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STOCHASTIC DIFFERENTIAL EQUALITIES AND GLOBAL LIMITS

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Dedicated to Professor Ştefan Mititelu
on the occasion of his seventieth birthday

Abstract. This paper is a completion of [1]. For some stochastic processes we define a new kind of limit, the global limit, which allows us to find stochastic differential equalities. We show that the temporal derivative and the B-derivatives defined in [1] are global limits and we use the global limit to establish the generalized Itô formula.

1. INTRODUCTION

Let $(\Omega, \mathcal{K}, P)$ be a probability space, $\mathbb{R}$ the set of real numbers and

$$B\colon [0,\infty)\times\Omega \to \mathbb{R}^m, \quad B = (B_1,\dots,B_m),$$

where for each $i = 1,\dots,m$, B_i is a stochastic process with continuous paths. In [1] we said that B satisfies 'the H hypothesis' if there exists $\mathcal{H}_B$, a class of stochastic processes defined on $[0,\infty)^2\times\Omega$ and $\mathbb{R}^m$ valued which has some properties (see [1]). We denoted $\mathcal{H}_B^n = \{(f_1,\dots,f_n) \mid f_1,\dots,f_n \in \mathcal{H}_B\}$ and for $f, g : [0,\infty)^2\times\Omega \to \mathbb{R}^n$ with $f-g \in \mathcal{H}_B^n$, we wrote $f \overset{B}{\equiv} g$. In particular, $h \in \mathcal{H}_B^n$ iff $h \overset{B}{\equiv} 0$.

If $F : \mathbb{R}^p \times \mathbb{R}^{q_1}\times\cdots\times\mathbb{R}^{q_k} \to \mathbb{R}^n$ is a function, $\theta\in[0,1]$ and $a_i\colon [0,\infty)\times\Omega\to\mathbb{R}$, $i = 1,\dots,p$, $X_j\colon [0,\infty)\times\Omega \to \mathbb{R}^{q_j}$, $j = 1,\dots,k$, are stochastic processes, we said that

$$F(a_1,\dots,a_p, dX_1,\dots,dX_k) \overset{\theta}{=} 0$$

if $h_\theta \overset{B}{\equiv} 0$, where

$$h_\theta\colon [0,\infty)^2\times\Omega\to\mathbb{R}^n, \quad h_\theta(t,s) = F(a_1(\theta_{ts}),\dots,a_p(\theta_{ts}), \Delta_{ts}X_1,\dots,\Delta_{ts}X_k),$$

$\theta_{ts} = \theta(t-s)+s$, $\Delta_{ts}X_i = X_i(t)-X_i(s)$. We called the equality above *stochastic differential θ-equality.*

If $\theta = 0$, we called it *Itô stochastic differential equality* and we wrote

$$F(a_1,\dots,a_p, dX_1,\dots,dX_k) \overset{I}{=} 0.$$

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If $\theta = \frac{1}{2}$, we called it *Stratonovich stochastic differential equality* and we wrote

$$F(a_1, \ldots, a_p, dX_1, \ldots, dX_k) \overset{S}{=} 0.$$

If $\theta \in [0,1]$ and $X\colon \infty \times \Omega \to \mathbb{R}^n$ is a stochastic process and there exist right-continuous functions $u\colon \infty \times \Omega \to \mathbb{R}^n$, $v\colon \infty \times \Omega \to \mathcal{M}_{n,m}(\mathbb{R})$ such that

$$dX \overset{\theta}{=} udt + vdB,$$

we said in [1] that X is *θ-differentiable*. We called u the *temporal θ-derivative* of X and we denoted it by $D_t^\theta X$ and we called v the *B-derivative* of X and we denoted it by $D_B X$.

If $\theta = 0$ we called X *Itô-differentiable* and we denoted the temporal derivative by $D_t^I X$. If $\theta = \frac{1}{2}$ we called X *Stratonovich-differentiable* and we denoted the temporal derivative by $D_t^S X$.

Using this point of view about stochastic differential equalities, unlike other authors (e.g. [2]), in paper [1] we could avoid the use of the stochastic integral to obtain some results from stochastic calculus, e.g. the Itô formula and θ-formula (generalized Itô formula).

In this paper we introduce a new kind of limit called *global limit* and denoted abbreviated by Lim. We preserve the notation lim for the classical limit. Then we show that the temporal derivative and the B-derivatives are limits of this new type and how to obtain stochastic differential equalities using Lim. A new proof for θ-formula is given here, also using the global limit.

We will denote $\Delta_{ts}^{\theta\sigma} = \theta_{ts} - \sigma_{ts}$ and $\Delta_{ts}^{\theta\sigma} X = X(\theta_{ts}) - X(\sigma_{ts})$ for $\theta, \sigma \in [0,1]$.

