $\kappa \neq 1$ and $\mu = 2(1-n)$ will provide an example of compact η-Einstein contact metric manifold which is not Sasakian (note that for $\kappa \neq 1$ no contact metric (κ, μ)-manifold can be K-contact). For instance, let N^m be an m-dimensional manifold of constant sectional curvature $c = m - 2$ such that the corresponding tangent sphere bundle T_1N^m is compact. Then, by the above results, T_1N^m with its canonical contact metric structure is a contact metric (κ, μ)-manifold with $\kappa = c(2-c) = -(m-2)(m-4)$ and $\mu = -2c = -2(m-2)$. Since

$\dim(T_1N^m) = 2m - 1 = 2(m-1) + 1 = 2n + 1$, where we have put $n = m - 1$, one has easily that $\mu = 2(n-1)$, so that T_1N^m is η-Einstein and it is non-Sasakian for $m > 3$.

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