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MORE ON SELF-CONCORDANT FUNCTIONS ON RIEMANNIAN MANIFOLDS

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Dedicated to Professor Ştefan Mititelu
on the occasion of his seventieth birthday

ABSTRACT. The aim of this work is to bring together some contributions to the development of the class of self-concordant functions on Riemannian manifolds, emphasizing on the class of self-concordant barriers. Due to their properties, this class of functions is intensively studied to develop interior point methods for convex and nonlinear optimization. The present study comes to complete some of our recent results, published in Balkan J. Geom. Appl., 2009, and Sci. Bull. UPB, Series A: Appl. Math. Phys., 2010 (see [1] and [3]).

1. INTRODUCTION

The origin of self-concordant function must be traced back to the works of Nesterov and Nemirovsky [18], which introduced such functions as barriers in interior point methods for optimization algorithms in Euclidean spaces. Recent works by famous scientists suggest that such kind of methods are better understood on manifolds rather than on Euclidean space. To have some illustrative examples, see [16] by Jiang, Moore and Ji, for Newton-type algorithms on smooth manifolds; [15] by D. den Hertog, for interior point methods with applications to linear, quadratic and convex programming; [26] by Udrişte, for a study of the Riemannian context of optimization methods. Therefore, the notion of self-concordant function has been introduced on Riemannian manifolds due to the necessity to develop a large class of optimization methods.

Due to its practical importance, this topic brought into our attention too, see [1] and [3] for our contribution to this field. In our paper [1], we introduced a new class of self-concordant functions, defined on Riemannian manifolds endowed with metrics of diagonal type. Later, in the work [3], we introduced a class of self-concordant functions which generate the given metric and we have given a strong motivation to

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the study of self-concordant functions on manifolds. It is the aim of this study to bring together some contributions to the development of the class of self-concordant functions and to complete some of our recent results, published in [1] and [3].

2. PRELIMINARIES

First of all, we have to introduce the general framework and the definition that will be used throughout the paper.

On the Riemannian manifold (M, g), we denote by ∇ the Levi-Civita connection induced by the metric g. Consider a function $f\colon M \to \mathbb{R}$, defined on an open domain, as closed mapping, that is $\{(f(P), P),\ P \in \mathrm{Dom}\,(f)\}$ is a closed set in the product manifold $\mathbb{R} \times M$. Suppose f be at least three times differentiable.

According to [16] and [24], we introduce

Definition 2.1. The function f is said to be *k-self-concordant*, $k \geq 0$, with respect to the Levi-Civita connection ∇ defined on M if the following condition holds:

$$\left|\nabla^3 f(x)(X_x, X_x, X_x)\right| \leq 2k \left(\nabla^2 f(x)(X_x, X_x)\right)^{\frac{3}{2}}, \quad \forall x \in M,\ \forall X_x \in T_x M.$$

As we can easily see, the self-concordant function is introduced in Definition 2.1 on a differential manifold in such a way that the properties of self-concordant functions in Euclidean space are preserved. We underline that the self concordance is preserved under addition, affine transformations, and scalar multiplication by a value greater than one.

3. ILLUSTRATIVE EXAMPLES

The main goal of this section, is to give a proper justification regarding the significance of the research of self-concordant function on manifolds [3], [8]. That is, we give answer to this problem:

$\star$ PROBLEM. Can we find self-concordant functions on Riemannian manifold, which cannot be represented as self-concordant functions in Euclidean space?

• Let $\mathbb{R}$ be the set of all real numbers and x a point on $\mathbb{R}$. According to [28] a Riemannian metric on $\mathbb{R}$ is a function $g\colon \mathbb{R} \to (0, \infty)$, which is assumed of C^∞-class. This metric yields the linear connection $\Gamma(x) = \dfrac{d}{dx} \ln \sqrt{g(x)}$. Given a function f, at least two times differentiable, then the Hessian has the form $\nabla^2 f = f'' - \Gamma f'$.

Case 1. Consider $M = \mathbb{R}$ endowed with the metric $g(x) = \exp(2x)$. In this case $\Gamma(x) = 1$, for all real x. For the real function $f(x) = \exp(x)$, it follows $\nabla^2 f = 0$ and $\nabla^3 f = 0$, therefore f is self-concordant. If we consider on $M = \mathbb{R}$ the Euclidean metric $g(x) = 1$, then $\Gamma(x) = 0$, for all real x, and the self-concordance condition for the same function f leads to $1 \leq 4k^2 \exp(x)$, which is not true for all x.