2. THE GLOBAL LIMIT

Definition 2.1. Let it be the functions $f\colon \big([0,\infty)^2 \setminus \{(s,s) \mid s \in [0,\infty)\}\big) \times \Omega \to \mathbb{R}^n$, $g\colon [0,\infty) \times \Omega \to \mathbb{R}^n$ and $h\colon [0,\infty)^2 \times \Omega \to \mathbb{R}^n$, $h(t,s) = (t-s)(f(t,s) - g(s))$, $t \neq s$, $h(s,s) = 0$. If $h \overset{B}{\equiv} 0$, then we say that the *global limit* of f is g and we denote $\operatorname{Lim} f = g$.

Example 2.1. $\operatorname{Lim} \dfrac{(\Delta_{ts} B_i)^2}{t-s} = 1$ for each $i = 1, \ldots, m$.

If we consider $f_i\colon \big([0,\infty)^2 \setminus \{(s,s) \mid s \in [0,\infty)\}\big) \times \Omega \to \mathbb{R}$, $f_i(t,s) = \dfrac{(\Delta_{ts} B_i)^2}{t-s}$, $g_i\colon [0,\infty) \times \Omega \to \mathbb{R}$, $g_i(s) = 1$ for each s and $h_i(t,s) = (\Delta_{ts} B_i)^2 - (t-s)$ for each $t, s \in [0,\infty)$, then $h_i(t,s) = (t-s)(f_i(t,s) - g_i(s))$ for $t \neq s$ and $h_i(s,s) = 0$. Since $h_i \overset{B}{\equiv} 0$, the global limit of f_i is g_i.

Example 2.2. $\operatorname{Lim} \dfrac{\Delta_{ts} B_i \Delta_{ts} B_j}{t-s} = 0$ for each $i \neq j$, $i, j \in \{1, \ldots, m\}$.

If we consider $f_{ij}\colon ([0,\infty)^2\backslash\{(s,s)\,|\,s\in[0,\infty)\})\times\Omega\to\mathbb{R}$, $f_{ij}(t,s)=\dfrac{\Delta_{ts}B_i\Delta_{ts}B_j}{t-s}$, $g_{ij}\colon [0,\infty)\times\Omega\to\mathbb{R}$, $g_{ij}(s)=0$ for each s and $h_{ij}(t,s)=\Delta_{ts}B_i\Delta_{ts}B_j$ for each $t,s\in[0,\infty)$, then $h_{ij}(t,s)=(t-s)(f_{ij}(t,s)-g_{ij}(s))$ for $t\neq s$ and $h_{ij}(s,s)=0$. Since $h_{ij}\overset{B}{\equiv}0$, the global limit of f_{ij} is g_{ij}.

Example 2.3. If $f(t,s)=\left(\dfrac{(\Delta_{ts}B_1)^2}{t-s},\dfrac{(\Delta_{ts}B_2)^2}{t-s},\dfrac{\Delta_{ts}B_1\Delta_{ts}B_2}{t-s}\right)$, $t\neq s$, then the global limit of f is $(1,1,0)$ (i.e. the function $g:\infty\times\Omega\to\mathbb{R}^3$, $g(s)(\omega)=(1,1,0)$ for each $(s,\omega)\in[0,\infty)\times\Omega$).

It is an immediate consequence of 2.1 and 2.2.

Example 2.4. Let $X,Y\colon [0,\infty)\times\Omega\to R$ be θ-differentiable stochastic processes. If $f(t,s)=\dfrac{\Delta_{ts}X\Delta_{ts}Y}{t-s}$, $t\neq s$, then $\mathrm{Lim}f=\sum\limits_{k=1}^{m}D_{B_k}XD_{B_k}Y.$

Since

$$\Delta_{ts}X\overset{B}{\equiv}D_t^\theta X(\theta_{ts})(t-s)+\sum_{\ell=1}^{m}D_{B_\ell}X(\theta_{ts})\Delta_{ts}B_\ell$$

and

$$\Delta_{ts}Y\overset{B}{\equiv}D_t^\theta Y(\theta_{ts})(t-s)+\sum_{\ell=1}^{m}D_{B_\ell}Y(\theta_{ts})\Delta_{ts}B_\ell,$$

$$\begin{aligned}(t-s)\left(f(t,s)-\sum_{k=1}^{m}D_{B_k}X(s)D_{B_k}Y(s)\right)&=\Delta_{ts}X\Delta_{ts}Y\\ &-(t-s)\sum_{k=1}^{m}D_{B_k}X(s)D_{B_k}Y(s)\overset{B}{\equiv}\sum_{k=1}^{m}\big(D_{B_k}X(\theta_{ts})D_{B_k}Y(\theta_{ts})\\ &-D_{B_k}X(s)D_{B_k}Y(s)\big)\left(\Delta_{ts}B_k\right)^2\\ &+\sum_{k=1}^{m}D_{B_k}X(s)D_{B_k}Y(s)\left((\Delta_{ts}B_k)^2-(t-s)\right)\overset{B}{\equiv}0.\end{aligned}$$

Proposition 2.1. *Let us suppose that*

$$f,f_1,f_2\colon \big([0,\infty)^2\setminus\{(s,s)\mid s\in[0,\infty)\}\big)\times\Omega\to\mathbb{R}^n$$

and $g,g_1,g_2\colon [0,\infty)\times\Omega\to\mathbb{R}^n$.

