

A NEW SETTING FOR HIGHER ORDER LAGRANGIANS IN THE TIME DEPENDENT CASE

MARCELA POPESCU AND PAUL POPESCU

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ABSTRACT. A new setting that allows a new definition of a time dependent Lagrangian of higher order is considered. Canonical constructions of some connections and of a semispray, associated with a such Lagrangian, are performed in an analogously manner as in the time independent case.

1. INTRODUCTION

The aim of this paper is to construct a general setting of time dependent higher order spaces, that allows to consider some new classes of time dependent Lagrangians. Most authors study the time dependent case considering the product manifold $M \times \mathbb{R}$ or $M \times I$, where $I \subset \mathbb{R}$ is an interval, when the coordinates on M can be taken independently of the real coordinate $t \in \mathbb{R}$ or $t \in I$. We study here a more case, replacing $M \times I$ by the total space M_I of a fibered manifold $M_I \rightarrow I$. It allows to consider also fibered manifolds that are not locally trivial ones. This fibered manifold lifts to a higher order manifold $\mathcal{T}^k M_I$, that can play the role of a higher order tangent space. It allows to consider some connections and a canonical k -order semispray, as in the time independent case (see [7] and [13]).

2. THE AFFINE SETTING OF THE TIME DEPENDENT CASE, USING FIBERED MANIFOLDS

A *fibered manifold* is a surjective submersion $E \xrightarrow{\pi} M$. An *affine bundle* $E \xrightarrow{\pi} M$ is a fibered manifold which the change rules of the local coordinates on E have the

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form

$$x^{i'} = x^{i'}(x^j), y^{\alpha'} = g_{\alpha'}^{\alpha}(x^j)y^{\alpha} + v^{\alpha'}(x^j), \quad (2.1)$$

where the coordinates (x^i) are on the base M and the coordinates (y^{α}) are on the fibers. An *affine section* in the bundle E is a differentiable map $M \xrightarrow{s} E$ such that $\pi \circ s = id_M$. The local components (s^{α}) of s change according to the rule $s^{\alpha'}(x^{i'}) = g_{\alpha'}^{\alpha}(x^j)s^{\alpha}(x^j) + v^{\alpha'}(x^j)$. We denote by $\bar{E} \xrightarrow{\bar{\pi}} M$ the canonically vector bundle associated with the affine bundle $E \xrightarrow{\pi} M$. More precisely, using local coordinates, the coordinates change on \bar{E} following the rules $x^{i'} = x^{i'}(x^j)$, $z^{\alpha'} = g_{\alpha'}^{\alpha}(x^j)z^{\alpha}$, when the coordinates on E change according the formulas (2.1).

Every vector bundle is an affine bundle, called a *central affine bundle*. In this case $v^{\alpha}(x^j) = 0$.

In the time dependent case we consider a particular fibered manifold $M_I \xrightarrow{\pi} I$, where I is a one dimensional manifold (i.e., according to the classification of one dimensional manifolds, it is an open real interval $I \subset \mathbb{R}$ or a circle S^1); for sake of simplicity we take $I = \mathbb{R}$. It is possible that M_I be not a local trivial fibration (or a bundle); for example, $M_I = \mathbb{R}^2 \setminus \{(x, 0) : x < 0\}$ and $\pi(x, y) = x$. If M is a manifold, then M_I can be a bundle $M_I \rightarrow I$, with a typical fiber M , for example the trivial bundle $M \times I \rightarrow I$.

