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ON CLARKE-MITITELU SUBDIFFERENTIAL

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Dedicated to Professor Ştefan Mititelu
on the occasion of his seventieth birthday

Abstract. Based on a generalization of the classical notion of differential, the subdifferential is a relatively new notion in Functional Analysis. It results in a natural way from the need for finding appropriate treatments of some classes of problems in Optimization Theory. The aim of this work is to survey the concept of subdifferential emphasizing the contribution of Professor Ştefan Mititelu to the development of this field.

1. INTRODUCTION AND PRELIMINARIES

The first model of generalized gradient is the subdifferential of convex functions, introduced by Moreau [15], and founded by Rockafellar [19].

Generalized gradient models were developed taking into account practical reasons, see [18] by Pshenichnyi, [3] by Clarke, [20] by Rockafellar, [6] by Demyanov and Dixon, [8] by Hiriart-Urruty, [11] by Mititelu.

The first model called *quasidifferential* is introduced by Pshenichnyi [18] in the framework given in the following.

Let E be a topological linear space and E^* its dual. The function $f\colon E \to \mathbb{R}$ is called *quasidifferentiable* at the point $x^0 \in E$ if the direction derivative

$$f'(x^0; v) = \lim_{\lambda \downarrow 0} \frac{f(x^0 + \lambda v) - f(x^0)}{\lambda}$$

exists for all direction $v \in E$. The quasidifferential (in the sense of Pshenichnyi) of the function f at the point x^0 is a weakly compact set $M_f(x^0)$ of E^*, defined by

$$f'(x^0; v) = \max_{\xi \in M_f(x^0)} < \xi, v >,$$

or, in equivalent form,

$$M_f(x^0) = \{\xi \in E^* \mid f'(x^0; v) \geq < \xi, v >, \quad \forall v \in E\}.$$

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The following two remarks are in order:

1) if the function f is Gâteaux differentiable at the point x^0, then the function f is quasidifferentiable at x^0 and $M_f(x^0) = \{\nabla f(x^0)\}$;

2) the quasidifferential M_f is defined by analogy with the subdifferential of convex functions [11], [15], [19].

One of the most important model of generalized gradients is called *subdifferential* and is introduced by Clarke [3], for Lipschitz functions.

Using various types of direction derivatives, many subsequent theories have developed the concept of subdifferential for nonsmooth functions. For an overall view, the reader is referred to the next monographs: [3] by Clarke, [6] by Demyanov and Dixon, [9] by Kusraev and Kutateladze, [11] by Mititelu, [17] by Penot, and [20] by Rockafellar, as well as to the following research papers: [7] by Demyanov and Rubinov, [12]÷[14] by Mititelu, [16] by Morzhin, [21] by Tolstonogov.

2. CLARKE STRUCTURE

Let be given $f : \mathbb{R}^n \to \mathbb{R}$ a locally Lipschitz function around a point x^0. The *Clarke derivative* of the function f at the point x^0 on the direction of v in $\mathbb{R}^n$ is the direction derivative

$$f^0(x^0; v) = \limsup_{x \to x^0,\, \lambda \downarrow 0} \frac{f(x + \lambda v) - f(x)}{\lambda}.$$

The set

$$\partial f(x^0) = \{\xi \in \mathbb{R}^n \mid f^0(x^0; v) \geq < \xi, v >, \quad \forall v \in \mathbb{R}^n\}$$

is called *Clarke subdifferential* or *generalized gradient* of the function f at the point x^0.

Since any locally Lipschitz is almost everywhere differentiable in the sense of Lebesgue measure, the subdifferential admits the equivalent representation [4]

$$\partial f(x^0) = \mathrm{conv}\{\lim_{i \to \infty} \nabla f(x_i) \mid x_i \to x^0, \quad x_i \in D\},$$

where D is the set in $\mathbb{R}^n$ where f is differentiable and ∇f is its derivative.

If the function is lower semicontinuous, then [3], [4]

$$\partial f(x) = \{\xi \in \mathbb{R}^n \mid (\xi, -1) \in N_{\mathrm{epi} f}(x, f(x))\},$$

where $N_{\mathrm{epi} f}$ is the Clarke normal cone to the epigraph of the function f at the point $(x, f(x)) \in \mathrm{epi} f$.

According to Clarke [5], some specific properties of a locally K-Lipschitz function in a neighborhood of a point x are:

$$\begin{aligned} &1) \quad |f^0(x; v)| \leq K\|v\|; \\ &2) \quad |f^0(x; v_1) - f^0(x; v_2)| \leq K\|v_1 - v_2\|; \\ &3) \quad \|\xi\| \leq K, \quad \forall \xi \in \partial f(x). \end{aligned}$$

This theory allowed the development of nonsmooth analysis, with wide applications in Mathematical Programming, Variational Problems, Numerical Analysis. For some details, see [5], [9], [22], [23].