a) *If* $\mathrm{Lim}\,f=g_1$ *and* $\mathrm{Lim}\,f=g_2$, *then* $g_1=g_2$.

b) *If* $\lim\limits_{t\searrow s}f(t,s)=g(s)$ *for each* s *(as a limit of a real function), then* $\mathrm{Lim}\,f=g$.

c) *If* $\mathrm{Lim}\,f_1=g_1$ *and* $\mathrm{Lim}\,f_2=g_2$, *then Lim* $(f_1+f_2)=g_1+g_2$.

d) *If* $\lim\limits_{t\searrow s}f_1(t,s)=g_1(s)$ *for each* s *and* $\mathrm{Lim}\,f_2=g_2$, *then* $\mathrm{Lim}\,(f_1f_2)=g_1g_2$.

e) *Let* $f = (f_1, \ldots, f_n)$ *and* $g = (g_1, \ldots, g_n)$. *Then* $\mathrm{Lim}\, f = g$ *if and only if* $\mathrm{Lim}\, f_i = g_i$ *for each* $i = 1, \ldots, n$.

Proof. a) Since $h_1 \overset{B}{\equiv} 0$ and $h_2 \overset{B}{\equiv} 0$, where $h_1(t,s) = (t-s)(f(t,s) - g_1(s))$, $t \neq s$, $h_1(s,s) = 0$ and $h_2(t,s) = (t-s)(f(t,s) - g_2(s))$, $t \neq s$, $h_2(s,s) = 0$, it follows that $h \overset{B}{\equiv} 0$, where $h(t,s) = h_1(t,s) - h_2(t,s) = (t-s)(g_2(s) - g_1(s))$. But this is possible only if $g_1(s) = g_2(s)$ for each s.

b) Because $\lim\limits_{t \searrow s} (f(t,s) - g(s)) = 0$ for each s, $h \overset{B}{\equiv} 0$, where

$$h(t,s) = (f(t,s) - g(s))\,(t-s), \quad t \neq s, \quad h(s,s) = 0.$$

It means that $\mathrm{Lim} f = g$.

c) Let

$$\begin{aligned} h(t,s) &= (t-s)\,(f_1(t,s) + f_2(t,s) - g_1(s) - g_2(s)) \\ &= (t-s)\,(f_1(t,s) - g_1(s)) + (t-s)\,(f_2(t,s) - g_2(s)) \\ &= h_1(t,s) + h_2(t,s), \end{aligned}$$

$t \neq s$, $h(s,s) = 0$ and it follows that $h \overset{B}{\equiv} 0$.

d) Let

$$\begin{aligned} h(t,s) &= (t-s)\,(f_1(t,s) f_2(t,s) - g_1(s) g_2(s)) \\ &= \left[(f_1(t,s) - g_1(s))\, g_2(s)\right](t-s) + f_1(t,s)\left[(t-s)\,(f_2(t,s) - g_2(s))\right], \end{aligned}$$

$t \neq s$, $h(s,s) = 0$. If we put $h_1(t,s) = \left[(f_1(t,s) - g_1(s))\, g_2(s)\right](t-s)$, $t \neq s$, $h_1(s,s) = 0$ and $h_2(t,s) = f_1(t,s)\left[(t-s)\,(f_2(t,s) - g_2(s))\right]$, $t \neq s$, $h_2(s,s) = 0$, then $h = h_1 + h_2$ and because $h_1 \overset{B}{\equiv} 0$, $h_2 \overset{B}{\equiv} 0$ it follows that $h \overset{B}{\equiv} 0$.

e) This is a direct consequence of the definition of global limit. □

3. TEMPORAL DERIVATIVE AND B-DERIVATIVE AS GLOBAL LIMITS

Theorem 3.1. *Let* $X\colon [0,\infty) \times \Omega \to \mathbb{R}$ *be a stochastic process.*

a) *If* X *is* θ*-differentiable and*

$$r_{B_i} X(t,s) = \frac{\Delta_{ts} X \Delta_{ts} B_i}{t-s}, i = 1, \ldots, m, \; r_t^{\theta} X(t,s) = \frac{\Delta_{ts} X - D_B X(\theta_{ts}) \Delta_{ts} B}{t-s}, \; t \neq s,$$

then

$$\mathrm{Lim}\, r_{B_i} X = D_{B_i} X, \; i = 1, \ldots, m$$

and

$$\mathrm{Lim}\, r_t^{\theta} X = D_t^{\theta} X.$$

b) *If there exist right-continuous functions* $u\colon \infty \times \Omega \to \mathbb{R}$, $v\colon [0,\infty) \times \Omega \to \mathbb{R}^m$ *such that* $\mathrm{Lim}\, r_v = u$, *where* $r_v(t,s) = \dfrac{\Delta_{ts} X - v(\theta_{ts}) \Delta_{ts} B}{t-s}$, $t \neq s$, *then* X *is* θ*-differentiable and* $D_t^{\theta} X = u$, $D_B X = v$.