Considering the natural projections $TM_I \xrightarrow{\pi_1} M_I$ and $TM_I \xrightarrow{\pi_2} TI = I \times \mathbb{R}$, then for every $z \in M_I$, $\pi(z) = t$ one define $\mathcal{T}_z M_I = T_z M_I \cap \pi_2^{-1}(t, 1)$. Then the natural projection $\pi_I: \mathcal{T}M_I \rightarrow TM_I$ is the projection of an affine bundle. Considering local coordinates $\{x^i, t\}_{i=1, m}$ on M_I , which change according to the rules $x^{i'} = x^{i'}(x^i, t)$, $t' = t$. The local coordinates induced on $(TM_I)_t$ and $\mathcal{T}M$ change according to the rules $x^{i'} = x^{i'}(x^i, t)$, $y^{i'} = y^i \frac{\partial x^{i'}}{\partial x^i}(x^i, t) + \frac{\partial x^{i'}}{\partial t}(x^i, t)$. These formulas prove the following interpretation.

Proposition 2.1. *The affine bundles $\mathcal{T}M_I \rightarrow M_I$ and $J^1\pi \rightarrow M_I$ are isomorphic.*

It is the classical setting used, for example, in [2, Sect. 33]. We say that $\pi_I: J^1\pi = \mathcal{T}M_I \rightarrow TM_I$ is the *t-tangent bundle* of M_I .

In the case when $M_I = M \times I$, where M is a manifold, some local coordinates on M which change according to the rules $x^{i'} = x^{i'}(x^i)$ induce a local changes of coordinates on $\mathcal{T}M_I$ given by, $x^{i'} = x^{i'}(x^i)$, $t' = t$, thus $\pi: \mathcal{T}M_I \rightarrow TM_I$ is a local vector bundle. In this case we say that $M \times I \rightarrow M$ is the trivial time bundle of the manifold M . The t-tangent bundle $\mathcal{T}M_I$ is in this case the vector bundle $\pi_0^* TM$, where $M_I = M \times I \xrightarrow{\pi_Q} M$ is the natural projection.

In the general case, the t-tangent bundle $\mathcal{T}M_I$ is an affine bundle. Thus we can consider its affine dual bundle, that is an affine bundle $\mathcal{T}^{\dagger}M_I \rightarrow \mathcal{T}M_I$ with one

dimensional fiber. The manifold T^*M_I is called in [2, Sect. 33] the *extended phase space*; $T^*M_I \cong \mathcal{T}^\dagger M_I$.

Proposition 2.2. *The total space of the cotangent bundle T^*M_I is canonically diffeomorphic with the total space $\mathcal{T}^\dagger M_I$ of the affine dual of $\mathcal{T}M_I \rightarrow M_I$.*

Proof. The coordinates (t, x^i, y^0, y^i) on TM_I change according to the rules:

$$\begin{cases} t' = t, \\ x^{i'} = x^{i'}(x^i, t), \\ y^{0'} = y^0, \\ y^{i'} = \frac{\partial x^{i'}}{\partial x^i} y^i + \frac{\partial x^{i'}}{\partial t} y^0, \end{cases}$$

(t, x^i, p_0, p_i) on T^*M_I and (t, x^i, Ω, p_i) on $\mathcal{T}^\dagger M_I$ change according to the rules:

$$\begin{cases} t' = t, \\ x^{i'} = x^{i'}(x^i, t), \\ \frac{\partial x^{i'}}{\partial t'} p_{i'} + p_{0'} = p_0, \\ \frac{\partial x^{i'}}{\partial x^i} p_{i'} = p_i \end{cases} \quad \text{and} \quad \begin{cases} t' = t, \\ x^{i'} = x^{i'}(x^i, t), \\ \frac{\partial x^{i'}}{\partial t'} p_{i'} + \Omega' = \Omega, \\ \frac{\partial x^{i'}}{\partial x^i} p_{i'} = p_i. \end{cases}$$

respectively, thus the conclusion follows. □

It follows that we can consider an affine Hamiltonian h on $\mathcal{T}M_I$ as $h: \mathcal{T}M_I \rightarrow \mathcal{T}^\dagger M_I = T^*M_I$.