However, the subdifferential of Clarke may be generalized to a wider class of functions. That is why, this theory was extended to the class of non-Lipschitzian functions, a new structure being introduced by Ştefan Mititelu in his works [12], [13] and [14].

3. MITITELU STRUCTURE

The structure proposed by Ştefan Mititelu is an extension of the subdifferential in the sense of Clarke to general nonsmooth functions.

Let be given $f\colon A \to \mathbb{R}$ an arbitrary function, defined on the open set A in $\mathbb{R}^n$. The following properties are remarkable. For a complete study of subdifferentials of the functions obtained by algebraic operations with almost everywhere Fréchet differentiable functions, we address the reader to the monograph [11], as well as to the research works [12], [13] and [14].

Proposition 3.1. *Suppose the function $f^0(x;v)$ is finite.*

1) *The function $f^0(\cdot;v)$ is upper semicontinuous at the point x.*

2) *If f is continuous around x, the function f^0 is upper semicontinuous at the point $(x;v)$.*

Proposition 3.2. *The function $f^0(x;\cdot)$ is sublinear, that is subadditive and positive homogeneous on $\mathbb{R}^n$.*

Proposition 3.3. *Suppose the point x belongs to A and $f^0(x;\cdot)$ is finite. Then, there exists $f'(x;v)$, that is the direction derivative of the function f at the point x, on the direction of v. Moreover, $f'(x;v) = f^0(x;v)$, for all $v \in \mathbb{R}^n$.*

In [12], Mititelu introduced

Definition 3.1. Let be given $f\colon A \to \mathbb{R}$ an arbitrary function, where A is open in $\mathbb{R}^n$, and the point x in A. The set

$$\partial f(x) = \{\xi \in \mathbb{R}^n \mid f^0(x;v) \geq <\xi, v>, \quad \forall v \in \mathbb{R}^n\}$$

is called *subdifferential* or *generalized gradient* of the function f at the point x.

If $\partial f(x) \neq \emptyset$, then the function f is called *subdifferentiable* at the point x. The elements of $\partial f(x)$ are called *subgradients*.

Denote by $\mathcal{D}(f^0_x)$ the *effective domain* of the function $f^0(x;\cdot)$, that is the set

$$\mathcal{D}(f^0_x) = \{v \in \mathbb{R}^n \mid f^0(x;v) < \infty\}.$$

When $f^0(x;\cdot) \neq -\infty$, then f is proper convex. It follows that the domain $\mathcal{D}(f^0_x)$ is a nonempty convex set.

We underline that if $f^0(x;v) = -\infty$, for all $v \in \mathbb{R}^n$, then $\partial f(x) = \emptyset$. That is why, to give a complete characterization of the model proposed by Mititelu, in the following we shall suppose $\partial f(x) \neq \emptyset$.

In [12], Şt. Mititelu shows that the infinite values of the function $f^0(x;\cdot)$ are not essential in the determination of the subdifferential $\partial f(x)$, essential being its finite values only. More exactly, we have the following two results.

Theorem 3.1. *If the function f is subdifferentiable at the point x, then $\partial f(x)$ admits the representation*

$$\partial f(x) = \{\xi \in \mathbb{R}^n \mid f^0(x;v) \geq <\xi, v>, \quad \forall v \in \mathcal{D}(f_x^0)\}.$$

Theorem 3.2. *The following statements hold true:*

If $f^0(x;v) \neq -\infty$, for all $v \in \mathbb{R}^n$, then f is subdifferentiable at $x \in A$, the subdifferential $\partial f(x)$ is a nonempty closed convex set, and

$$f^0(x;v) = \sup\{<\xi, v> \mid \xi \in \partial f(x)\}, \quad \forall v \in \mathbb{R}^n.$$

Moreover, if $f^0(x;\cdot)$ is finite, then $\partial f(x)$ is compact and

$$f^0(x;v) = \max\{<\xi, v> \mid \xi \in \partial f(x)\}, \quad \forall v \in \mathcal{D}(f_x^0).$$

The subdifferential $\partial f(x)$ of an almost everywhere Fréchet differentiable function $f\colon A \to \mathbb{R}$, around a point $x \in A$, admits an equivalent representation as convex hull of the limit normals of f at the point x. This is proved by the following two results.

