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MINIMAL SURFACES BETWEEN TWO POINTS

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ABSTRACT. We formulate and study two-time optimal controlled problems which lead us to minimal surfaces. The proper technique to solve such problems is the two-time maximum principle invented by our research group.

Section 1 studies the minimal surfaces as optimal evolutions via the two-time maximum principle. The evolution PDE is of 2-flow type and the adjoint PDE is of divergence type. Section 2 analyzes the minimal surfaces evolving around an obstacle. Sections 3 and 4 reconsider the same problem for touching an obstacle, detailing the results for the cylinder and the sphere.

1. MINIMAL SURFACES AS OPTIMAL EVOLUTIONS

The minimal surfaces are characterized by zero mean curvature. These become an area of intense mathematical and scientific study over the past 15 years, specifically in the areas of molecular engineering and materials sciences due to their anticipated nanotechnology applications. The most extensive meeting ever held on the subject, in its 250-year history, was organized in 2001 at Clay Mathematics Institute. In spite of all these efforts, the old thinking about minimality was not changed.

Recently, the first author changes the traditional geometrical viewpoint (contained in [1]-[5]), looking at a minimal surface as solution in a two-time optimal control system via the multitime maximum principle (see [6]-[23]). On the other hand, our study shows, on a classical problem, the efficiency of multitime optimal control.

Let $\Omega_{0\tau}$ be a two dimensional interval fixed by the diagonal opposite points $0,\ \tau \in \mathbb{R}^2_+$. Looking for surfaces $x^i(t) = x^i(t^1, t^2)$, $(t^1, t^2) \in \Omega_{0\tau}$, $i = 1, 2, 3$, of minimum area, that evolve between two points $x(0)$, $x(\tau)$ and rely transversally on two curves Γ_0 and Γ_1, let us show that such a surface (2-sheet) is a solution of a special PDE system, via the optimal control theory (multitime maximum principle).

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In $\mathbb{R}^3$ we introduce the two-time controlled dynamics

$$\text{(PDE)} \qquad \frac{\partial x^i}{\partial t^\alpha}(t) = u^i_\alpha(t),$$

$$t = (t^1, t^2) \in \Omega_{0\tau},\ i = 1, 2, 3;\ \alpha = 1, 2,\ x^i(0) = x^i_0,\ x^i(\tau) = x^i_1,$$

where $u = (u_\alpha) = (u^i_\alpha) : \Omega_{0\tau} \to \mathbb{R}^6$ represents two open-loop C^1 control vectors, linearly independent, eventually fixed on the boundary $\partial\Omega_{0\tau}$. The complete integrability conditions of the (PDE) system, restrict the set of controls to

$$\mathcal{U} = \left\{ u = (u_\alpha) = (u^i_\alpha) \,\middle|\, \frac{\partial u^i_1}{\partial t^2}(t) = \frac{\partial u^i_2}{\partial t^1}(t) \right\}.$$

A solution of the (PDE) system is a surface (2-sheet) $\sigma : x^i = x^i(t^1, t^2)$. Suppose $x(0) = x_0$ belongs to the image Γ_0 of a curve in $\mathbb{R}^3$ and $\tau = (\tau^1, \tau^2)$ is the two-time when the 2-sheet $x(t^1, t^2)$ reaches the curve Γ_1 in $\mathbb{R}^3$, at $x(\tau) = x_1$, with Γ_0 and Γ_1 transversal to σ. Using the area, we introduce the cost functional

$$(J) \qquad J(u(\cdot)) = -\iint_{\Omega_{0\tau}} \sqrt{||u_1||^2||u_2||^2 - < u_1, u_2 >^2}\ dt^1 dt^2.$$

Of course, the maximization of $J(u(\cdot))$ is equivalent to the minimization of the area, under the constraint (PDE).