Let us find a way to construct non-trivial affine Hamiltonians using Hamiltonians on M_I . Let $\bar{H}: T^*M_I \rightarrow \mathbb{R}$ be a Hamiltonian on M_I . We suppose that there is $\alpha_0 \in \mathbb{R}$ such that $\frac{\partial \bar{H}}{\partial p_0}(t, x^i, p_0, p_i) \neq 0$, $(\forall) P(t, x^i, p_0, p_i) \in T^*M_I$ and $\bar{H}(P) = \alpha_0$. According to the implicit functions theorem, it follows that the local equation $\bar{H}(t, x^i, p_0, p_i) = \alpha_0$ has local solutions $p_0 = H_0(t, x^i, p_i)$, that define a global section $h_0: \mathcal{T}^*M_I \rightarrow \mathcal{T}^\dagger M_I = T^*M_I$, $h_0(t, x^i, p_i) = (t, x^i, p_i, H_0(t, x^i, p_i))$ i.e. an affine Hamiltonian on $\mathcal{T}M_I$.

Let us find under which conditions on a Hamiltonian $\bar{H}: T^*M_I \rightarrow \mathbb{R}$, the solutions of its Hamilton equation comes from the sections of the fibred manifold $T^*M_I \xrightarrow{\pi'} I$, where π' is obtained as the composition $T^*M_I \xrightarrow{\pi'} M_I \rightarrow I$. In order to

see this, let us write the Hamilton equations of \bar{H} in the form

$$\left\{ \begin{array}{l} \frac{dx^i}{dt} = \frac{\partial \bar{H}}{\partial p_i}, \\ \frac{dt}{dt} = \frac{\partial \bar{H}}{\partial p_0}, \\ \frac{dp_i}{dt} = -\frac{\partial \bar{H}}{\partial x^i}, \\ \frac{dp_0}{dt} = -\frac{\partial \bar{H}}{\partial t}. \end{array} \right.$$

The second equation imposes that \bar{H} fulfils $\frac{\partial \bar{H}}{\partial p_0} = 1$, thus it has the local form $\bar{H}(t, x^i, p_0, p_i) = p_0 + H_0(t, x^i, p_i)$. The equation $\bar{H}(t, x^i, p_0, p_i) = 0$ has the solution $p_0 = -H_0(t, x^i, p_i)$, thus the formula $(t, x^i, p_i) \xrightarrow{h} (t, x^i, p_i, -H_0(t, x^i, p_i))$ define an affine Hamiltonian on $\mathcal{T}M_I$. The above Hamilton equations become the Hamilton equations of h (see [13]).

3. A GENERALIZATION OF THE HIGHER TANGENT SPACES

In the general case, the t-tangent bundle $\mathcal{T}M_I$ is an affine bundle. We are going to define, for $k \geq 1$, the k -acceleration t-bundle $\mathcal{T}^k M_I$, as an affine bundle over $\mathcal{T}^{k-1} M_I$. In the case of a trivial time bundle of a manifold M , we find $\mathcal{T}^k M_I = I \times T^k M$, thus we recover the well known bundle of accelerations of order k .

Let us denote $M_I = \mathcal{T}^0 M_I$, $\pi_1 = \pi : \mathcal{T}M_I \rightarrow \mathcal{T}^0 M_I$ and $\mathcal{T}^1 M_I = \mathcal{T}M_I$ and consider the vector pseudofield $\Gamma^{(1)} = \frac{\partial}{\partial t} + y^i \frac{\partial}{\partial x^i}$ on the affine bundle $\mathcal{T}^1 M_I \xrightarrow{\pi} \mathcal{T}^0 M_I$. Let us suppose that the vector pseudofields $\Gamma^{(r)}$ on $T^r M$ and the r -acceleration bundles $\mathcal{T}^r M_I$ have been defined for $1 \leq r \leq k-1$, as affine bundles over $\mathcal{T}^{r-1} M_I$. Then $\mathcal{T}^k M_I$ is defined according to the change rules of the local coordinates given by formulas (3.2) below. The vector pseudofield $\Gamma^{(k)}$ on $\mathcal{T}^k M_I$ is defined by its action $\Gamma_U^{(k)} = \Gamma_U^{(k-1)} + ky^{(k)i} \frac{\partial}{\partial y^{(k-1)i}}$, where $\Gamma_U^{(k-1)}$ is considered as a (local) vector field on $T^k M$ and U is the domain which corresponds to the coordinates (x^i) . Notice that