Proof. a) Let $h_i(t,s) = (t-s)\left(\dfrac{\Delta_{ts}X\Delta_{ts}B_i}{t-s} - D_{B_i}X(s)\right)$, $t \neq s$, $h_i(s,s) = 0$, $i = 1,\dots,m$, and $H(t,s) = (t-s)\left(\dfrac{\Delta_{ts}X - D_BX(\theta_{ts})\Delta_{ts}B}{t-s} - D_t^\theta X(s)\right)$, $t \neq s$, $H(s,s) = 0$. We have to show that $h_1 \overset{B}{\equiv} 0,\dots,h_m \overset{B}{\equiv} 0$ and $H \overset{B}{\equiv} 0$.

Because $\Delta_{ts}X = D_t^\theta X(\theta_{ts})\Delta_{ts} + \sum_{k=1}^m D_{B_k}X(\theta_{ts})\Delta_{ts}B_k + h_X(t,s)$, where $h_X \overset{B}{\equiv} 0$, we obtain

$$h_i(t,s) = D_t^\theta X(\theta_{ts})\Delta_{ts}\Delta_{ts}B_i + \sum_{k=1}^m D_{B_k}X(\theta_{ts})\Delta_{ts}B_k\Delta_{ts}B_i + h_X(t,s)\Delta_{ts}B_i$$

$$-D_{B_i}X(s)\Delta_{ts} = D_t^\theta X(\theta_{ts})\Delta_{ts}\Delta_{ts}B_i + \sum_{k=1,k\neq i}^m D_{B_k}X(\theta_{ts})\Delta_{ts}B_k\Delta_{ts}B_i$$

$$+D_{B_i}X(\theta_{ts})((\Delta_{ts}B_i)^2 - \Delta_{ts}) + (D_{B_i}X(\theta_{ts}) - D_{B_i}X(s))\,\Delta_{ts} + \Delta_{ts}B_i h_X(t,s)$$

and $H(t,s) = \left(D_t^\theta X(\theta_{ts}) - D_t^\theta X(s)\right)\Delta_{ts} + h_X(t,s)$. Now it is obvious that $h_i \overset{B}{\equiv} 0$, $i = 1,\dots,m$, and $H \overset{B}{\equiv} 0$.

b) If we take $h(t,s) = (t-s)\left(\dfrac{\Delta_{ts}X - v(\theta_{ts})\Delta_{ts}B}{t-s} - u(s)\right)$, $t \neq s$, $h(s,s) = 0$, then $h \overset{B}{\equiv} 0$ and consequently if

$$\widetilde{h}(t,s) = (t-s)\left(\frac{\Delta_{ts}X - v(\theta_{ts})\Delta_{ts}B}{t-s} - u(\theta_{ts})\right) = h(t,s) - (u(\theta_{ts}) - u(s))\,(t-s),$$

$t \neq s$, $\widetilde{h}(s,s) = 0$, then $\widetilde{h} \overset{B}{\equiv} 0$. Further we obtain

$$\Delta_{ts}X = u(\theta_{ts})(t-s) + v(\theta_{ts})\,(\Delta_{ts}B)^T + \widetilde{h}(t,s)$$

and that is $dX \overset{\theta}{=} udt + vdB$. □

Remark 3.1. Theorem 3.1 says that if we want to see if X is θ-differentiable and to find the B-derivatives and the temporal derivative, we have to compute $D_{B_i}X = \mathrm{Lim}\, r_{B_i}X$, $i = 1,\dots,m$, and $D_t^\theta X = \mathrm{Lim}\, r_t^\theta X$.

If $\theta = 0$ we denote $r_t^I X$ and if $\theta = \dfrac{1}{2}$ we denote $r_t^S X$.

Example 3.1. Let $m = 1$ and $X(t) = \dfrac{1}{2}B^2(t)$.

We have $r_BX(t,s) = \dfrac{\Delta_{ts}\left(\frac{1}{2}B^2\right)\Delta_{ts}B}{t-s} = f_1(t,s)\cdot f_2(t,s)$, $t \neq s$, where $f_1(t,s) = \dfrac{B(t)+B(s)}{2}$, $f_2(t,s) = \dfrac{(\Delta_{ts}B)^2}{t-s}$. Since $\lim_{t\searrow s} f_1(t,s) = B(s)$ for each s and $\mathrm{Lim} f_2 = 1$, it follows that $D_B X = \operatorname{Lim} r_B X = B \cdot 1 = B$.

$$r_t^\theta X(t,s) = \frac{\Delta_{ts}\left(\frac{1}{2}B^2\right) - B(\theta_{ts})\Delta_{ts}B}{t-s} = \frac{\Delta_{ts}\left(\frac{1}{2}B^2\right) - B(s)\Delta_{ts}B}{t-s} - \frac{\Delta_{ts}^{\theta 0}B\Delta_{ts}B}{t-s}$$

$$= \frac{1}{2}\cdot\frac{(\Delta_{ts}B)^2}{t-s} - \theta\frac{\left(\Delta_{ts}^{\theta 0}B\right)^2}{\theta_{ts}-s} - \frac{\Delta_{ts}^{\theta 0}B\Delta_{ts}^{1\theta}B}{t-s},$$