Two-time optimal control problem of minimal surfaces:

$$\max_{u(\cdot)} J(u(\cdot))$$

subject to

$$\frac{\partial x^i}{\partial t^\alpha}(t) = u^i_\alpha(t),\ i = 1, 2, 3;\ \alpha = 1, 2,$$

$$u(t) \in \mathcal{U},\ t \in \Omega_{0\tau};\ x(0) = x_0,\ x(\tau) = x_1.$$

To solve the previous problem, we apply the multitime maximum principle [6]-[23]. In general notations, we have

$$x = (x^i),\ u = (u_\alpha),\ u_\alpha = (u^i_\alpha),\ p = (p^\alpha),\ p^\alpha = (p^\alpha_i),\ \alpha = 1, 2;\ i = 1, 2, 3$$

$$u_1 = (u^1_1, u^2_1, u^3_1),\ u_2 = (u^1_2, u^2_2, u^3_2),\ p^1 = (p^1_1, p^1_2, p^1_3),\ p^2 = (p^2_1, p^2_2, p^2_3)$$

$$X_\alpha(x(t), u(t)) = u_\alpha(t),\ X^0(x(t), u(t)) = -\sqrt{|u_1||^2||u_2||^2 - < u_1, u_2 >^2}$$

and the control Hamiltonian is

$$H(x, p, u) = p^\alpha_i X^i_\alpha(x, u) + p_0 X^0(x, u).$$

Taking $p_0 = 1$, the adjoint dynamics says

$$\text{(ADJ)} \qquad \frac{\partial p^\alpha_i}{\partial t^\alpha} = -\frac{\partial H}{\partial x^i} = 0.$$

On the other hand, we have to maximize $H(x,p,u)$ with respect to the control u, hence

$$\frac{\partial H}{\partial u^i_\alpha} = \frac{\partial X^0}{\partial u^i_\alpha} + p^\alpha_i = 0.$$

Of course, $x^i_\alpha = u^i_\alpha$ and X^0 (or H) does not depend on x^i. We eliminate p^α_i using the adjoint PDE. It follows the two-time Euler-Lagrange PDEs

$$\frac{\partial}{\partial t^\alpha}\left(\frac{\partial X^0}{\partial x^i_\alpha}\right) = 0.$$

Summarizing, we obtain

Theorem 1.1. *The solution of the previous optimal control problem is a minimal surface.*

Since explicitly

$$H(x,p,u) = p^\alpha_i u^i_\alpha - \sqrt{||u_1||^2||u_2||^2 - < u_1, u_2 >^2},$$

the critical points u of H are solutions of the algebraic system

$$\begin{aligned} p^1_i + \frac{1}{X^0}\left(||u_2||^2 u^i_1 - < u_1, u_2 > u^i_2\right) &= 0 \\ p^2_i + \frac{1}{X^0}\left(||u_1||^2 u^i_2 - < u_1, u_2 > u^i_1\right) &= 0. \end{aligned} \tag{1.1}$$

Lemma 1.1. *The dual vectors p^1 and p^2 determine the areal energy density $||p^1||^2||p^2||^2 - < p^1, p^2 >^2 = (X^0)^3$.*

System (1.1) is equivalent to the system

$$u^i_1 p^1_i = -X^0,\ u^i_1 p^2_i = 0,\ u^i_2 p^1_i = 0,\ u^i_2 p^2_i = -X^0. \tag{1.2}$$

In this way, u_1 is orthogonal to p^2, and u_2 is orthogonal to p^1. Also

$$\delta^{ij} p^1_i p^1_j = -||u_2||^2 X^0,\ \delta^{ij} p^1_i p^2_j = < u_1, u_2 > X^0,\ \delta^{ij} p^2_i p^2_j = -||u_1||^2 X^0. \tag{1.3}$$

System (1.1) is equivalent to

$$-X^0 u^i_1 = ||u_1||^2 p^1_i + < u_1, u_2 > p^2_i,\ -X^0 u^i_2 = ||u_2||^2 p^2_i + < u_1, u_2 > p^1_i$$

or, via relations (1.3), we obtain the unique solution

$$u^i_1 = \frac{||p^2||^2}{(X^0)^2} p^1_i - \frac{< p^1, p^2 >}{(X^0)^2} p^2_i,\ u^i_2 = \frac{||p^1||^2}{(X^0)^2} p^2_i - \frac{< p^1, p^2 >}{(X^0)^2} p^1_i.$$