$$\Gamma_U^{(k)} = \frac{\partial}{\partial t} + y^{(1)i} \frac{\partial}{\partial x^i} + 2y^{(2)i} \frac{\partial}{\partial y^{(1)i}} + \dots + ky^{(k)i} \frac{\partial}{\partial y^{(k-1)i}}.$$

On the intersection of two domains corresponding to U and U' , we have:

$$\Gamma_{U'}^{(k)} = \Gamma_U^{(k)} - \Gamma_U^{(k)}(y^{(k)i'}) \frac{\partial}{\partial y^{(k)i'}}. \quad (3.1)$$

Indeed, according to the recursive definitions of $\Gamma_U^{(r)}$ and $\Gamma_{U'}^{(r)}$ we have $\Gamma_{U'}^{(k)}(x^{i'}) = \Gamma_U^{(k)}(x^{i'})$, $\Gamma_{U'}^{(k)}(y^{(r)i'}) = \Gamma_U^{(k)}(y^{(r)i'})$, for $r = \overline{1, r-1}$ and $\Gamma_{U'}^{(k)}(y^{(k)i'}) = 0$, $(\forall) i' = \overline{1, m}$;

thus

$$\begin{aligned} \Gamma_U^{(k)} &= \Gamma_U^{(k)}(t') \frac{\partial}{\partial t'} + \Gamma_U^{(k)}(x^{i'}) \frac{\partial}{\partial x^{i'}} + \Gamma_U^{(k)}(y^{(1)i'}) \frac{\partial}{\partial y^{(k)i'}} + \dots + \Gamma_U^{(k)}(y^{(k-1)i'}) \frac{\partial}{\partial y^{(k-1)i'}} \\ &+ \Gamma_U^{(k)}(y^{(k)i'}) \frac{\partial}{\partial y^{(k)i'}} = \Gamma_{U'}^{(k)} + \Gamma_U^{(k)}(y^{(k)i'}) \frac{\partial}{\partial y^{(k)i'}}. \end{aligned}$$

Proposition 3.1. *The fibered manifold $(\mathcal{T}^k M_I, \pi_k, \mathcal{T}^{k-1} M_I)$ is an affine bundle, for $k \geq 1$.*

In the case of a trivial time bundle of a manifold M , we find that $\mathcal{T}^k M_I = I \times T^k M$, where $T^k M$ is the bundle of accelerations of order k of M , studied by R. Miron in [7]-[9].

The vector bundle canonically associated with the affine bundle $(\mathcal{T}^k M_I, \pi_k, \mathcal{T}^{k-1} M_I)$ is the vector bundle $p_{k-1}^* VTM_I$, where $p_{k-1}: \mathcal{T}^k M_I \rightarrow M_I$ is $p_{k-1} = \pi_1 \circ \pi_2 \circ \dots \circ \pi_{k-1}$ and $VTM_I \rightarrow M_I$ is the vertical vector bundle of the fibered manifold $M_I \rightarrow M$. The total space of the dual $p_{k-1}^* (VTM_I)^*$ of the vector bundle $p_{k-1}^* VTM_I$ is the total space of the fibered manifold $(\mathcal{T}^{k-1} M_I \times_{M_I} (VTM_I)^*, r_k, M_I)$. As in the time independent case, it can be used in the study of the dual geometrical objects of order k on M which are time dependent, in particular to define time dependent Hamiltonians of order k on M . In the sequel we denote $p_{k-1}^* (VTM_I)^* = \mathcal{T}^{k*} M_I$; it can be viewed as a vector bundle over $\mathcal{T}^{k-1} M_I$ as well as a fibered manifold over M_I .