CASE 2. Suppose that $M = (0, \infty)$ is endowed with the metric $g(x) = \dfrac{1}{x^2}$. In this case $\Gamma(x) = -\dfrac{1}{x}$, for all real $x \in (0, \infty)$. On $(0, \infty)$, consider the real function $f(x) = -\ln x$. It follows $\nabla^2 f = 0$ and $\nabla^3 f = 0$, therefore f is k-self-concordant. If we consider on $M = (0, \infty)$ the Euclidean metric $g(x) = 1$, then the self-concordance condition for the same function f leads to $k \geq 1$. Therefore, the function f is not k-self-concordant with respect to the Euclidean metric if $k \in (0, 1)$.

Remark that the previous two examples are thought to be elementary, the functions under consideration being linear geodesic.

- Let us consider the Poincaré plane

$$H = \left\{(x, y) \in \mathbb{R}^2 \mid y > 0\right\}, \qquad g_{ij} = \frac{1}{y^2}\delta_{ij}.$$

The components of the Riemannian connection are

$$\Gamma^1_{11} = \Gamma^1_{22} = \Gamma^2_{12} = \Gamma^2_{21} = 0, \quad \Gamma^1_{12} = \Gamma^1_{21} = \Gamma^2_{22} = -\frac{1}{y}, \quad \Gamma^2_{11} = \frac{1}{y}.$$

If we consider a function f on (H, g_{ij}), and we denote

$$f_{,ij} = \frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma^s_{ij}\frac{\partial f}{\partial x^s}, \quad f_{,ijk} = f_{,ij,k} = \partial_k f_{,ij} - f_{,sj}\Gamma^s_{ki} - f_{,si}\Gamma^s_{kj}$$

then we have

$$\nabla^2 f(x)(X_x, X_x) = f_{,ij}X^i X^j, \qquad \nabla^3 f(x)(X_x, X_x, X_x) = f_{,ijk}X^i X^j X^k.$$

If $f(x, y) = \dfrac{1}{y}$ and $X_x = (u, v)$, after some calculations, we find

$$\nabla^2 f(x, y)(X_x, X_x) = \frac{u^2 + v^2}{y^3}, \quad \nabla^3 f(x, y)(X_x, X_x, X_x) = -\frac{v(u^2 + v^2)}{y^4}.$$

Directly from Definition 2.1, we get

Proposition 3.1. *On the set* $A = \{(x, y) \mid x \in \mathbb{R}, \quad 0 < y \leq 4k^2\}$, *the function* $f(x, y) = \dfrac{1}{y}$ *is k-self-concordant.*

- Consider the Euclidean plane

$$M = \left\{(x, y) \in \mathbb{R}^2 \mid y > 0\right\}, \qquad g_{ij} = \delta_{ij}.$$

The connection components are $\Gamma^k_{ij} = 0$.

If $f(x, y) = \dfrac{1}{y}$ and $X_x = (u, v)$, after some calculations, we find

$$\nabla^2 f(x, y)(X_x, X_x) = \frac{2v^2}{y^3}, \quad \nabla^3 f(x, y)(X_x, X_x, X_x) = -\frac{6v^3}{y^4}.$$

Using Definition 2.1, we can see that the function f is k-self-concordant if $y \leq \dfrac{8k^2}{9}$, and we have

Proposition 3.2. *On the set* $A = \left\{(x,y) \,|\, x \in \mathbb{R},\quad \dfrac{8k^2}{9} < y \leq 4k^2\right\}$*, the function* $f(x,y) = \dfrac{1}{y}$ *is k-self-concordant on the Poincaré plane but is not k-self-concordant on the Euclidean plane.*

The results in this section (for example, Proposition 3.1 for the Poincaré plane) show that there exist self-concordant functions on Riemannian manifolds which cannot be represented as self-concordant functions in Euclidean spaces.

4. SELF SELF-CONCORDANT FUNCTIONS

Given the metric g, and inspired by the linearity of the set of self-concordant functions [18], in our work [1], we used decomposable functions $f\colon \mathbb{R}^n_+ \to \mathbb{R}$, of the form

$$f(x^1, x^2, \dots, x^n) = f_1(x^1) + f_2(x^2) + \cdots + f_n(x^n), \tag{4.1}$$

to find a new class of self-concordant functions. Here $f_i\colon \mathbb{R}_+ \to \mathbb{R}$ are differentiable functions.

On the Riemannian manifold $M = \mathbb{R}^n_+$, endowed with the diagonal metric

$$g(x^1, x^2, \dots, x^n) = \operatorname{diag}\left(\frac{1}{g_1^2(x^1)}, \frac{1}{g_2^2(x^2)}, \cdots \frac{1}{g_n^2(x^n)}\right), \tag{4.2}$$

where the functions $\dfrac{1}{g_i}$ admit upper bounded primitives, we considered the case of differential equality in Definition 2.1, and taking into account the form of function f in (4.1), we found

