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OPEN PROBLEMS RISEN BY CONSTANTIN UDRIŞTE AND HIS RESEARCH COLLABORATORS

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Invited paper to celebrate Professor Constantin Udrişte, on the occasion of his seventies

ABSTRACT. We bring into the attention of mathematical world some classes of open problems risen by Professor Constantin Udrişte and his research collaborators, as they appear in the works [4]÷[11]. This work pay attention to the following research directions: extrema theory; vector fields theory; dynamical systems; optimal control.

1. OPEN PROBLEMS IN EXTREMA THEORY

Let us comment some surprises that appear in the theory of extrema of real-valued functions, focusing especially to the case when the constraints are integral manifolds of a Pfaff system, [12], [16], [17], [18].

Let $D \subseteq \mathbb{R}^n$ be an open set and

$$\omega^j(x) = \sum_{i=1}^{n} \omega_i^j(x)dx^i = 0, \quad j = \overline{1,p},\ p < n \qquad (*)$$

a Pfaff system on D, where $\operatorname{rank}[\omega_i^j(x)] = p, \quad \forall x \in D$.

An *integral manifold at* x_0 of the Pfaff system $(*)$ is a regular parametrized manifold $r = (x^1, \dots, x^n)$ passing through x_0 which satisfies

$$\sum_{i=1}^{n} \omega_i^j(r(u)) \frac{\partial x^i}{\partial u^\ell} = 0, \quad \forall u \in I \subseteq \mathbb{R}^m,\ \ell = \overline{1,m},\ j = \overline{1,p}.$$

For $m = 1$ the manifold is called *integral curve.*

Definition 1.1. Let $f\colon D \to \mathbb{R}$ be a real-valued function. The point $x_0 \in D$ is called *a minimum (maximum) point for f constrained by the Pfaff system* $(*)$ if for

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every integral manifold $r\colon I \to D$ at x_0, there exists a neighborhood $V_{x_0} \subseteq D$ such that

$$f(x) \geq (\leq) f(x_0), \quad \forall x \in V_{x_0} \cap r(I).$$

If the neighborhood V_{x_0} does not depend on the integral manifold r, then we should say that x_0 is a minimum (maximum) point of f uniformly constrained by $(*)$.

In [16] it is stated the following result:

Theorem 1.1. *If $x_0 \in D$ is a minimum point for the restriction of real-valued function f of any C^1 curve at x_0, then x_0 is a minimum point for f.*

From this theorem it follows that the notions of extremum constrained by integral manifolds and extremum constrained by integral curves are equivalent.

Open problem 1.1. Does Theorem 1.1 hold if "C^1 curves" at x_0 are replaced by "C^k curves", $2 \leq k \leq \infty$, at x_0?

Open problem 1.2. (continuation). In $\mathbb{R}^n$, it is possible that a point x_0 be a minimum point for the restriction of f to any straight line at x_0, without being a minimum point for f. Can it also be possible for the geodesics at x_0 on a Riemannian manifold?

According to [16], in the case that the Pfaff system $(*)$ is completely integrable, there is no difference between "uniformly constrained extremum" and "constrained extremum".

Open problem 1.3. Is it possible to exist constrained extrema points which are not uniformly constrained?

Clearly, such kind of points should be found only in the case of the constraints described by a Pfaff system which is not completely integrable.

For a point $x_0 \in D$, let M_{x_0} be the union of all integral curves at x_0 of the Pfaff system $(*)$.

Conjecture 1.1. For a point $x_0 \in D$, let M_{x_0} be the union of all integral curves at x_0 of the Pfaff system $(*)$. Let x_0 be an extremum point of $f\colon D \to \mathbb{R}$ constrained by $(*)$, which is not a free extremum. Then x_0 is an extremum point uniformly constrained by $(*)$ if and only if M_{x_0} is an integral manifold at x_0 of the system $(*)$.

