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AUXILIARY PRINCIPLE TECHNIQUE FOR SOLVING GENERAL MIXED VARIATIONAL INEQUALITIES

MUHAMMAD ASLAM NOOR

Dedicated to Professor Ştefan Mititelu
on the occasion of his seventieth birthday

Abstract. It is well known that the minimum of the sum of two nonconvex function can be characterized by a class of general mixed variational inequalities. In this paper, we use the auxiliary principle technique to prove the existence of a unique solution of the general mixed variational inequality. We also discuss some special cases.

1. INTRODUCTION

It is well known that the variational inequalities represent the optimality condition for the differentiable convex functions on the convex sets in normed space. Variational inequality theory provides us with a unified and general framework to study a large class of problems with applications in industry and applied sciences, see [1, 22] and the references therein. In the recent years, the concept of convexity has been generalized in several directions, see, for example, [3] and the references therein. One can introduce [2] the nonconvex set, which is called the g-convex set, using the idea of a segmental type of non-connected convexity for sets by taking into account only convex combinations of special types of points as well as g-convex function. It has been shown [14]-[16] that these nonconvex functions enjoy some nice properties which convex function have. We would like to emphasize that the g-convex set and g-convex functions may not be convex sets and convex functions. Noor et al [20] have shown that the minimum of a sum of differentiable and non-differentiable g-convex function on the g-convex set can be characterized by a class of variational inequalities, which is called the general mixed variational inequality. Using the resolvent operator and resolvent equations technique, they have studied

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the existence of a solution of the general mixed variational inequalities involving the nonconvex semicontinuos function on the whole space. If the nonlinear term is not differentiable, then it is not possible to extend the projection and resolvent method for proving the existence of a solution of the general mixed variational inequality. This fact motivated us to apply the auxiliary principle technique for solving the general mixed variational inequalities. In this technique, one usually considers an auxiliary variational inequality problem and proves that the associated mapping has a fixed point, which is the solution of the original problem. In fact, this technique also enables to suggest and analyze some iterative methods for solving the variational inequalities and related optimization problems. In this paper, we again use this technique to study the existence of the solution of general mixed variational inequalities and this is the main motivation of this paper. Several special cases are also discussed.

2. PRELIMINARIES

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ respectively. Let K be a nonempty closed convex set in H.

For given nonlinear operators $T, g\colon H \to H$, consider the problem of finding $u \in K$ such that

$$\langle Tu, g(v) - u\rangle + \varphi(g(v)) - \varphi(u) \geq 0, \qquad \forall v \in H : g(v) \in K. \tag{2.1}$$

Inequality of type (2.1) is called the *general mixed variational inequality involving three operators.* We would like to emphasize that the problem (2.1) is quite different than the problem considered by Noor et al [20].

To convey an idea of the applications of the general mixed variational inequalities (2.1), Noor et al [20] have shown that the minimum of the sum of differentiable and nondifferentiable g-convex functions on a nonconvex set K in H can be characterized by the general mixed variational inequality of type (2.1). For the sake of completeness, we include all the details. For this purpose, we recall the following well known concepts, see [3], [14]-[16].

Definition 2.1 ([3])**.** Let K be any set in H. The set K is said to be g-convex if there exist a function $g : H \longrightarrow H$, such that

$$u + t(g(v) - u) \in K, \quad \forall u, v \in H : u, g(v) \in K, \quad t \in [0, 1].$$

Note that every convex set is g-convex, but the converse is not true, see [3]. If $g = I$, then the g-convex set K is called the convex set.

Definition 2.2 ([14]-[16])**.** The function $F\colon K \longrightarrow H$ is said to be g-convex, if there exists a function g such that

$$F(u) + t(g(v) - u)) \leq (1 - t)F(u) + tF(g(v)), \quad \forall u, v \in H : u, g(v) \in K, \quad t \in [0, 1].$$

Clearly every convex function is g-convex, but the converse is not true. For the properties and various classes of the g-convex functions, see Noor [12].

We note that if the g-convex function is differentiable, then

$$F(g(g)) - F(u) \geq \langle F'(u), g(v) - u\rangle, \quad u, v \in H : u, g(v) \in K,$$

and the converse is also true.

For given differentiable g-convex function F and a nondifferentiable g-convex function φ, we consider the functional of the type

$$I[v] = F(v) + \varphi(v), \quad \forall v \in K. \tag{2.2}$$

We now prove that the minimum of the functional $I[v]$ on the g-convex set K can be characterized by a class of variational inequalities and this is the main motivation of our next result.

Lemma 2.1 ([20])**.** *Let F be a differentiable g-convex function and φ be a nondifferentiable g-convex function on the g-convex set K. Then $u \in K$ is the minimum of $I[v]$, defined by* (2.2)*, on $K \subset g(H)$, if and only if $u \in K$ satisfies the inequality*

$$\langle F'(u), g(v) - u\rangle + \varphi(g(v)) - \varphi(u) \geq 0, \quad \forall v \in H : g(v) \in K, \tag{2.3}$$

where $F'(u)$ is the differential of F at $u \in K$.