Notice that the local coordinates $(y^{(k)i})$ change according to the rules:

$$ky^{(k)i'} = k \frac{\partial x^{i'}}{\partial x^i} y^{(k)i} + \Gamma_U^{(k-1)}(y^{(k-1)i'})$$

and the local coordinates $(t, x^i, y^{(1)i}, \dots, y^{(k)i})$ on $\mathcal{T}^k M_I$ change according to the rules:

$$\begin{aligned} t' &= t, \\ x^{i'} &= x^{i'}(t, x^i), \\ y^{(1)i'} &= \frac{\partial x^{i'}}{\partial t} + y^{(1)i} \frac{\partial x^{i'}}{\partial x^i}, \\ 2y^{(2)i'} &= \frac{\partial y^{(1)i'}}{\partial t} + y^{(1)i} \frac{\partial y^{(1)i'}}{\partial x^i} + 2y^{(2)i} \frac{\partial y^{(1)i'}}{\partial y^{(1)i}}, \\ &\vdots \\ ky^{(k)i'} &= \frac{\partial y^{(k-1)i'}}{\partial t} + y^{(1)i} \frac{\partial y^{(k-1)i'}}{\partial x^i} + \dots + ky^{(k)i} \frac{\partial y^{(k-1)i'}}{\partial y^{(k-1)i}}. \end{aligned} \tag{3.2}$$

The local vector field $\frac{\partial}{\partial t}$ changes according to the formula

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t'} + \frac{\partial x^{i'}}{\partial t} \frac{\partial}{\partial x^{i'}} + \frac{\partial y^{(1)i'}}{\partial t} \frac{\partial}{\partial y^{(1)i'}} + \cdots + \frac{\partial y^{(k)i'}}{\partial t} \frac{\partial}{\partial y^{(k)i'}}. \quad (3.3)$$

As in the time independent case, there is an affine morphism $\chi_k: \mathcal{T}^k M_I \rightarrow T\mathcal{T}^{k-1} M_I$, that it is injective on fibers. It has the local form

$$(t, x^i, y^{(1)i}, \dots, y^{(k)i}) \xrightarrow{h_k} \frac{\partial}{\partial t} + y^{(1)i} \frac{\partial}{\partial x^i} + 2y^{(2)i} \frac{\partial}{\partial y^{(1)i}} + \cdots + ky^{(k)i} \frac{\partial}{\partial y^{(k-1)i}};$$

notice that in the point $y^{(k-1)} \in \mathcal{T}^{k-1} M_I$ of coordinates $(t, x^i, y^{(1)i}, \dots, y^{(k-1)i})$, all the local vectors in the right side belong to $T_{y^{(k-1)}} \mathcal{T}^{k-1} M_I$.

Proposition 3.2. *The local definition of $\chi_k: \mathcal{T}^k M_I \rightarrow T\mathcal{T}^{k-1} M_I$ does not depend on coordinates. Then χ_k is an affine morphism of affines bundles over $\mathcal{T}^{k-1} M_I$, that is injective on fibers.*

The composition $\pi^k = \pi_0 \circ \cdots \circ \pi_k: \mathcal{T}^k M_I \rightarrow M_I$ defines a fibered manifold on $\mathcal{T}^k M_I$ with base M_I , that it is not an affine bundle. The vector bundle $V\mathcal{T}^k M_I = \ker \pi_*^k$ is the vertical bundle of $\mathcal{T}^k M_I$.