Theorem 4.1. *Let us suppose that the manifold* $M = \mathbb{R}^n_+$ *is endowed with the diagonal metric* (4.2)*, and for each* $i = \overline{1,n}$ *the function* g_i *satisfies the inequalities* $\displaystyle\int \frac{1}{g_i(x^i)} dx^i < 0$ *(no summation). If*

$$f_i(x^i) = \frac{1}{k^2} \int \left[\frac{1}{g_i(x^i)} \int \frac{1}{g_i(x^i)\left(\displaystyle\int \frac{1}{g_i(x^i)} dx^i\right)^2} dx^i\right] dx^i, \qquad i = \overline{1,n}. \tag{4.3}$$

then the decomposable function f, defined by (4.1)*, is k-self-concordant.*

We remark that the metrics in (4.2) are particular cases of Hessian metrics, [2], [5], [24]. Indeed, the decomposable function $H = \sum_{i=1}^{n} H_i(x^i)$, satisfies

$$\frac{\partial^2 H}{\partial x^i \partial x^j} = H_i''(x^i)\delta_{ij}, \quad i = \overline{1,n},\ j = \overline{1,n}.$$

Such kind of metrics are used by Papa Quiroz [20] and Rapcsák [21], [22] to solve wide classes of problems arising from linear optimizations and nonlinear optimizations, respectively. Moreover, Hessian type metrics are useful tools in solving specific problems of WDVV PDEs of string theory [6].

Now, we would like to find decomposable functions f as in (4.1) which satisfy Theorem 4.1 and generate the metric g in (4.2), that is to have

$$\frac{1}{g_i} = f_i'', \text{ for all } i = \overline{1,n},$$

(we call such kind of function *self self-concordant*).

After a technical computation, we get the criterion in

Theorem 4.2 (UDRIŞTE ODE). *Suppose the functions f_i, $i = \overline{1,n}$, are given by formulas* (4.3). *Then a sufficient condition for the function f in* (4.1) *to be self self-concordant is that each f_i is the solution of the following* ODE

$$\left(f_i' + \alpha\right)\exp\left(k^2 f_i\right) - \beta \exp\left(\gamma f_i'\right) = 0, \qquad i = \overline{1,n} \tag{4.4}$$

where α, β, and γ are real constants.

Theorem 4.2 indicates a wide class of self self-concordant functions. As the reader can see, for $\alpha = \gamma = 0$ and $\beta = 1$ one obtains Theorem 2.4 in [1] which claims that the Shannon entropy [23] is a self self-concordant function.

- CASE STUDY (LINEAR APPROXIMATION OF 4.4).

If in equation (4.4), we consider the expansion of exp function up to $0(x)$, after calculation, we find

$$\left(1 - \beta\gamma + k^2 y\right) y' + \alpha k^2 y + \alpha - \beta = 0.$$

In a very special case, $\alpha = \beta$, and $f_i > 0$, $i = \overline{1,n}$, we get

$$(1 - \alpha\gamma)\ln f_i + k^2 f_i = -\alpha k^2 x_i + k_0, \quad i = \overline{1,n},$$

k_0 being an arbitrary constant.

We can see that for higher order approximations in (4.4), we obtain forms which can be studied either by asymptotic methods or by numerical methods using computer experiments. To make a computer aided study of this equation we can perform symbolic computations too. In this respect, we recommend the MAPLE software package [7], [27].

5. A SPECIAL CASE STUDY

Let us consider the manifold $M = \mathbb{R}^2$, endowed with the Riemannian metric $\bar{g}$ of diagonal type, $\bar{g}(x,y) = \operatorname{diag}(g(x)\ ,\ 1)$. Here g is a positive real function of C^∞-class. It follows that $\bar{g}$ produces a Riemannian metric [28].

Given a function $f\colon M \to \mathbb{R}$, we find sufficient conditions for f to be k-self-concordant on the manifold $(M, \bar{g})$. We indicate procedures for finding such kind of functions on the manifold $(M, \bar{g})$. This new class of functions improves and generalizes the results in [3].

After long calculations, we find the criterion in

Proposition 5.1. *Let us consider* $f(x,y) := f(x)$ *and denote* $p = \dfrac{1}{f'}\left(\dfrac{f'^2}{g}\right)'$. *The function* f *is* k*-self-concordant if and only if* $p > 0$ *and* ${p'}^2 \leq 2k^2 g p^3$.

The proof of this result assumes technical calculations and will be given in a forthcoming work.

Corollary 5.1 in the following, assumes in Proposition 5.1 that $p(x) = c$, where c is a positive real constant and I a real interval such that $f(x) + a > 0$.

In this case, from the condition in Proposition 5.1, we write

$$\frac{f'^2(x)}{g(x)} = c(f(x) + a). \tag{5.1}$$

We think (5.1) as an equation in g, and we obtain

Corollary 5.1. *Any real function* f*, whose values belongs to* I*, is* k*-self-concordant with respect to the metric* $\bar{g}$*, for*

$$g(x) = \frac{f'^2(x)}{c(f(x) + a)}.$$

We now think the equation (5.1) as an equation in f. In this case, we studied two particular cases.