$t \neq s$. Let $F_1(t,s) = \frac{(\Delta_{ts}B)^2}{t-s}$, $F_2(t,s) = \frac{\left(\Delta_{ts}^{\theta 0}B\right)^2}{\theta_{ts}-s}$ and $F_3(t,s) = \frac{\Delta_{ts}^{\theta 0}B\Delta_{ts}^{1\theta}B}{t-s}$. Since $\operatorname{Lim} F_1 = 1$, $\operatorname{Lim} F_2 = 1$ and $\operatorname{Lim} F_3(t,s) = 0$, it follows that $D_t^\theta X = \operatorname{Lim} r_t^\theta X = \frac{1}{2} - \theta$. Then

$$dX \overset{\theta}{=} \left(\frac{1}{2} - \theta\right)dt + BdB.$$

Example 3.2. Let $X(t) = tB(t)$.

$r_B X(t,s) = \frac{(tB(t) - sB(s))\Delta_{ts}B}{t-s} = t\frac{(\Delta_{ts}B)^2}{t-s} + B(s)\Delta_{ts}B$, $t \neq s$, and $D_B X = \operatorname{Lim} r_B X = s \cdot 1 + 0 = s$.

$r_t^\theta X(t,s) = \frac{tB(t) - sB(s) - \theta_{ts}\Delta_{ts}B}{t-s} = B(t) - \theta\Delta_{ts}B$, $t \neq s$, and $D_t^\theta X = \operatorname{Lim} r_t^\theta X = B$.

Then

$$dX \overset{\theta}{=} Bdt + tdB.$$

Example 3.3. Let $m = 1$ and $X = (B_1^2 + \cdots + B_m^2)^{\frac{1}{2}}$.

$$r_{B_i} X(t,s) = \frac{\Delta_{ts}X\Delta_{ts}B_i}{t-s} = \frac{\left(X^2(t) - X^2(s)\right)\Delta_{ts}B_i}{(t-s)\left(X(t)+X(s)\right)} = \frac{\sum_{k=1}^m \left(B_k^2(t) - B_k^2(s)\right)\Delta_{ts}B_i}{(t-s)\left(X(t)+X(s)\right)}$$

$$= \sum_{k=1,k\neq i}^m \frac{B_k(t)+B_k(s)}{X(t)+X(s)}\cdot\frac{\Delta_{ts}B_k\Delta_{ts}B_i}{t-s} + \frac{B_i(t)+B_i(s)}{X(t)+X(s)}\cdot\frac{(\Delta_{ts}B_i)^2}{t-s},$$

$t \neq s$.

$$\operatorname{Lim} r_{B_i} X = \sum_{k=1, k\neq i}^{m} \frac{B_k}{X} \cdot 0 + \frac{B_i}{X} \cdot 1 = \frac{B_i}{X},\ i = 1, \ldots, m.$$

$$r_t^I X(t,s) = \frac{X(t) - X(s) - \sum_{k=1}^{m} \frac{B_k(s)}{X(s)} \Delta_{ts} B_k}{t-s} = \frac{X(t)X(s) - \sum_{k=1}^{m} B_k(t)B_k(s)}{(t-s)\,X(s)}$$

$$= \frac{X^2(t)X^2(s) - \left(\sum_{k=1}^{m} B_k(t)B_k(s)\right)^2}{(t-s)\left(X(t)X(s) + \sum_{k=1}^{m} B_k(t)B_k(s)\right) X(s)}$$

$$= \frac{\sum_{k=2}^{m}\sum_{i=1}^{k-1} (B_i(t)B_k(s) - B_k(t)B_i(s))^2}{(t-s)\left(X(t)X(s) + \sum_{k=1}^{m} B_k(t)B_k(s)\right) X(s)}$$

$$= \frac{\sum_{k=2}^{m}\sum_{i=1}^{k-1} (\Delta_{ts}B_i B_k(s) - \Delta_{ts}B_k B_i(s))^2}{(t-s)\left(X(t)X(s) + \sum_{k=1}^{m} B_k(t)B_k(s)\right) X(s)}$$

$$= \sum_{k=2}^{m}\sum_{i=1}^{k-1} \frac{1}{\left(X(t)X(s) + \sum_{k=1}^{m} B_k(t)B_k(s)\right) X(s)} \cdot$$

$$\cdot \left[B_k^2(s) \frac{(\Delta_{ts}B_i)^2}{t-s} + B_i^2(s) \frac{(\Delta_{ts}B_k)^2}{t-s} - 2B_k(s)B_i(s) \frac{\Delta_{ts}B_i \Delta_{ts}B_k}{t-s} \right],$$

$t \neq s$.