In [12], it was shown that theory of sufficient conditions for extrema of f constrained by equalities, can be generalized almost word by word to the case of extrema of f constrained by the system $(*)$.

Theorem 1.2. *Let $f\colon D \subseteq \mathbb{R}^n \to \mathbb{R}$ be a C^2 function and $x_0 \in D$ be a solution of the system*

$$\frac{\partial f}{\partial x^i}(x_0) + \sum_{k=1}^{p} \lambda_k \omega_i^k(x_0) = 0, \quad i = \overline{1,n}.$$

Suppose that the quadratic form

$$d^2f(x_0) + \frac{1}{2}\sum_{k=1}^{p}\lambda_k \sum_{i,j=1}^{n}\left(\frac{\partial \omega_j^k}{\partial x^i} + \frac{\partial \omega_i^k}{\partial x^j}\right)(x_0)dx^i dx^j$$

constrained by

$$\sum_{i=1}^{n}\omega_i^k(x_0)dx^i = 0, \quad k = \overline{1,p}$$

is positive (negative) definite. Then x_0 is a minimum (maximum) point of f constrained by the Pfaff system $(*)$.

Open problem 1.4. Are the conditions of Theorem 1.2 sufficient in order that a point x_0 be an extremum point of f uniformly constrained by the Pfaff system $(*)$? If not, which are these conditions?

2. OPEN PROBLEMS IN THE THEORY OF VECTOR FIELDS

Consider $X = (X_1, \ldots, X_n)$ be a C^1-class vector field, defined on an open and connected set D in $\mathbb{R}^n$. Let $f\colon D \to \mathbb{R}$ be a C^1-class scalar field and $M\colon f(x) = c$ a level hypersurface associated to the function f. If the restriction of X to M is a vector field tangent to M, that is $(X, \text{grad} f) = 0$ or $D_X f = 0$, then M is called a field hypersurface of X, [8].

Let $m \in \{1, 2, \ldots, n-2\}$. A hypersurface in $\mathbb{R}^n$, $n \geq 3$, which can be generated by the motion of an m-plane G, relying on a submanifold Γ with $n-m-1$ dimensions, is called a *ruled hypersurface*; the m-plane G is called a generator (ruling), and the submanifold Γ is called a director submanifold.

Theorem 2.1 ([10])**.** *The field hypersurfaces of torse forming vector fields are ruled hypersurfaces.*

Open problem 2.1. Is the converse of Theorem 2.1 true or not?

Consider now $X(x, a) = (X_1(x, a), \ldots, X_n(x, a))$, $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ be a C^∞-class vector field, which depends on the parameter $a = (a_1, \ldots, a_m) \in \mathbb{R}^m$. The submanifolds orthogonal to the field lines of X are solutions of the Pfaff equation

$$X_1(x, a)dx_1 + \cdots + X_n(x, a)dx_n = 0.$$

Open problem 2.2. Has the Hopf bifurcation of the flow generated by $X(x, a)$ any influence upon the set of submanifolds orthogonal to field lines?

Open problem 2.3. For the set of submanifolds orthogonal to the field lines of $X(x, a)$, is there a bifurcation phenomenon?

In his paper [9], Professor Constantin Udrişte introduces the new notion of *geometric dynamics*, showing that on a semi-Riemannian manifold (M, g), the orbits of a vector field X are solutions of a second-order differential equation

$$\ddot{x} = \text{grad} f + F(\dot{x})$$

(Euler-Lagrange prolongation of $\dot{x} = X(x)$), where $f = \frac{1}{2}g(X,X)$ is the energy, and $g(F(v),w) = d(g(X))(p)(v,w)$ for tangent vectors v and w at the point p in M. This dynamical system is conservative with the Hamiltonian being given by $H(v) = \frac{1}{2}g(v,v) - f(p)$. Also, Professor Udrişte formulates these facts in the language of Lagrangian dynamics and Jacobi manifolds, and applies this theory to electromagnetic dynamical systems, see also [23].