Let $s: \mathcal{T}^k M_I \rightarrow \mathcal{T}^{k+1} M_I$ be an affine section of the affine bundle $\pi_{k+1}: \mathcal{T}^{k+1} M_I \rightarrow \mathcal{T}^k M_I$. The composition $\mathcal{T}^k M_I \xrightarrow{s} T^{k+1} M \xrightarrow{\chi_{k+1}^k} \mathcal{T}^{k+1} M_I$, i.e. $S = \chi_{k+1} \circ s$ is a vector field on $\mathcal{T}^k M_I$, called a *k-t-semispray* on M . Since χ_{k+1} is injective on fibers, the following property is true.

Proposition 3.3. *There is a one to one correspondence between the sections of the affine bundle $\pi_{k+1}: \mathcal{T}^{k+1} M_I \rightarrow \mathcal{T}^k M_I$ and the k-t-semisprays on M .*

Notice that the local form of a *k-t-semispray* is

$$S = \frac{\partial}{\partial t} + y^{(1)i} \frac{\partial}{\partial x^i} + \cdots + ky^{(k)i} \frac{\partial}{\partial y^{(k-1)i}} + (k+1) S^i \frac{\partial}{\partial y^{(k)i}}$$

and the local functions $(S^i(x^i, y^{(1)i}, \dots, y^{(k)i}))$ change according to the rules:

$$\begin{aligned} (k+1) S^{i'} &= \frac{\partial y^{(k)i'}}{\partial t} + y^{(1)i} \frac{\partial y^{(k)i'}}{\partial x^i} + \cdots + ky^{(k)i} \frac{\partial y^{(k)i'}}{\partial y^{(k-1)i}} + (k+1) S^i \frac{\partial x^{i'}}{\partial x^i} \\ &= \Gamma_U^{(k)} \left(y^{(k)i'} \right) + (k+1) S^i \frac{\partial x^{i'}}{\partial x^i}. \end{aligned}$$

Let us consider the local vector field $\Delta = \frac{\partial}{\partial t} + y^{(1)i} \frac{\partial}{\partial y^{(1)i}} + \cdots + ky^{(k)i} \frac{\partial}{\partial y^{(k)i}}$.

Lemma 3.1. *The local vector fields Δ define a global vector field.*

Thus the local vector fields

$$y^{(1)i} \frac{\partial}{\partial y^{(1)i}} + \cdots + ky^{(k)i} \frac{\partial}{\partial y^{(k)i}} \text{ and } \frac{\partial}{\partial t} + y^{(1)i} \frac{\partial}{\partial y^{(1)i}} + \cdots + ky^{(k)i} \frac{\partial}{\partial y^{(k)i}}$$

have not in general global forms on $\mathcal{T}^k M_I$, thus no one of them can play the role of a Liouville vector field.

In the particular case when $M_I = M \times I$, the two vector fields have global forms, recovering the Liouville vector field of order k on M , used in the time independent case.

The endomorphism $J: T\mathcal{T}^k M_I \rightarrow T\mathcal{T}^k M_I$ locally defined by

$$\begin{aligned} J\left(\frac{\partial}{\partial x^i}\right) &= \frac{\partial}{\partial y^{(1)i}}, \quad J\left(\frac{\partial}{\partial y^{(1)i}}\right) = \frac{\partial}{\partial y^{(2)i}}, \dots, \quad J\left(\frac{\partial}{\partial y^{(k-1)i}}\right) = \frac{\partial}{\partial y^{(k)i}}, \\ J\left(\frac{\partial}{\partial y^{(k)i}}\right) &= 0, \quad J\left(\frac{\partial}{\partial t}\right) = -\left(y^{(1)i} \frac{\partial}{\partial y^{(1)i}} + \cdots + ky^{(k)i} \frac{\partial}{\partial y^{(k)i}}\right), \end{aligned}$$

has a global form. Formulas (3.3) and (3.2) show that $J\left(\frac{\partial}{\partial t}\right)$ is well defined. Notice that $J^{k+1} = 0$ and $J(\Gamma_U^{(k)}) = 0$. We call J the $(k+1)$ -tangent structure on $\mathcal{T}^k M_I$. The tensorial form of J is