$\operatorname{Lim} r_t^I X = \sum_{k=2}^{m}\sum_{i=1}^{k-1} \frac{B_k^2 + B_i^2}{2X^3} = \frac{m-1}{2X}.$

Consequently

$$dX \overset{I}{=} \frac{m-1}{2X} dt + \sum_{i=1}^{m} \frac{B_i}{X} dB_i.$$

Example 3.4. Let $X = \exp(Y)$, $Y(t) = \mathrm{ct} + \sum_{k=1}^{m} \alpha_k B_k(t)$.

$$r_{B_i} X(t,s) = \frac{\Delta_{ts} X \Delta_{ts} B_i}{t-s} = \frac{\Delta_{ts}\exp(Y)}{\Delta_{ts} Y} \cdot \frac{\Delta_{ts} Y \Delta_{ts} B_i}{t-s}$$

$$= \frac{\Delta_{ts}\exp(Y)}{\Delta_{ts} Y}\left(c\Delta_{ts} B_i + \sum_{k=1}^{m} \alpha_k \frac{\Delta_{ts} B_k \Delta_{ts} B_i}{t-s}\right), \qquad t \neq s.$$

$\operatorname{Lim} r_{B_i} X = \exp(Y)\left(c \cdot 0 + \sum_{k=1,\, k\neq i}^{m} \alpha_k \cdot 0 + \alpha_i\right) = \alpha_i X.$

$$r_t^I X(t,s) = \frac{X(t) - X(s) - \sum_{k=1}^{m} \alpha_k X(s) \Delta_{ts} B_k}{t-s} = X(s)\frac{\exp(\Delta_{ts} Y) - 1 - \Delta_{ts} Y + c\Delta_{ts}}{t-s},$$

$t \neq s$. Using Taylor formula, we get

$$r_t^I X(t,s) = cX(s) + \frac{1}{2}X(s)\frac{(\Delta_{ts} Y)^2}{t-s} + \frac{1}{2}X(s)\frac{(\Delta_{ts} Y)^2 \omega(\Delta_{ts} Y)}{t-s},$$

where ω is continuous and $\lim_{x\to 0} \omega(x) = 0$. But

$$(\Delta_{ts} Y)^2 = \left(c\Delta_{ts} + \sum_{k=1}^{m} \alpha_k \Delta_{ts} B_k\right)^2 = c^2 \Delta_{ts}^2 + \sum_{k=1}^{m} \alpha_k^2 \left(\Delta_{ts} B_k\right)^2$$

$$+2c\Delta_{ts} \sum_{k=1}^{m} \alpha_k \Delta_{ts} B_k + 2\sum_{i\neq j} \alpha_i \alpha_j \Delta_{ts} B_i \Delta_{ts} B_j$$

and

$$r_t^I X(t,s) = cX(s) + \frac{1}{2}X(s)(c^2\Delta_{ts} + \sum_{k=1}^{m} \alpha_k^2 \frac{(\Delta_{ts} B_k)^2}{t-s} + 2c\sum_{k=1}^{m} \alpha_k \Delta_{ts} B_k$$

$$+2\sum_{i\neq j} \alpha_i \alpha_j \frac{\Delta_{ts} B_i \Delta_{ts} B_j}{t-s})\left(1 + \omega(\Delta_{ts} Y)\right), \qquad t \neq s.$$

$\operatorname{Lim} r_t^I X = \left(c + \frac{1}{2}\sum_{k=1}^{m} \alpha_k^2\right) X$. We obtain

$$dX \stackrel{I}{=} \left(c + \frac{1}{2}\sum_{k=1}^{m} \alpha_k^2\right) X dt + \sum_{k=1}^{m} \alpha_k X dB_k.$$

Lemma 3.1. *Let $X\colon [0,\infty) \times \Omega \to \mathbb{R}$ be a differentiable stochastic process, $\theta, \sigma \in [0,1]$, $\theta > \sigma$, and $f(t,s) = \frac{\Delta_{ts}^{\theta\sigma} X \Delta_{ts} B_k}{t-s}$, $t \neq s$. Then* $\operatorname{Lim} f = (\theta - \sigma) D_{B_k} X$.

Proof. The following equalities hold true

$$\begin{aligned}(t-s)\left(f(t,s)-(\theta-\sigma)D_{B_k}X\right) &= \Delta_{ts}^{\theta\sigma}X\Delta_{ts}B_k-(\theta-\sigma)(t-s)D_{B_k}X\\ &= D_t^I X(s)(\theta_{ts}-\sigma_{ts})\Delta_{ts}B_k+\sum_{\ell=1}^{m}D_{B_\ell}X(s)\Delta_{ts}^{\theta\sigma}B_\ell\Delta_{ts}B_k\\ &\quad +h(t,s)\Delta_{ts}B_k-(\theta-\sigma)(t-s)D_{B_k}X,\end{aligned}$$