Open problem 2.4. The theory in [9] shows that every dynamical system of order one can be prolonged to an Euler-Lagrange dynamical system of order two whose trajectories are geodesics of a Lagrangian defined by the velocity vector field (Lagrange structure of order one). In a similar manner, every dynamical system of order two can be prolonged to an Euler-Lagrange dynamical system of order four whose trajectories are geodesics of a Lagrangian defined by velocity and acceleration vector fields (Lagrange structure of order two). Can this viewpoint create better examples for higher order Lagrange spaces [3]?

3. OPEN PROBLEMS IN MAGNETIC DYNAMICAL SYSTEMS

Let D be an open connected set of $\mathbb{R}^3$, with a piecewise smooth boundary ∂D. If consider P a point of $\bar{D}$, and denote by $\vec{J}$ the current density, that is a C^∞-class vector field on $\bar{D} = D \cup \partial D$, then the vector field

$$\vec{H}(M) = \frac{1}{4\pi}\int_D \frac{\vec{J}(P) \times \vec{PM}}{PM^3} d\nu_P, \quad M \in \mathbb{R}^3,$$

is called the *Biot-Savart-Laplace vector field*, [1].

Denote by $\vec{H} = H_x\vec{\imath} + H_y\vec{\jmath} + H_z\vec{k}$ a C^∞-class magnetic field defined on $\mathbb{R}^3$, and by $\vec{r} = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$ the position vector of the point $M(x,y,z)$. A magnetic line α starting from $M_0(x_0,y_0,z_0)$ at the moment $t = 0$ is the oriented curve $\vec{r} = \vec{r}(t)$, $t \in (-\varepsilon, \varepsilon)$ satisfying the Cauchy problem

$$\frac{d\vec{r}}{dt} = \vec{H}(\vec{r}), \quad \vec{r}(0) = \vec{r}_0.$$

The *magnetic surface* $\sum$: $h(x,y,z) = c$, relying on a curve $\beta\colon (a,b) \to \mathbb{R}^3$ is the solution of the Cauchy problem

$$(\vec{H}, \nabla h) = 0, \quad H(\beta(u)) = h(\beta(0)), \ \forall u \in (a,b).$$

A magnetic surface is generated by magnetic lines, and in the absence of symmetries, a magnetic line is an open curve. Sometimes, the image of an open field line is dense in the magnetic surface.

Open problem 3.1. [7] Let A be a subdomain of D, and $\vec{H}_A$, $\vec{H}_D$ be the Biot-Savart-Laplace vector field on A and D respectively. Do there exist domains A in D with the property that $\vec{H}_A$ and $\vec{H}_D$ have the same phase portrait on $\mathbb{R}^3 \setminus \bar{D}$?

Let
$$f\colon \mathbb{R}^3 \to \mathbb{R}, \quad f = \frac{1}{2}\left(H_x^2 + H_y^2 + H_z^2\right)$$
be the *energy* of $\vec{H}$, leaving aside a multiplicative factor μ, [15]. According to [10], let α be a magnetic line included in $\mathrm{int}(\mathbb{R}^3 \setminus D)$. Differentiate $\frac{d\vec{r}}{dt} = \vec{H}$ along α, consider $\mathrm{rot}\vec{H} = 0$, and replace $\frac{d\vec{r}}{dt}$ by $\vec{H}$. We find the prolongation
$$\frac{d^2\vec{r}}{dt^2} = \nabla f \tag{3.1}$$
which is a potential dynamical system of order two. We concentrate the result in

Theorem 3.1. *Any magnetic line in* $\mathrm{int}(\mathbb{R}^3 \setminus D)$ *is a trajectory of a potential dynamical system with three degrees of freedom associated to the potential* $-f$ *in* $\mathrm{int}(\mathbb{R}^3 \setminus D)$.