Consequently, we have the following

Theorem 1.2. *A minimal surface is a solution of the PDEs*

$$\text{(PDE)}\qquad \begin{aligned}\frac{\partial x^i}{\partial t^1} &= \frac{||p^2||^2}{(X^0)^2}p^1_i - \frac{<p^1,p^2>}{(X^0)^2}p^2_i,\\ \frac{\partial x^i}{\partial t^2} &= \frac{||p^1||^2}{(X^0)^2}p^2_i - \frac{<p^1,p^2>}{(X^0)^2}p^1_i,\end{aligned}$$

$$(X^0)^3 = ||p^1||^2||p^2||^2 - <p^1,p^2>^2,$$

$$x(0) = x_0,\ x(\tau) = x_1;$$

$$\text{(ADJ)}\qquad \frac{\partial p^\alpha_i}{\partial t^\alpha} = 0,\ B(p(t))|_{\partial\Omega_{0\tau}} = 0,$$

where B means boundary condition.

The simplest minimal surface is a planar quadrilateral fixed by the starting point $x^i(0) = x^i_0$ on Γ_0 and the terminal point $x^i(\tau) = x^i_1$ on Γ_1. In this case the vectors $u_\alpha(\cdot) = u_{\alpha 0} = (u^i_{\alpha 0})$ are constant in time, and consequently, the parametric representation

$$x^i(t) = u^i_{10}t^1 + u^i_{20}t^2 + x^i_0,\ i = 1,2,3,$$

depends on six arbitrary constants. We fix these constants by the following conditions: $x^i(\tau) = x^i_1$ which implies $\det[u_1, u_2, x_0 - x_1] = 0$, and the transversality conditions

$$(\tau)\qquad p(0) \perp T_{x_0}\Gamma_0, \quad p(\tau) \perp T_{x_1}\Gamma_1,$$

which show that the optimal plane is orthogonal to Γ_0 and Γ_1, and so the tangent lines $T_{x_0}\Gamma_0$ and $T_{x_1}\Gamma_1$ are parallel.

2. MINIMAL EVOLUTION PASSING THROUGH TWO POINTS

2.1. **Two-time minimal evolution avoiding an obstacle.** Let us apply the multitime maximum principle to search the surface of minimum area satisfying the following conditions: it contains two diagonal points $x(0)$, $x(\tau)$ and it avoids an obstacle A whose boundary ∂A is a surface. For that we start with the controlled dynamics problem and its solution in Section 1. Suppose $x(t) \notin \partial A$ for $t \in \Omega_{0\tau}$. In this hypothesis, the multitime maximum principle applies, and hence the initial dynamics (PDE) and the adjoint dynamics (ADJ) are those in Section 1.

To simplify the problem, we accept as obstacle a 2-dimensional cylinder (that supports a global tangent frame $\{u_1, u_2\}$), respectively a 2-dimensional sphere (that does not support a global tangent frame because any continuous vector field on such sphere vanishes somewhere).

2.2. **Two-time evolution touching an obstacle.** The points $0 \leq s_0 \leq s_1 \leq \tau$ determine a decomposition of the two dimensional interval $\Omega_{0\tau}$ in Ω_{0s_0}, $\Omega_{0s_1} \setminus \Omega_{0s_0}$, $\Omega_{0\tau} \setminus \Omega_{0s_1}$. To simplify the problem, suppose the sheet $x(t) \notin \partial A$ for $t \in \Omega_{0s_0} \cup (\Omega_{0\tau} \setminus \Omega_{0s_1})$ is a union of two planar quadrilaterals (one starting from $x(0)$ and ending in $x(s_0)$ and the other starting from $x(s_1)$ and ending at $x(\tau)$). If $x(t) \in \partial A$ for $t \in \Omega_{0s_1} \setminus \Omega_{0s_0}$, then we need the study in Section 3 and Section 4, which emphasizes the controls and the dual variables capable of keeping the evolution on the obstacle. Furthemore, suitable smoothness conditions on boundaries are necessary.

3. TOUCHING, APPROACHING AND LEAVING A CYLINDER

3.1. **Touching a cylinder.** Let us take the cylinder $C : (x^1)^2 + (x^2)^2 \leq r^2$, $x = (x^1, x^2, x^3)$, as obstacle. Suppose $x(t) \in \partial C$ for $t \in \Omega_{0s_1} \setminus \Omega_{0s_0}$. In this case we use the modified version of two-time maximum principle.