THE CASE $g = f'$. From equation (5.1), we immediately find $[\ln(f(x) + a)]' = c$. After integration, we get $f(x) = \exp(cx + b) - a$, for some real constants a, b, and c, with $a > 0$.

Corollary 5.2. *The function* $f(x) = \exp(cx + b) - a$, $x \in \mathbb{R}$, *is* k*-self-concordant with respect to the metric* $\bar{g}$*, for* $g(x) = c\exp(cx + b)$, $c > 0$.

THE CASE $g = \dfrac{1}{f'}$. With (5.1), we obtain $(f'(x))^3 = c(f(x) + a)$. Denote $s(x) = f(x) + a$ and the previous relation becomes $s'(x)s^{-\frac{1}{3}}(x) = \sqrt[3]{c}$. After an integration, we find

Corollary 5.3. *The function*

$$f\colon \left(-\frac{b}{\sqrt[3]{c}}, \infty\right) \to \mathbb{R}, \quad f(x) = \left(\frac{2}{3}\left(\sqrt[3]{c}x + b\right)\right)^{\frac{3}{2}} - a$$

is k-self-concordant with respect to the metric $\bar{g}$, for $g(x) = \dfrac{\sqrt{3}}{\sqrt[3]{c}\sqrt{2\left(\sqrt[3]{c}x + b\right)}}$.

Corollary 5.4, holds for $\dfrac{f'^2}{g} = p$ in Proposition 5.1 and assumes $k \geq \dfrac{1}{\sqrt{2}}$.

Corollary 5.4. *Any real function f, whose values belongs to $\mathbb{R}$ is k-self-concordant with respect to the metric $\bar{g}$, for*

$$g(x) = f'^2(x)\exp(-f(x)).$$

We underline the importance of metric $\bar{g}$ in the study of iterative 2D Hessian structures [4].

6. SELF-CONCORDANT BARRIERS

In constrained optimization, a barrier function is a continuous function whose value at a point increases to infinity as the point approaches the boundary of the feasible region [19]. It is used as a penalizing term for violations of constraints. The two most common types of barrier functions are inverse barrier functions and logarithmic barrier functions, see [10] and [25].

According to [1], [11], [25] we have

Definition 6.1. The function f is said to be *c-self-concordant barrier*, $c \geq 0$, with respect to the Levi-Civita connection ∇ defined on M, if the following condition holds:

$$(df(x)(X_x))^2 \leq c\,\nabla^2 f(x)(X_x, X_x), \quad \forall x \in M,\ \forall X_x \in T_xM.$$

Self-concordant barrier functions are intensively studied to develop the barrier functions used in interior point methods for convex and nonlinear optimization. We have in mind the works by Duistermaat [9] on the boundary behavior of the Riemannian structure of a self-concordant barrier function and the works by Güler on interior point methods for convex programming [12], [13].

After a straightforward calculation, we obtain

Proposition 6.1. *On the set* $A = \left\{(x, y)\,|\, x \in \mathbb{R},\quad \dfrac{8k^2}{9} < y \leq 4k^2\right\}$, *the function* $f(x, y) = \dfrac{1}{y}$ *is $64k^6$-self-concordant barrier.*

Also, in [2] is given with proof the following result:

Proposition 6.2. *The function* $f : (0,1) \times (0,1) \to \mathbb{R}$,

$$f(x,y) = x \arctan\, x - \frac{1}{2} \ln(1 + x^2) + y \arctan\, y - \frac{1}{2} \ln(1 + y^2)$$

is a 2*-self-concordant function and a* $\frac{\pi^2}{2}$*-self-concordant barrier.*

The results in Proposition 6.1 and Proposition 6.2 are deep ones since finding self-concordant barriers is an actual problem of Optimization Theory. We emphasize the theories of Nesterov and Nemirovsky [18] which use this class of functions for developing linear and convex quadratic programs with convex quadratic constraints, and Udrişte's works [26], [28] which develop barrier methods for smooth convex programming on Riemannian manifolds. For other advances on this field, we address the reader to [17] and [24].

7. CONCLUSIONS

In our previous work [1], we introduced and studied two wide classes of self-concordant functions, defined on Riemannian manifolds endowed with metrics of diagonal type. Next, in our note [3], we have further developed the results published in [1] by introducing a class of self-concordant functions which generates the given metric (UDRIŞTE ODE). Also, we have given the motivation for the study of self-concordant functions on manifolds. In this work, we unified and further developed these results, suggesting relevant links between Differential Geometry and Applied Sciences, see also [4], [8] and [29].

OPEN PROBLEM. Find other classes of self-concordant functions (barriers) with respect to given metrics. Apply them to appropriate optimization problems.

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