If denote $\frac{d\vec{r}}{dt} = -\vec{v}$, the differential system of order two (3.1) can be written on the phase space $\mathbb{R}^6$ as a Hamiltonian system
$$\frac{d\vec{r}}{dt} = -\vec{v}, \quad \frac{d\vec{v}}{dt} = -\nabla f,$$
with the Hamiltonian $\mathcal{H}(\vec{r}, \vec{v}) = \frac{1}{2}\|\vec{v}\|^2 - f(\vec{r})$.

We underline that the Hamiltonian flow conserves the phase space volume, while the Hamiltonian $\mathcal{H}$ is a first integral of the Hamiltonian dynamical system. Moreover, the general theory of Hamiltonian systems shows that the preceding prolongation is a new Lorentz law, based on the geometrical structure that incorporates the magnetic field, [10].

Theorem 3.2. *The trajectory of the dynamical system*
$$\frac{d^2\vec{r}}{dt^2} = \nabla f, \quad \vec{r}(0) = \vec{r}_0 \in \mathbb{R}^3 \setminus D$$
having $\mathcal{H} > -f$ *as constant total energy and laying in* $\mathbb{R}^3 \setminus D$ *is a reparametrized geodesic of the Riemann-Jacobi metric*
$$g_{ij} = (\mathcal{H} + f)\delta_{ij}, \quad i, j = 1, 2, 3.$$

Let us suppose that $\vec{r}(0) = \vec{r}_0 \in D$ and α lies in D. Differentiating $\frac{d\vec{r}}{dt} = \vec{H}$ along α, and replacing $\frac{d\vec{r}}{dt}$ by $\vec{H}$ in convenient terms, we get a prolongation
$$\frac{d^2\vec{r}}{dt^2} = \nabla f + \vec{J} \times \frac{d\vec{r}}{dt}, \tag{3.2}$$
which is a non-potential dynamical system of order two. Moreover, if we take the scalar product with $\frac{d\vec{v}}{dt}$, we find $\frac{d}{dt}\mathcal{H} = 0$, that is the total energy $\mathcal{H}$ is conserved.

Theorem 3.3. *Any magnetic line included in D is a trajectory of a non-potential dynamical system with three degrees of freedom for which the energy*

$$\mathcal{H} = \frac{1}{2}\|\vec{v}\|^2 - f(x, y, z)$$

is conserved.

In [6] and [15] it is described a suitable geometrical structure which shows that the conservative non-potential dynamical system of order two (3.2) describes a new Lorentz world-force law (nonclassical magnetic dynamics).

Theorem 3.4 (LORENTZ-UDRIŞTE WORLD-FORCE LAW)**.** *The trajectory of the dynamical system*

$$\frac{d^2\vec{r}}{dt^2} = \nabla f + \vec{J} \times \frac{d\vec{r}}{dt}, \quad \vec{r}(0) = \vec{r}_0 \in D$$

having $\mathcal{H} > -f$ as constant total energy and lying in D is a reparametrized horizontal geodesic of the Riemann-Jacobi-Lagrange structure

$$g_{ij} = (\mathcal{H} + f)\delta_{ij}\,, \quad N^i{}_j = \Gamma^i_{jk} y^k + F^i{}_j\,, \quad i, j, k = 1, 2, 3,$$

where Γ^i_{jk} is the Riemann connection determined by the metric g_{ij}, $N^i{}_j$ is a nonlinear connection, and $F_{ij} = (\mathrm{rot}\vec{H})_{ij}$, $F^i{}_j = g^{ih} F_{hj}$.

It is clear that the Riemann-Jacobi-Lagrange structure incorporates the magnetic field both by the Riemann-Jacobi metric and by the nonlinear connection.

Remark that the trajectories of the preceding conservative (potential or non-potential) dynamical systems divides into three classes:

1) the class of original magnetic lines that correspond to the energy $\mathcal{H} = 0$;
2) the class of trajectories for the energy $\mathcal{H} = \text{constant} < 0$;
3) the class of trajectories for the energy $\mathcal{H} = \text{constant} > 0$.