We introduce the set $N = \mathbb{R}^3 \setminus C : \ f(x) = r^2 - ((x^1)^2 + (x^2)^2) \leq 0$, and we build the functions

$$c_\alpha(x,u) = \frac{\partial f}{\partial x^i}(x) X^i_\alpha(x,u), \ \alpha = 1, 2,$$

i.e., $c_\alpha(x,u) = -2(x^1 u^1_\alpha + x^2 u^2_\alpha)$. Let us use the two-time maximum principle adding the constraints

$$c_1(x,u) = -2(x^1 u^1_1 + x^2 u^2_1) = 0, \ c_2(x,u) = -2(x^1 u^1_2 + x^2 u^2_2) = 0.$$

Then the adjoint equations

$$\frac{\partial p^\alpha_i}{\partial t^\alpha}(t) = -\frac{\partial H}{\partial x^i} + \lambda^\gamma(t) \frac{\partial c_\gamma}{\partial x^i}$$

are reduced to

$$\text{(ADJ}'\text{)} \qquad \frac{\partial p^\alpha_1}{\partial t^\alpha}(t) = \lambda^\gamma(t)(-2u^1_\gamma), \ \frac{\partial p^\alpha_2}{\partial t^\alpha}(t) = \lambda^\gamma(t)(-2u^2_\gamma), \ \frac{\partial p^\alpha_3}{\partial t^\alpha}(t) = 0.$$

The critical point condition with respect to the control u is

$$\frac{\partial H}{\partial u} = \lambda^\gamma \frac{\partial c_\gamma}{\partial u},$$

i.e.,

$$\frac{\partial H}{\partial u^i_1} = \lambda^1 \frac{\partial c_1}{\partial u^i_1}, \ \frac{\partial H}{\partial u^i_2} = \lambda^2 \frac{\partial c_2}{\partial u^i_2},$$

or

$$\begin{aligned} p^1_i = \lambda^1(-2x^i) - \frac{1}{X^0}\left(||u_2||^2 u^i_1 - < u_1, u_2 > u^i_2\right), \ i = 1,2 \\ p^1_3 = -\frac{1}{X^0}\left(||u_2||^2 u^3_1 - < u_1, u_2 > u^3_2\right), \\ p^2_i = \lambda^2(-2x^i) - \frac{1}{X^0}\left(||u_1||^2 u^i_2 - < u_1, u_2 > u^i_1\right), \ i = 1,2, \\ p^2_3 = -\frac{1}{X^0}\left(||u_1||^2 u^3_2 - < u_1, u_2 > u^3_1\right). \end{aligned} \tag{3.1}$$

We recall that $x(t) \in \partial C$ means $(x^1)^2 + (x^2)^2 = r^2$. Consequently,

$$x^i p^1_i = \lambda^1(-2r^2),\ x^i p^2_i = \lambda^2(-2r^2),\ i = 1, 2.$$

To develop further our ideas, we accept that the cylinder C is represented by the parametrization $x^1 = r\cos t^1$, $x^2 = r\sin t^1$, $x^3 = t^2$. We use the partial velocities (orthogonal vectors)

$$\frac{\partial x^1}{\partial t^1} = -r \sin t^1,\ \frac{\partial x^2}{\partial t^1} = r \cos t^1,\ \frac{\partial x^3}{\partial t^1} = 0$$

$$\frac{\partial x^1}{\partial t^2} = 0,\ \frac{\partial x^2}{\partial t^2} = 0,\ \frac{\partial x^3}{\partial t^2} = 1.$$

Then the area formula is

$$r^2 \int_0^{\tau^1}\int_0^{\tau^2} dt^1 dt^2 = r^2 \tau^1 \tau^2.$$

On the other hand, the evolution PDEs emphasize the controls

$$u_1 :\ u^1_1 = -r \sin t^1,\ u^2_1 = r\cos t^1,\ u^3_1 = 0$$

$$u_2 :\ u^1_2 = 0,\ u^2_2 = 0,\ u^3_2 = 1,$$

with $< u_1, u_2 >= 0$, $||u_1|| = r$, $||u_2|| = 1$. Using equalities (3.1), we obtain $u^i_1 p^1_i = 2$, $u^i_1 p^2_i = 0$, $u^i_2 p^1_i = 0$, $u^i_2 p^2_i = 2$, which confirm that u_1 is orthogonal to p^2 and u_2 is orthogonal to p^1. The relations $p^1_i = \lambda^1(-2x^i) + 2u^i_1$, $i = 1, 2,\ p^1_3 = 0$ produce

$$p^1_1 = -2\lambda^1 r \cos t^1 - 2r \sin t^1,\ p^1_2 = -2\lambda^1 r \sin t^1 + 2r \cos t^1,\ p^1_3 = 0.$$