where $h\overset{B}{\equiv}0$. Since $D_t^I X(s)(\theta_{ts}-\sigma_{ts})\Delta_{ts}B_k = D_t^I X(s)(\theta-\sigma)(t-s)\Delta_{ts}B_k\overset{B}{\equiv}0$, $\sum\limits_{\ell=1,\,\ell\neq k}^{m}D_{B_\ell}X(s)\Delta_{ts}^{\theta\sigma}B_\ell\Delta_{ts}B_k\overset{B}{\equiv}0$ and $h(t,s)\Delta_{ts}B_k\overset{B}{\equiv}0$, we get

$$\begin{aligned}(t-s)\left(f(t,s)-(\theta-\sigma)D_{B_k}X\right) &\overset{B}{\equiv} D_{B_k}X(s)\Delta_{ts}^{\theta\sigma}B_k\Delta_{ts}B_k-(\theta-\sigma)(t-s)D_{B_k}X\\ &= D_{B_k}X(s)\Delta_{ts}^{\theta\sigma}B_k\Delta_{ts}^{1\theta}B_k+D_{B_k}X(s)\Delta_{ts}^{\theta\sigma}B_k\Delta_{ts}^{\sigma 0}B_k+D_{B_k}X(s)\left(\Delta_{ts}^{\theta\sigma}B_k\right)^2\\ &\quad -(\theta-\sigma)(t-s)D_{B_k}X\overset{B}{\equiv}D_{B_k}X(s)\left[\left(\Delta_{ts}^{\theta\sigma}B_k\right)^2-(\theta_{ts}-\sigma_{ts})\right]\overset{B}{\equiv}0,\end{aligned}$$

which completes the proof. □

Consequence 3.1. *Under the conditions of Lemma* 3.1

$$D_t^\sigma X = D_t^\theta X+(\theta-\sigma)\triangle_B X,$$

where $\triangle_B X=\sum\limits_{k=1}^{m}D_{B_k}\left(D_{B_k}X\right).$

If we take Lim in both sides of the equality

$$r_t^\sigma X(t,s)=r_t^\theta X(t,s)+\sum_{k=1}^{m}\frac{\Delta_{ts}^{\theta\sigma}\left(D_{B_k}X\right)\Delta_{ts}B_k}{t-s}$$

and we use Lemma 3.1, we obtain

$$D_t^\sigma X = D_t^\theta X+(\theta-\sigma)\sum_{k=1}^{m}D_{B_k}\left(D_{B_k}X\right).$$

4. GENERALIZED ITÔ FORMULA (θ-FORMULA)

In [1] we proved θ-formula. In this section, using the global limit, we give another proof.

Theorem 4.1 (θ-formula)**.** *Let* $\theta\in[0,1]$, $g\colon\mathbb{R}^n\to\mathbb{R}$, $g\in C^3(\mathbb{R}^n)$ *and let* $X\colon\infty\times\Omega\to\mathbb{R}^n$, $X=(X_1,\dots,X_n)$, *be differentiable stochastic process. Then* *the following stochastic differential equality holds*

$$dg(X) \overset{\theta}{=} \left(\nabla g(X) D_t^{\theta} X + \left(\frac{1}{2} - \theta \right) \sum_{i,j=1}^{n} \frac{\partial^2 g}{\partial x_i \partial x_j}(X) D_B X_i \cdot (D_B X_j)^T \right) dt$$
$$+ \nabla g(X) D_B X dB.$$

Proof. Using Taylor formula, we can write

$$\Delta_{ts} g(X) = \sum_{i=1}^{n} \frac{\partial g}{\partial x_i}(X(s)) \Delta_{ts} X_i + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 g}{\partial x_i \partial x_j}(X(s)) \Delta_{ts} X_i \Delta_{ts} X_j$$
$$+ \omega_s(X(t)) \sum_{i=1}^{n} (\Delta_{ts} X_i)^2,$$

where ω_s is continuous and $\lim\limits_{x \to X(s)} \omega_s(x) = 0$. X is differentiable, hence X is right-continuous and then $\lim\limits_{t \searrow s} \omega_s(X(t)) = 0$.

$$r_{B_k} g(X)(t,s) = \frac{\Delta_{ts} g(X) \Delta_{ts} B_k}{t-s} = \sum_{i=1}^{n} \frac{\partial g}{\partial x_i}(X(s)) r_{B_k} X_i(t,s)$$
$$+ \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 g}{\partial x_i \partial x_j}(X(s)) \Delta_{ts} X_i r_{B_k} X_j(t,s)$$
$$+ \omega_s(X(t)) \sum_{i=1}^{n} \Delta_{ts} X_i r_{B_k} X_i(t,s).$$

We obtain $v_k = \mathrm{Lim}\, r_{B_k} g(X) = \sum_{i=1}^{n} \frac{\partial g}{\partial x_i}(X) D_{B_k} X_i$, $k = 1, \ldots, m$.