OPEN PROBLEM 3.2. Find the physical significance for the trajectories of the preceding conservative dynamical systems, which correspond to positive or negative constant energy $\mathcal{H}$.

We also remark that the geometric magnetic dynamics is altered if we replace the magnetic field $\vec{H}$ with $-\vec{H}$, [4].

OPEN PROBLEM 3.3. Can we explain the migration of the earth's magnetic poles (pole shifts) in the context of geometric magnetic dynamics?

For an explanation of the pole shifts by means of Dirac delay in a Sabba Ştefănescu magnetic flow, we address the reader to the work [4].

Some properties of the magnetic field $\vec{H}$ are crucial for real problems in geophysics and plasma confinement, [20], [13], [14]. These properties could be derived from:

a) the geometry-topology of the field lines (magnetic lines), oriented curves, which are solutions of the differential system

$$\frac{dx}{dt} = H_x, \quad \frac{dy}{dt} = H_y, \quad \frac{dz}{dt} = H_z; \tag{3.3}$$

b) the geometry-topology of the field surfaces (magnetic surfaces), which are constant level sets attached to the solutions of the partial differential equation

$$H_x \frac{\partial f}{\partial x} + H_y \frac{\partial f}{\partial y} + H_z \frac{\partial f}{\partial z} = 0. \tag{3.4}$$

Remark that the field lines and surfaces are integral characteristics of the magnetic vector field, [5], [20], [21], [22].

Regarding properties of the lines of the magnetic field generated by piecewise rectilinear electric circuits, we emphasize the papers by Sabba Ştefănescu (for complete references, see [20]) and a series of manuscript notices that are still unpublished. The study of these works leads to the idea of knowing whether in the complicated structure of magnetic phase portraits [19], especially of those due to non planar electric circuits, one can detect simpler structural elements, [10]. The investigation undertaken in this direction leads to the identification of the existence of certain algebraic first integrals for describing magnetic lines. These first integrals determine pencils of algebraic surfaces on which there are wound magnetic lines, generally transcendent ones. The complete determination of the magnetic lines may be reduced in these cases to Abelian quadratures and in their turn, these quadratures, under entirely special conditions, are reducible to elliptical integrals and even to elementary transcendent ones.

The following statement is known as the Sabba Ştefănescu Conjecture.

Conjecture 3.1. The differential system (3.3) describing the lines of a magnetic field generated by a configuration of piecewise rectilinear electric circuits admits an algebraic first integral.

Open problem 3.4. Which are the Cartesian implicit equations and the shape of the magnetic lines around n ($n > 2$) equal rectilinear circuits placed arbitrarily in the space?

In [15], it is initiated a study of electromagnetic dynamical systems. To develop further our investigation, we need some mathematical ingredients of electromagnetism. Let $\vec{E} = \vec{E}(x,t)$, $\vec{E} = E_1\vec{\imath}_1 + E_2\vec{\imath}_2 + E_3\vec{\imath}_3$ be the electric vector field on the domain $U \times \mathbb{R}$. The electric line α which starts at the moment $s = 0$ from the point (x_0^1, x_0^2, x_0^3) is the oriented curve $\alpha\colon (-\varepsilon, \varepsilon) \to U$, $\alpha(s) = (x^1(s), x^2(s), x^3(s))$, a solution of the Cauchy problem

$$\frac{dx^i}{ds} = E_i, \quad x^i(0) = x_0^i, \quad i = 1, 2, 3.$$

The set of all images of maximal electric lines is called the *phase portrait* of the electric field $\vec{E}$. The parameter t can produce bifurcations in the flow generated by the field $\vec{E}(x,t)$.

Let

$$f\colon U \to \mathbb{R}, \quad f = \frac{1}{2}\left(E_1^2 + E_2^2 + E_3^2\right)$$

be the energy of $\vec{E}$, leaving aside a multiplicative factor.