Similarly, the relations $p^2_i = \lambda^2(-2x^i) + 2u^i_2$, $i = 1, 2,\ p^2_3 = 2$ give

$$p^2_1 = -2\lambda^2 r \cos t^1,\ p^2_2 = -2\lambda^2 r \sin t^1,\ p^2_3 = 2.$$

Adjoint equations (ADJ′) become

$$\frac{\partial \lambda^1}{\partial t^1} + \frac{\partial \lambda^2}{\partial t^2} = -1.$$

We use the particular solution $\lambda^1 = k^1 - t^1,\ \lambda^2 = k^2$ of this PDE.

The evolution (PDE) on the interval $\omega_0 \le t^1 \le \omega_1$ shows that the surface of evolution is a cylindric quadrilateral fixed by the initial point $x^i(\omega_0, t^2) = x^i_0$ generator and with the terminal point $x^i(\omega_1, t^2) = x^i_1$ generator.

3.2. **Approaching and leaving the cylinder.** Now we must put together the previous results. Suppose $x(t) \in N = \mathbb{R}^3 \setminus C$ for $t \in \Omega_{0s_0} \cup (\Omega_{0\tau} \setminus \Omega_{0s_1})$ and $x(t) \in \partial C$ for $t \in \Omega_{0s_1} \setminus \Omega_{0s_0}$.

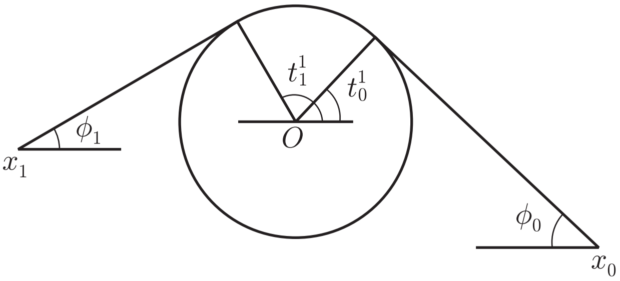

Figure 1

For $t \in \Omega_{0s_0} \cup (\Omega_{0\tau} \setminus \Omega_{0s_1})$, the 2-sheet of evolution $x(\cdot)$ consists in two pieces. To simplify, we accept it as a union of two planar sheets. Suppose the first planar sheet touches the cylinder at the point $x(s_0)$. In this case, we can take

$$p^1_1 = -2r\cos\phi_0,\ p^1_2 = 2r\sin\phi_0,\ p^1_3 = 0,$$

for the angle ϕ_0 as shown in Fig. 1, and

$$p^2_1 = 0,\ p^2_2 = 0,\ p^2_3 = 2.$$

By the jump conditions, the vectors $p^1(\cdot)$, $p^2(\cdot)$ are continuous when the evolution 2-sheet $x(\cdot)$ hits the boundary ∂C at the two-time s_0. In other words, we must have the identities

$$\begin{gathered} -2k^1 r\cos t^1_0 - 2r\sin t^1_0 + 2rt^1_0 \cos t^1_0 = -2r\cos\phi_0 \\ -2k^1 r\sin t^1_0 + 2r\cos t^1_0 + 2rt^1_0 \sin t^1_0 = 2r\sin\phi_0 \\ 2k^2 r\cos t^1_0 = 0,\ 2k^2 r\sin t^1_0 = 0, \end{gathered}$$

i.e., $k^1 = t^1_0$, $t^1_0 + \phi_0 = \dfrac{\pi}{2}$, $k^2 = 0$. The last two equalities show that the optimal (particularly, planar) 2-sheet is tangent to the cylinder along the generator $x^1 = x^1(s_0)$, $x^2 = x^2(s_0), x^3 \in R$.