Proof. Let $u \in K$ be a minimum of functional $I[v]$ on K. Then

$$I[u] \leq I[g(v)], \quad \forall v \in H : g(v) \in K. \tag{2.4}$$

Since K is a g-convex set, so, for all $u, v \in H : u, g(v) \in K$, $t \in [0, 1]$, $g(v_t) = u + t(g(v) - u) \in K$. Setting $g(v) = g(v_t)$ in (2.4), we have

$$I[u] \leq I[u + t(g(v) - u)],$$

which implies that

$$\begin{aligned} F(u) + \varphi(u) &\leq F(u + t(g(v) - u)) + \varphi(u + t(g(v) - u)) \\ &\leq F(u + t(g(v) - u)) + \varphi(u) + t[\varphi(g(v)) - \varphi(u)]. \end{aligned}$$

From which we have

$$F(u + t(g(v) - u)) - F(u) + t(\varphi(g(v)) - \varphi(u)) \geq 0, \quad \forall v \in K.$$

Dividing the above inequality by t and taking $t \longrightarrow 0$, we have

$$\langle F'(u), g(v) - u\rangle + \varphi(g(v)) - \varphi(u) \geq 0, \quad \forall v \in H : g(v) \in K,$$

which is the required result (2.3).

Conversely, let $u \in K$ satisfy the inequality (2.3). Since F is a g-convex function, $\forall u, v \in H : u, g(v) \in K, \quad t \in [0, 1], \quad u + t(g(v) - u) \in K$.

Consider

$$\begin{aligned} I[u] - I[g(v)] &= F(u) + \varphi(u) - F(v) + \varphi(g(v)) \\ &\leq \langle F'(u), u - g(v)\rangle + \varphi(g(v)) - \varphi(u) \\ &\leq 0, \quad \text{using (2.3)}, \end{aligned}$$

which implies that

$$F(u) \leq F(g(v)), \quad \forall v \in H : g(v) \in K$$

showing that $u \in K$ is the minimum of F on K in H. □

Lemma 2.1 implies that g-convex programming problem can be studied via the general mixed variational inequality (2.1) with $Tu = F'(u)$. In a similar way, one can show that the general variational inequality is the Fritz-John condition of the inequality constrained optimization problem.

For $g = I$, the identity operator, the general variational inequality (2.1) is equivalent to finding $u \in K$ such that

$$\langle Tu, v - u\rangle + \varphi(v) - \varphi(u) \geq 0, \qquad \forall v \in K, \tag{2.5}$$

which is known as the mixed variational inequality or variational inequality of the second type. For applications and numerical methods of mixed variational inequality (2.5), see [2]-[22] and the references therein.

If φ is the indicator function of a closed convex set K, then the general mixed variational inequality (2.1) is equivalent to finding $u \in K$ such that

$$\langle Tu, g(v) - u\rangle \geq 0, \quad \forall v \in H : g(v) \in K, \tag{2.6}$$

which is called the general variational inequality involving two nonlinear operators, introduced and studied by Noor [15]. Note the general variational inequality (2.6) is quite different from the general variational inequality of the type: Find $u \in H : g(u) \in K$ such that

$$\langle Tu, g(v) - g(u)\rangle \geq 0, \quad \forall v \in H : g(v) \in K, \tag{2.7}$$

which was introduced and studied by Noor [8] in 1988. It has been shown that a wide class of unrelated nonsymmetric and odd-order obstacle and unilateral probels, which arise in various branches of financial mathematical, economics, transportation, structural, mathematical and engineering sciences can be studied via general variational inequalities of the type (2.7). For more details, see [7]-[18]. and the references therein.

If $g = I$, the identity operator, then general variational inequality (2.6) is equivalent to finding $u \in K$ such that

$$\langle Tu, v - u\rangle \geq 0, \quad \forall v \in K,$$

which is known as the classical variational inequality and was introduced in 1964 by Stampacchia [22]. For recent applications, numerical methods, sensitivity analysis, dynamical systems and formulation of variational inequalities, see [1]-[22] and the references therein.

We also need the following concepts and results.

Definition 2.3. For all $u, v \in H$, an operator $T\colon H \to H$ is said to be:

(i) *strongly monotone*, if there exists a constant $\alpha > 0$ such that

$$\langle Tu - Tv, u - v\rangle \geq \alpha||u - v||^2;$$

(ii) *Lipschitz continuous*, if there exists a constant $\beta > 0$ such that

$$||Tu - Tv|| \leq \beta||u - v||.$$

From (i) and (ii), it follows that $\alpha \leq \beta$.

3. MAIN RESULTS

In this Section, we use the auxiliary principle technique of Glowinski, Lions and Tremoliers [5] to study the existence of a solution of the general mixed variational inequality (2.1).