In [15] are proved

Theorem 3.5. *Every electric line is the trajectory of a non-potential dynamical system with three degrees of freedom for which the energy*

$$\mathcal{H} = \frac{1}{2}\delta_{ij}\dot{x}^i\dot{x}^j - f(x^1, x^2, x^3)$$

is conserved.

As above denote by $\vec{H}$ the magnetic vector field (magnetizing force), and consider $\vec{B}$ be the magnetic induction.

Theorem 3.6. *The equations of motion of a particle moving in an electric field $\vec{E}$ are Hamiltonian, with respect to the energy*

$$\mathcal{H} = \frac{1}{2}\delta_{ij}\dot{x}^i\dot{x}^j - f(x^1, x^2, x^3),$$

and the simplectic form $\Omega = \delta_{ij}dx^i \wedge d\dot{x}^j - \partial_t\vec{B}$, where the magnetic induction $\vec{B}$ is viewed as a closed 2-form $\vec{B} = B_1 dx^2 \wedge dx^3 + B_2 dx^3 \wedge dx^1 + B_3 dx^1 \wedge dx^2$ associated to the vector field $\vec{B} = B_1\vec{\imath}_1 + B_2\vec{\imath}_2 + B_3\vec{\imath}_3$.

Open problem 3.5. Find the properties for the field lines of the Poynting vector field $\vec{S} = \vec{E} \times \vec{H}$.

Open problem 3.6. Study the electromagnetic geometric dynamics defined by a vector field in the distribution generated by the vector fields $\vec{E}$ and $\vec{H}$.

4. OPEN PROBLEMS IN OPTIMAL CONTROL

To formulate our open problem, we need to introduce a weak convergence definition, a weak compactness theorem and an extreme point theorem [11], all based on the curvilinear integral (see also [2]).

First of all, we start with the infinite dimensional space

$$L^\infty = L^\infty(\Omega_{0,t}; \mathbb{R}^k) = \left\{ u(\cdot) : \Omega_{0,t} \to \mathbb{R}^k | \ \operatorname{ess} \sup_{0 \le s \le t} |u(s)| < \infty \right\},$$

of the functions $u(\cdot)$, with the norm

$$||u||_{L^\infty} = \operatorname{ess} \sup_{0 \le s \le t} |u(s)|.$$

Definition 4.1. Let $u_n \in L^\infty$, $n \in \mathbb{N}$, be a sequence and $u \in L^\infty$ be a point. The sequence u_n is called *weakly convergent* to u if and only if

$$\int_{\gamma_{0,t}} u_n(s) \cdot v_\alpha(s) ds^\alpha \to \int_{\gamma_{0,t}} u(s) \cdot v_\alpha(s) ds^\alpha$$

as $n \to \infty$, for all completely integrable 1-forms $dv = v_\alpha(s)ds^\alpha$, $s \in \Omega_{0,t}$ satisfying $\left|\int_{\gamma_{0,t}} v_\alpha(s)ds^\alpha\right| < \infty$, where γ is an arbitrary piecewise C^1-class curve joining the points 0 and t in $\mathbb{R}^m_+$.

We denote the previous weak convergence by $u_n \rightharpoonup u$.

We introduce now a weak compactness theorem, [11], in L^∞ similar to the result of Alaoglu, given in [2].

Theorem 4.1. *For any sequence of controls $u_n \in \mathcal{U}$, $n \in \mathbb{N}$, there exists a subsequence u_{n_k} and $u \in \mathcal{U}$ such that $u_{n_k} \rightharpoonup u$.*

Definition 4.2. A point z in a convex set K is called *extreme* provided that there do not exist two points $x, \hat{x} \in K$ and $0 < \lambda < 1$ such that $z = \lambda x + (1 - \lambda)\hat{x}$.

Now, we can give the statement of an extreme point theorem, [11], similar to those of Krein-Milman, [2].

Theorem 4.2. *Each convex, nonempty subset of L^∞, which is compact in the weak topology, has at least one extreme point.*

Open problem 4.1. Give proofs for Theorems 4.1 and 4.2.

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