Let us analyse what happen with the evolution 2-sheet as it leaves the boundary ∂C at the point $x(s_1)$. Then we have

$$\begin{aligned} p^1_1(t^{1-}_1, t^2) &= -2rt^1_0\cos t^1_1 - 2r\sin t^1_1 + 2rt^1_1\cos t^1_1, \\ p^1_2(t^{1-}_1, t^2) &= -2rt^1_0\sin t^1_1 - 2r\cos t^1_1 + 2rt^1_1\sin t^1_1, \\ p^1_3(t^{1-}_1, t^2) &= 0; \end{aligned}$$

$$p_1^2(t_1^{1-},t^2)=0,\ p_2^2(t_1^{1-},t^2)=0,\ p_3^2(t_1^{1-},t^2)=2.$$

The jump theory gives

$$p^\alpha(t_1^{1+},t^2)=p^\alpha(t_1^{1-},t^2)-\lambda^\alpha(t_1^1,t^2)\nabla f(x(t_1^1,t^2))$$

for $f(x)=r^2-(x^1)^2-(x^2)^2$. Then

$$\lambda^1(t_1^1,t^2)\nabla f(x(t_1^1,t^2))=(t_1^1-t_0^1)\begin{pmatrix}2r\cos\,t_1^1\\ 2r\sin\,t_1^1\\ 0\end{pmatrix}.$$

In this way, $p_1^1(t_1^{1+},t^2)=-2r\sin\,t_1^1$, $p_2^1(t_1^{1+},t^2)=2r\cos\,t_1^1$, and so the planar 2-sheet of evolution is tangent to the boundary ∂C along the generator by the point $x(s_1)$. If we apply the usual two-time maximum principle after $x(\cdot)$ leaves the cylinder C, we find (see Fig. 1)

$$p_1^1=\text{const.}=-2r\cos\,\phi_1,\ p_2^1=\text{const.}=-2r\sin\,\phi_1;\ p_1^2=0,\ p_2^2=0.$$

Therefore $-\cos\,\phi_1=-\sin\,t_1^1$, $-\sin\,\phi_1=-\cos\,t_1^1$ and so $\phi_1+t_1^1=\pi$.

Open Problem. What happen when the surface in the exterior of the cylinder is a non-planar minimal sheet?

4. TOUCHING, APPROACHING AND LEAVING A SPHERE

4.1. **Touching a sphere.** Let us take as obstacle the sphere $B:\ f(x)=r^2-\delta_{ij}x^ix^j\geq 0$, $x=(x^1,x^2,x^3)$, $i,j=1,2,3$. Suppose $x(t)\in\partial B$ for $t\in\Omega_{0s_1}\setminus\Omega_{0s_0}$. In this case we use the modified version of two-time maximum principle.

We introduce the set $N=\mathbb{R}^3\setminus B:\ f(x)=r^2-\delta_{ij}x^ix^j\leq 0$ and we build the functions $c_\alpha(x,u)=\dfrac{\partial f}{\partial x^i}(x)X_\alpha^i(x,u)$, $\alpha=1,2$, i.e., $c_\alpha(x,u)=-2\delta_{ij}x^iu_\alpha^j$. Let us use the two-time maximum principle adding the constraints

$$c_1(x,u)=-2\delta_{ij}x^iu_1^j=0,\ c_2(x,u)=-2\delta_{ij}x^iu_2^j=0.$$

Then the condition

$$\frac{\partial p_i^\alpha}{\partial t^\alpha}(t)=-\frac{\partial H}{\partial x^i}+\lambda^\gamma(t)\frac{\partial c_\gamma}{\partial x^i}$$

is reduced to

$$\text{(ADJ}'')\qquad \frac{\partial p_i^\alpha}{\partial t^\alpha}(t)=\lambda^\gamma(t)(-2u_\gamma^i).$$

The condition of critical point for the control u becomes

$$\frac{\partial H}{\partial u}=\lambda^\gamma\frac{\partial c_\gamma}{\partial u},$$

i.e.,

$$\frac{\partial H}{\partial u_1^i}=\lambda^1\frac{\partial c_1}{\partial u_1^i},\ \frac{\partial H}{\partial u_2^i}=\lambda^2\frac{\partial c_2}{\partial u_2^i}$$

or

$$\begin{aligned} p^1_i &= \lambda^1(-2x^i) - \frac{1}{X^0}\left(||u_2||^2 u^i_1 - < u_1, u_2 > u^i_2\right) \\ p^2_i &= \lambda^2(-2x^i) - \frac{1}{X^0}\left(||u_1||^2 u^i_2 - < u_1, u_2 > u^i_1\right). \end{aligned} \tag{4.1}$$