Theorem 3.1. *Let T be a strongly monotone operator with constant $\alpha > 0$ and Lipschitz continuous with constant $\beta > 0$. Let g be a strongly monotone and Lipschitz continuous operator with constants $\sigma > 0$ and $\delta > 0$ respectively. If there exists a $\rho > 0$ such that*

$$|\rho - \frac{\alpha}{\beta^2}| < \frac{\sqrt{\alpha^2 - \beta^2 k(2-k)}}{\beta^2}, \quad \alpha > \beta\sqrt{k(2-k)}, \quad k < 1, \tag{3.1}$$

where

$$\theta = \sqrt{1 - 2\sigma + \delta^2} + \sqrt{1 - 2\rho\alpha + \rho^2\beta^2} = k + \sqrt{1 - 2\rho\alpha + \rho^2\beta^2} \tag{3.2}$$

$$k = \sqrt{1 - 2\sigma + \delta^2} \tag{3.3}$$

then the general mixed variational inequality (2.1) *has a unique solution.*

Proof. We use the auxiliary principle technique to prove the existence of a solution of (2.1). For a given $u \in K$ satisfying the general mixed variational inequality (2.1), we consider the problem of finding a solution $w \in K$ such that

$$\langle \rho Tu + w - g(u), g(v) - w\rangle + \varphi(g(v)) - \varphi(w) \geq 0, \quad \forall v \in H : g(v) \in K, \tag{3.4}$$

where $\rho > 0$ is a constant.

The inequality of type (3.4) is called the auxiliary variational inequality associated to problem (2.1). It is clear that relation (3.4) defines a mapping $u \longrightarrow w$. It is enough to show that the mapping $u \longrightarrow w$ defined by relation (3.4) has a fixed point belonging to H satisfying the general variational inequality (2.1). Let $w_1 \neq w_2$ be two solutions of (3.4) related to $u_1, u_2 \in H$ respectively. It is sufficient to show that for a well chosen $\rho > 0$,

$$\|w_1 - w_2\| \leq \theta \|u_1 - u_2\|,$$

with $0 < \theta < 1$, where θ is independent of u_1 and u_2. Taking $g(v) = w_2$ (respectively w_1) in (3.4), related to u_1 (respectively u_2), adding the resultant, we have

$$\langle w_1 - w_2, w_1 - w_2 \rangle \leq \langle g(u_1) - g(u_2) - \rho(Tu_1 - Tu_2), w_1 - w_2 \rangle,$$

from which we have

$$\begin{aligned} \|w_1 - w_2\| &\leq \|g(u_1) - g(u_2) - \rho(Tu_1 - Tu_2)\| \\ &\leq \|u_1 - u_2 - (g(u_1) - g(u_2))| + \|u_1 - u_2 - \rho(Tu_1 - Tu_2)\|. \end{aligned} \tag{3.5}$$

Since T is both strongly monotone and Lipschitz continuous operator with constants $\alpha > 0$ and $\beta > 0$ respectively, it follows that

$$\begin{aligned} \|u_1 - u_2 - \rho(Tu_1 - Tu_2)\|^2 &\leq \|u_2 - u_2\|^2 - 2\rho\langle u_1 - u_2, Tu_1 - Tu_2\rangle + \rho^2\|Tu_1 - Tu_2\|^2 \\ &\leq \left(1 - 2\rho\alpha + \rho^2\beta^2\right)\|u_1 - u_2\|^2. \end{aligned} \tag{3.6}$$

In a similar way, using the strongly monotonicity with constant $\sigma > 0$ and Lipschitz continuity with constant $\delta > 0$, we have

$$\|u_1 - u_2 - (g(u_1) - g(u_2))\| \leq \sqrt{1 - 2\sigma + \delta^2}\|u_1 - u_2\|. \tag{3.7}$$

From (3.2), (3.3), (3.5), (3.6) and (3.7), we have

$$\|w_1 - w_2\| \leq \left\{k + \sqrt{1 - 2\rho\alpha + \rho^2\beta^2}\right\}\|u_1 - u_2\| = \theta\|u_1 - u_2\|.$$

From (3.1) and (3.2), it follows that $\theta < 1$ showing that the mapping defined by (3.4) has a fixed point belonging to K, which is the solution of (2.1), the required result. □

Remark 3.1. We would like to point out that the auxiliary principle technique considered in this paper can be used to suggest both implicit and explicit iterative methods for solving the general mixed variational inequalities. This is another direction of the future research. We expect that the interested readers will be able to discover new applications of the general mixed variational inequalities in various areas of pure and applied sciences.

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COMSATS Institute of Information Technology,
Mathematics Department,
Islamabad, Pakistan.
College of Science, King Saud University
Mathematics Department,
Riyadh, Saudi Arabia
E-mail address: noormaslam@hotmail.com