For $t \neq s$,

$$r_v(t,s) = \frac{\Delta_{ts} g(X) - \sum_{k=1}^{m} v_k(\theta_{ts}) \Delta_{ts} B_k}{t-s}$$
$$= \frac{1}{t-s} (\Delta_{ts} g(X) - \sum_{k=1}^{m} \sum_{i=1}^{n} \frac{\partial g}{\partial x_i}(X(\theta_{ts})) D_{B_k} X_i(\theta_{ts}) \Delta_{ts} B_k)$$
$$= \frac{1}{t-s} (\sum_{i=1}^{n} \frac{\partial g}{\partial x_i}(X(s)) \Delta_{ts} X_i + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 g}{\partial x_i \partial x_j}(X(s)) \Delta_{ts} X_i \Delta_{ts} X_j$$
$$+ \omega_s(X(t)) \sum_{i=1}^{n} (\Delta_{ts} X_i)^2 - \sum_{i=1}^{n} \frac{\partial g}{\partial x_i}(X(\theta_{ts})) \sum_{k=1}^{m} D_{B_k} X_i(\theta_{ts}) \Delta_{ts} B_k)$$

$$= \sum_{i=1}^{n} \frac{\partial g}{\partial x_i}(X(s)) r_t^{\theta} X_i(t,s) - \sum_{i=1}^{n} \Delta_{ts}^{\theta 0} \left(\frac{\partial g}{\partial x_i}(X) \right) \sum_{k=1}^{m} D_{B_k} X_i(\theta_{ts}) \frac{\Delta_{ts} B_k}{t-s}$$
$$+ \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 g}{\partial x_i \partial x_j}(X(s)) \frac{\Delta_{ts} X_i \Delta_{ts} X_j}{t-s} + \omega_s(X(t)) \sum_{i=1}^{n} \frac{(\Delta_{ts} X_i)^2}{t-s}.$$

Using again Taylor formula, we get

$$\Delta_{ts}^{\theta 0} \left(\frac{\partial g}{\partial x_i}(X) \right) = \sum_{j=1}^{n} \frac{\partial^2 g}{\partial x_i \partial x_j}(X(s)) \Delta_{ts}^{\theta 0} X_j + \frac{1}{2} \sum_{j,\ell=1}^{n} \frac{\partial^3 g}{\partial x_i \partial x_j \partial x_\ell}(X(s)) \Delta_{ts}^{\theta 0} X_j \Delta_{ts}^{\theta 0} X_\ell$$
$$+ \omega_s^i(X(\theta_{ts})) \sum_{j=1}^{n} \left(\Delta_{ts}^{\theta 0} X_j \right)^2,$$

where $\lim_{t \searrow s} \omega_s^i(t) = 0$ and

$$r_v(t,s) = \sum_{i=1}^{n} \frac{\partial g}{\partial x_i}(X(s)) r_t^{\theta} X_i(t,s) + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 g}{\partial x_i \partial x_j}(X(s)) \frac{\Delta_{ts} X_i \Delta_{ts} X_j}{t-s}$$
$$- \sum_{i,j=1}^{n} \frac{\partial^2 g}{\partial x_i \partial x_j}(X(s)) \sum_{k=1}^{m} D_{B_k} X_i(\theta_{ts}) \frac{\Delta_{ts}^{\theta 0} X_j \Delta_{ts} B_k}{t-s}$$
$$- \frac{1}{2} \sum_{i,j,\ell=1}^{n} \frac{\partial^3 g}{\partial x_i \partial x_j \partial x_\ell}(X(s)) \Delta_{ts}^{\theta 0} X_j \Delta_{ts}^{\theta 0} X_\ell \sum_{k=1}^{m} D_{B_k} X_i(\theta_{ts}) \frac{\Delta_{ts} B_k}{t-s}$$
$$+ \sum_{i=1}^{n} \omega_s^i(X(\theta_{ts})) \sum_{j=1}^{n} \left(\Delta_{ts}^{\theta 0} X_j \right)^2 \sum_{k=1}^{m} D_{B_k} X_i(\theta_{ts}) \frac{\Delta_{ts} B_k}{t-s}$$
$$+ \omega_s(X(t)) \sum_{i=1}^{n} \frac{(\Delta_{ts} X_i)^2}{t-s}.$$

If $f(t,s) = \dfrac{\Delta_{ts} X_i \Delta_{ts} X_j}{t-s}$ and $g(t,s) = \dfrac{\Delta_{ts}^{\theta 0} X_j \Delta_{ts} B_k}{t-s}$, then

$$\operatorname{Lim} f = \sum_{k=1}^{m} D_{B_k} X_i D_{B_k} X_j$$

(Example 2.4), $\operatorname{Lim} g = \theta D_{B_k} X_j$ (Lemma 3.1), and we get

$$\operatorname{Lim} r_v = \sum_{i=1}^{n} \frac{\partial g}{\partial x_i}(X) D_t^{\theta} X_i + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 g}{\partial x_i \partial x_j}(X) \sum_{k=1}^{m} D_{B_k} X_i D_{B_k} X_j$$
$$- \sum_{i,j=1}^{n} \frac{\partial^2 g}{\partial x_i \partial x_j}(X) \theta \sum_{k=1}^{m} D_{B_k} X_i D_{B_k} X_j,$$

which completes the proof. □

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