We recall that $x(t) \in \partial B$ means $\delta_{ij}x^i x^j = r^2$. Consequently,

$$x^i p^1_i = \lambda^1(-2r^2),\ x^i p^2_i = \lambda^2(-2r^2).$$

To develop further our ideas, we accept that the sphere B is represented by the parametrization

$$x^1 = r\cos t^1 \cos t^2,\ x^2 = r\sin t^1 \cos t^2,\ x^3 = r\sin t^2.$$

We use the partial velocities (orthogonal vectors)

$$\frac{\partial x^1}{\partial t^1} = -r\sin t^1 \cos t^2,\ \frac{\partial x^2}{\partial t^1} = r\cos t^1 \cos t^2,\ \frac{\partial x^3}{\partial t^1} = 0$$

$$\frac{\partial x^1}{\partial t^2} = -r\cos t^1 \sin t^2,\ \frac{\partial x^2}{\partial t^2} = -r\sin t^1 \sin t^2,\ \frac{\partial x^3}{\partial t^2} = r\cos t^2.$$

Then the area formula is

$$\iint_{\sigma\subset\partial B} d\sigma = r^2 \int_0^{\tau^1}\int_0^{\tau^2} \cos t^2 dt^1 dt^2 = r^2\tau^1 \sin\tau^2.$$

On the other hand, the evolution PDEs emphasize the controls

$$u_1 :\ u^1_1 = -r\sin t^1 \cos t^2,\ u^2_1 = r\cos t^1 \cos t^2,\ u^3_1 = 0$$

$$u_2 :\ u^1_2 = -r\cos t^1 \sin t^2,\ u^2_2 = -r\sin t^1 \sin t^2,\ u^3_2 = r\cos t^2,$$

with $< u_1, u_2 >= 0$, $||u_1|| = r\cos t^2$, $||u_2|| = r$. Using equalities (4.1), we obtain $u^i_1 p^1_i = -X^0$, $u^i_1 p^2_i = 0$, $u^i_2 p^1_i = 0$, $u^i_2 p^2_i = -X^0$, which confirm that u_1 is orthogonal to p^2 and u_2 is orthogonal to p^1.

We have

$$p^1_i = \lambda^1(-2x^i) - \frac{1}{X^0}\,||u_2||^2 u^i_1,\ p^2_i = \lambda^2(-2x^i) - \frac{1}{X^0}\,||u_1||^2 u^i_2,\ i = 1, 2, 3,$$

or explicitly

$$\begin{gathered} p^1_1 = \lambda^1(-2r\ \cos t^1 \cos t^2) + r\sin t^1 \\ p^1_2 = \lambda^1(-2r\ \sin t^1 \cos t^2) - r\cos t^1 \\ p^1_3 = \lambda^1(-2r\ \sin t^2) \\ p^2_1 = \lambda^2(-2r\ \cos t^1\ \cos t^2) + r\cos t^1\ \sin t^2\ \cos t^2 \\ p^2_2 = \lambda^2(-2r\ \sin t^1\ \cos t^2) + r\sin t^1\ \sin t^2\ \cos t^2 \\ p^2_3 = \lambda^2(-2r\ \sin t^2) - r(\cos t^2)^2. \end{gathered}$$

Adjoint equations (ADJ$''$) become

$$\frac{\partial \lambda^1}{\partial t^1} + \frac{\partial \lambda^2}{\partial t^2} = \cos t^2.$$

We use the particular solution $\lambda^1 = 0,\ \lambda^2 = \sin t^2 + k^2$ of this PDE.

4.2. **Approaching and leaving the sphere.** We must now put together the previous results. So suppose $x(t) \in N = \mathbb{R}^3 \backslash B$ for $t \in \Omega_{0s_0} \cup (\Omega_{0\tau} \setminus \Omega_{0s_1})$ and $x(t) \in \partial C$ for $t \in \Omega_{0s_1} \setminus \Omega_{0s_0}$.