Proposition 3.4. *Let x be given a point in A and the sequences (x_k) and (ξ_k) in A and $\mathbb{R}^n$ respectively. If $x_k \to x$ in A and $\xi_k \to \xi$ in $\mathbb{R}^n$ such that $\xi_k \in \partial f(x_k)$, then $\xi \in \partial f(x)$.*

Denote by $\mathcal{D}_f$ the set of all x where f is almost everywhere Fréchet differentiable around the point x.

Theorem 3.3 (Mititelu, [12])**.** *Suppose the function f is almost everywhere Fréchet differentiable around the point x and $f^0(x;\cdot)$ is finite. Then f is subdifferentiable around the point x and the subdifferential $\partial f(x)$ admits the representation*

$$\partial f(x) = \text{conv}\{\lim_{x_k \to x} \nabla f(x_k) \mid x_k \in \mathcal{D}_f\}.$$

Mititelu structure allowed to obtain new versions of some classical results in smooth analysis, [13]. To emphasize these results, we need

Definition 3.2 (Mititelu, [13])**.** Let be given $f\colon A \to \mathbb{R}$ an arbitrary function, defined on the open set A in $\mathbb{R}^n$. The point $x \in A$ is called *Clarke critical point* of the function f if $f^0(x;v) \geq 0$, for all $v \in \mathbb{R}^n$.

Remark that the inequality in Definition 3.2 is equivalent to $0 \in \partial f(x)$.

The following results are versions of Fermat Theorem and Rolle Theorem, respectively, in nonsmooth framework.

Theorem 3.4 (Mititelu, [13])**.** *If $x \in A$ is an extremum point of the function f, then x is a Clarke critical point of f.*

Consider a and b be two distinct points in the set A. Introduce the set $[a, b]$ (closed interval in A) by $[a, b] = \{\lambda a + (1 - \lambda)b \,|\, 0 \leq \lambda \leq 1\}$.

Theorem 3.5 (Mititelu, [13])**.** *If the function f satisfies the conditions:*

$$1)\ f(a) = f(b); \quad 2)\ f \text{ is continuous on } [a, b],$$

there exists a point $c \in (a, b)$ such that either $f^0(c; b - a) \geq 0$ or $f^0(c; a - b) \geq 0$.

Consider again $f \colon A \to \mathbb{R}$ an arbitrary function, defined on the open set A in $\mathbb{R}^n$ and the compound function $\varphi \colon [0, 1] \to \mathbb{R}$, $\varphi(t) = f(x_t)$, where $x_t = a + t(b - a)$. In this setting, we have

Lemma 3.1. *If the function f is subdifferentiable on (a, b) and φ is subdifferentiable on $(0, 1)$, then for all $t \in (0, 1)$,*

$$\partial\varphi(t) \subseteq < \partial f(x_t), b - a > = \{< \xi, b - a > \ | \xi \in \partial f(x_t)\}.$$

Using Lemma 3.1 and Theorem 3.5, we obtain a generalization of the well known mean theorem of Lagrange. This is given in

Theorem 3.6 (Mititelu, [13])**.** *Suppose the function f is continuous on $[a, b]$. Then there exist a point $c \in (a, b)$, where f is subdifferentiable, and a subgradient $\xi \in \partial f(c)$ such that*

$$f(b) - f(a) = < \xi, b - a > \in < \partial f(c), b - a > .$$

The following result is a version of Cauchy Theorem in nonsmooth framework.

Theorem 3.7 (Mititelu, [13])**.** *Suppose the functions f and g satisfy the following two conditions:*

$$1)\ f \text{ and } g \text{ are continuous on } [a, b], \quad 2)\ g^0(x; b - a) < 0, \ for\ all\ x \in (a, b).$$

Then, there exist a point $c \in (a, b)$, where f and g are subdifferentiable, and the subgradients $\xi \in \partial f(c)$ and $\eta \in \partial g(c)$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{< \xi, b - a >}{< \eta, b - a >}.$$

Proof. For a proof, see [13]. □

Looking at the result in Theorem 3.7, we can easily conclude why l'Hôspital rules from smooth real analysis cannot be applied to the functions of several variables.

