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STABILITY OF PICARD AND MANN ITERATION FOR A GENERAL CLASS OF FUNCTIONS

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Dedicated to Professor Ştefan Mititelu
on the occasion of his seventieth birthday

Abstract. In this paper we establish some stability results for Picard and Mann iteration for a general class of functions. Our results generalize and unify some of the results in the literature, especially those of Imoru and Olatinwo [3], Berinde [1], and Osilike [4].

1. INTRODUCTION

Let (X, d) be a metric space, T a selfmap of X with a fixed point p. Let $x_0 \in X$, and assume that $x_{n+1} = f(T, x_n)$ is some iteration procedure which converges to p. Let $\{y_n\}$ be an arbitrary sequence in X, and define $\epsilon_n = d(y_{n+1}, f(T, y_n))$. If $\lim \epsilon_n = 0$ implies that $\lim y_n = p$, then the iteration procedure $x_{n+1} = f(T, x_n)$ is said to be T-stable. This definition is due to Ostrowski, who, in 1964 proved that maps T satisfying the Banach contraction principle are T-stable for Picard iteration. Picard iteration is the name given to function iteration; i.e., $x_{n+1} = Tx_n$.

Harder and Hicks [2] showed that Picard iteration is T-stable for several other contractive conditions, including that of Zamfirescu [6]. The second author [5] used a contractive condition independent of [6] and obtained stability results for other iteration processes, such as Mann, Kirk and Massa. Each map satisfying one of the contractive conditions considered in these papers possessed a unique fixed point.

Employing a new idea, Osilike [4] considered maps T having a fixed point and satisfying the condition

$$d(Tx, Ty) \le Ld(x, Tx) + ad(x, y) \tag{1.1}$$

for all x and y, where $L \ge 0$ and $0 \le a < 1$. He established T-stability for such maps with respect to Picard, Kirk, Mann, and Ishikawa iterations.

Imoru and Olatinwo generalized the condition of Osilike by replacing (1.1) with

$$d(Tx, Ty) \le \varphi(d(x, Tx)) + ad(x, y), \tag{1.2}$$

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where $0 \le a < 1$ and $\varphi\colon \mathbb{R}^+ \to \mathbb{R}^+$ is monotone increasing with $\varphi(0) = 0$.

By replacing L in (1.1) with more complicated expressions, the process of "generalizing" (1.1) could continue ad infinitum. In this paper we make an obvious assumption implied by (1.1), and one which renders all generalizations of the form (1.2) pointless.

We shall need the following Lemma, which appears in [1].

Lemma 1.1. *Let δ be a real number satisfying $0 \le \delta < 1$, and $\{\epsilon_n\}$ a positive sequence satisfying $\lim \epsilon_n = 0$. Then, for any positive sequence $\{u_n\}$ satisfying*

$$u_{n+1} \le \delta u_n + \epsilon_n,$$

it follows that $\lim u_n = 0$.

2. MAIN RESULTS

Theorem 2.1. *Let (X, d) be a complete metric space, T a selfmap of X, with a fixed point p, satisfying*

$$d(p, Ty) \le ad(p, y) \text{ for some } 0 \le a < 1 \text{ and for each } y \in X. \tag{2.1}$$

Then Picard iteration is T-stable.

Proof. Define $\{x_n\}$ by $x_{n+1} = Tx_n$, and assume that $x_n \to p$. Then, for any sequence $\{y_n\}$, using (2.1),

$$\begin{aligned} d(y_{n+1}, p) &\le d(y_{n+1}, Ty_n) + d(Ty_n, p) \\ &= \epsilon_n + d(Tp, Ty_n) \le \epsilon_n + ad(p, y_n). \end{aligned}$$

Taking the limit as $n \to \infty$ of both sides of the above inequality, and using Lemma 1.1, $\lim y_n = p$, and Picard iteration is T-stable. □

In a normed space E, Mann iteration is defined by $u_0 \in E$,

$$u_{n+1} = (1 - \alpha_n)u_n + \alpha_n Tu_n,$$

where $0 \le \alpha_n \le 1$ and $\sum \alpha_n = \infty$.

Theorem 2.2. *Let E be a Banach space, T a selfmap of E with a fixed point p and satisfying*

$$\|p - Ty\| \le a\|p - y\| \text{ for some } 0 \le a < 1 \text{ and for each } y \in X. \tag{2.2}$$

Then Mann iteration, with

$$0 < \alpha \le \alpha_n \text{ for all } n \tag{2.3}$$

is T-stable.

Proof. Suppose that $\{u_n\}$ converges to p. Let $\{y_n\}$ be an arbitrary sequence in E. Then, using (2.2)

$$\begin{aligned}\|y_{n+1} - p\| &\leq \|y_{n+1} - (1-\alpha_n)y_n - \alpha_n T y_n\| \\ &\quad + \|(1-\alpha_n)y_n + \alpha_n T y_n - p\| \\ &\leq \epsilon_n + (1-\alpha_n)\|y_n - p\| + \alpha_n a\|p - y_n\| \\ &\leq \epsilon_n + (1-\alpha_n(1-a))\|p - y_n\| \\ &\leq \epsilon_n + (1-\alpha(1-a))\|p - y_n\|.\end{aligned}$$

From Lemma 1.1, $\lim \|y_n - p\| = 0$, and Mann iteration satisfying (2.3) is T-stable. □

A special case of Mann iteration is that of Krasnoselskij, which is Mann iteration with each $\alpha_n = \lambda$ for some $0 < \lambda < 1$.

Corollary 2.1. *Let E and T be as in Theorem* 2.1. *Then Krasnoselskij iteration is T-stable.*

Proof. In Theorem 2.2, set each $\alpha_n = \lambda$. □

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