For $t \in \Omega_{0s_0} \cup (\Omega_{0\tau} \setminus \Omega_{0s_1})$, the 2-sheet of evolution $x(\cdot)$ consists in two pieces. To simplify, we accept a union of two planar sheets. Suppose the first planar sheet touches the sphere at the point $x(s_0)$. In this case, we can take

$$p_1^1 = -r\cos\,\phi_0,\ p_2^1 = r\sin\,\phi_0,\ p_3^1 = 0,$$

for the angle ϕ_0 as shown in Fig. 1, and

$$p_1^2 = 0,\ p_2^2 = 0,\ p_3^2 = -r.$$

By the jump conditions, the vectors $p^1(\cdot)$, $p^2(\cdot)$ are continuous when the evolution 2-sheet $x(\cdot)$ hits the boundary ∂B at the two-time s_0. In other words, we must have the identities

$$\begin{gathered} r\sin t_0^1 = -r\cos\,\phi_0,\ \ -r\cos t_0^1 = r\sin\,\phi_0 \\ (\sin t_0^2 + k^2)(-2r\,\cos t_0^1\,\cos t_0^2) + r\cos t_0^1\,\sin t_0^2\,\cos t_0^2 = 0 \\ (\sin t_0^2 + k^2)(-2r\,\sin t_0^1\,\cos t_0^2) + r\sin t_0^1\,\sin t_0^2\,\cos t_0^2 = 0 \\ (\sin t_0^2 + k^2)(-2r\,\sin t_0^2) - r(\cos t_0^2)^2 = -r, \end{gathered}$$

i.e., $t_0^1 + \phi_0 = \dfrac{\pi}{2},\ k^2 = 0,\ t_0^2 = 0$. The last two equalities show that the optimal (particularly, planar) 2-sheet is tangent to the sphere at the point $(x^1(s_0), x^2(s_0), x^3(s_0))$.

Let us analyse what happen with the evolution 2-sheet as it leaves the boundary ∂B at the point $x(s_1)$. We then have

$$\begin{gathered} p_1^1(t_1^{1-}, t_1^{2-}) = r\sin t_1^1,\ \ p_2^1(t_1^{1-}, t_1^{2-}) = -r\cos t_1^1,\ \ p_3^1(t_1^{1-}, t_1^{2-}) = 0 \\ p_1^2(t_1^{1-}, t_1^{2-}) = 0,\ p_2^2(t_1^{1-}, t_1^{2-}) = 0,\ p_3^2(t_1^{1-}, \theta_1^{2-}) = -r. \end{gathered}$$

The jump theory gives

$$p^\alpha(t_1^{1+}, t_1^{2+}) = p^\alpha(t_1^{1-}, t_1^{2-}) - \lambda^\alpha(t_1^1, t_1^2)\nabla f(x(t_1^1, t_1^2))$$

for $f(x) = r^2 - \delta_{ij}x^i x^j$. Then

$$\lambda^2(t_1^1, t_1^2)\nabla f(x(t_1^1, t_1^2)) = (t_1^2 - t_0^2)\begin{pmatrix} r\cos t_1^1 \cos t_1^2 \\ r\sin t_1^1 \cos t_1^2 \\ r\sin t_1^2 \end{pmatrix}.$$

In this way, $p^1_1(t^{1+}_1, t^{2+}_1) = -r\sin t^1_1$, $p^1_2(t^{1+}_1, t^{2+}_1) = r\cos t^1_1$, and so the planar 2-sheet of evolution is tangent to the boundary ∂B at the point $x(s_1)$. If we apply the usual two-time maximum principle after $x(\cdot)$ leaves the sphere C, we find (see Fig. 1)

$$p^1_1 = \text{const.} = -r\cos\,\phi_1,\;\; p^1_2 = \text{const.} = -r\sin\,\phi_1;\;\; p^2_1 = 0,\;\; p^2_2 = 0.$$

Therefore $-\cos\,\phi_1 = -\sin\,t^1_1,\; -\sin\,\phi_1 = -\cos\,t^1_1$ and so $\phi_1 + \theta^1_1 = \pi$.

Open Problem. What happen when the surface in the exterior of the sphere is a non-planar minimal sheet?

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