We continue our considerations by pointing out a result on the subdifferential of composite functions. In this respect, consider $L(\mathbb{R}^n, \mathbb{R}^m)$ the space of linear functionals defined on $\mathbb{R}^n$ with values on $\mathbb{R}^m$. Denote by $F^0(x; v)$ the Clarke directional derivative of a vector function $F \in L(\mathbb{R}^n, \mathbb{R}^m)$, at a point x in A, where $A \in \mathbb{R}^n$, on the direction of $v \in \mathbb{R}^n$. This one is defined by analogy to $f^0(x;v)$. Moreover, if $F = (F_1, \dots, F_m)$, then $F^0(x;v) = \big(F_1^0(x;v), \dots, F_m^0(x;v)\big)$ and

$$\partial F(x) = (\partial F_1(x), \dots, \partial F_m(x)) = \partial F_1(x) \times \cdots \times \partial F_m(x).$$

is the subdifferential of the vector function F at the point x.

Using this background, we are now in position to introduce

Definition 3.3. The vector function $F\colon A \to \mathbb{R}^n$ is *upper differentiable* at the point $x \in A$ if the Clarke directional derivative $F^0(x;\cdot)$ is linear on $\mathbb{R}^n$, that is there exists $D^sF(x) \in L(\mathbb{R}^n, \mathbb{R}^m)$ such that

$$F^0(x;v) = < D^sF(x), v >, \quad \forall v \in \mathbb{R}^n.$$

The element $D^sF(x)$ in Definition 3.3 is called the *upper derivative* of the function F at the point x.

Theorem 3.8 (Mititelu, [14]). *Given the functions $F\colon A \to \mathbb{R}^m$ and $g\colon \mathbb{R}^m \to \mathbb{R}$, consider the composite function $g \circ F\colon A \to \mathbb{R}$ such that the following conditions are fulfilled:*

1) *The function F is continuous around the point x in A and admits upper derivative $D^sF(x)$;*

2) *The function g is continuous around the point $F(x)$ in $\mathbb{R}^m$.*

On these conditions, the subdifferential $\partial(g \circ F)(x)$ is not empty and satisfies the inclusion

$$\partial(g \circ F)(x) \subseteq \partial g(F(x)) \circ D^sF(x) = \{z \circ D^sF(x) \,|\, z \in \partial g(F(x))\}.$$

Moreover, if F is one to one function, the above inclusion becomes equality.

Proof. For a proof in detail, see the monograph [11] and the research work [14]. □

The following two results are direct consequences of Theorem 3.8.

The first one is an Euler type theorem. This is given in

Corollary 3.1. *Let $\mathbb{C}$ be an open cone in $\mathbb{R}^n$ and $f\colon \mathbb{C} \to \mathbb{R}$ a subdifferentiable function. Suppose that the function f is homogeneous of degree m. Then*

$$mf(x) \in < \partial f(x), x >, \quad \forall x \in \mathbb{C}.$$

The second one is a local inversion type theorem. This is given in

Corollary 3.2. *Let be given the open sets A and I in $\mathbb{R}^n$ and $\mathbb{R}$ respectively. Consider the function $f\colon A \to I$, its inverse $f^{-1}\colon I \to A$, and the points $x \in A$ and $y = f(x) \in I$.*

1) *If the function f is continuous around x and its inverse f^{-1} admits upper derivative $D^sf^{-1}(y)$, then $1 \in < \partial f(x), D^sf^{-1}(y) >$.*

2) *If the function f is continuous around the point x and admits upper derivative $D^sf(x)$ and its inverse f^{-1} is subdifferentiable at y, then $I_n \in < \partial f^{-1}(y), D^sf(x) >$, where I_n is the unit matrix.*

A remarkable particular case of Corollary 3.2 is obtained for $n = 1$. In this case, if the function f is differentiable at the point x, with $f'(x) \neq 0$, then its inverse is differentiable at the point y and the following formula holds:

$$\left(f^{-1}\right)'(y) = \frac{1}{f'(x)},$$

We underline that the result listed above is the well-known *local inversion formula* from smooth analysis.

For a geometrical interpretation of the subdifferential calculus, we address the reader to the monograph [11]. Here it is shown that the geometry of tangent and normal cones at a point, proper to Lipschitzian calculus, is preserved for arbitrary nonsmooth functions too.

4. CONCLUSION

The generalized subdifferential introduced by Mititelu set up conditions for developing a complete nonsmooth analysis. According to Mititelu, this concept is called Clarke-extended subdifferential. However, to the best of our knowledge, we think it can be called Clarke-Mititelu subdifferential.

This work emphasizes the role of certain scientists in designing algorithms research based on generalized gradients, more accurately to the development of nonsmooth analysis [3]÷[20]. Our note suggests various links between this topic and Geometrical Methods in Statistics [1], Mathematical Modeling in Ecology [2], Economics [9], Optimal Control [10], [16], [21], Generalized Convexity [11], as well as Optimization Methods on Manifolds [22], [23].

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