$$\begin{aligned} J &= \left(-\Delta + \frac{\partial}{\partial t}\right) \otimes dt + \frac{\partial}{\partial y^{(1)i}} \otimes dx^i + 2\frac{\partial}{\partial y^{(2)i}} \otimes dy^{(1)i} + \cdots + k\frac{\partial}{\partial y^{(k)i}} \otimes dy^{(k-1)i} \\ &= \frac{\partial}{\partial y^{(1)i}} \otimes \left(dx^i - y^{(1)i} dt\right) + \frac{\partial}{\partial y^{(2)i}} \otimes \left(2dy^{(1)i} - y^{(2)i} dt\right) + \cdots \\ &\quad \cdots + \frac{\partial}{\partial y^{(k)i}} \otimes \left(kdy^{(k-1)i} - y^{(k)i} dt\right). \end{aligned}$$

where $\Delta = \frac{\partial}{\partial t} + y^{(1)i} \frac{\partial}{\partial y^{(1)i}} + \cdots + ky^{(k)i} \frac{\partial}{\partial y^{(k)i}}$.

The adjoint endomorphism $J^*: T\mathcal{T}^k M_I \rightarrow T\mathcal{T}^k M_I$ is locally given by

$$J^*\left(\frac{\partial}{\partial t}\right) = J^*(dx^i) = 0, \quad J^*(dy^{(1)i}) = dx^i - y^{(1)i} dt, \dots, \quad J^*(dy^{(k)i}) = dy^{(k-1)i} - y^{(k)i} dt.$$

For a (local) real function $f \in \mathcal{F}(\mathcal{T}^k M_I)$ we denote

$$\begin{aligned} \omega_J(f) &= \frac{\partial f}{\partial y^{(1)i}} \left(dx^i - y^{(1)i} dt\right) + \frac{\partial f}{\partial y^{(2)i}} \left(dy^{(1)i} - y^{(2)i} dt\right) + \cdots \\ &\quad \cdots + \frac{\partial f}{\partial y^{(k)i}} \left(dy^{(k-1)i} - y^{(k)i} dt\right). \end{aligned}$$

Since $J: T\mathcal{T}^k M_I \rightarrow T\mathcal{T}^k M_I$ is a global endomorphism, it follows that the local form $\omega_J(f)$ is well defined (i.e. its definition does not depend of coordinates).

In the sequel we study a class of Ehresmann connections on the fibered manifold $p_{k-1}: \mathcal{T}^k M_I \rightarrow M_I$, which we call non-linear k - t -connections on M_I . Notice that the vertical bundle is $V\mathcal{T}^k M_I = \ker (p_{k-1})_* \subset T\mathcal{T}^k M_I$.

A non-linear k - t -connection on M is given by

- a 1-dimensional distribution $H_0\mathcal{T}^k M_I \subset T\mathcal{T}^k M_I$ (called the t -horizontal distribution) that projects isomorphically on fibers by the composition of differentials $T\mathcal{T}^k M_I \xrightarrow{(p_{k-1})_*} TM_I \xrightarrow{\pi_*} T\mathbb{R}$, and
- an m -dimensional distribution $H_1\mathcal{T}^k M_I \subset T\mathcal{T}^k M_I$ (called the h -horizontal distribution) that projects isomorphically on the fibers of $VM_I = \ker \pi_*$, by the differential $\mathcal{T}^k M_I \xrightarrow{(p_{k-1})_*} TT M_I$.

The vector bundle $H_0\mathcal{T}^k M_I \oplus H_1\mathcal{T}^k M_I$ is denoted by $H\mathcal{T}^k M_I$ and it is called the *horizontal distribution*. Taking account into the dimension it follows that $T\mathcal{T}^k M_I = V\mathcal{T}^k M_I \oplus H\mathcal{T}^k M_I$ (Whitney sum), thus $H\mathcal{T}^k M_I$ is an Ehresmann connection.

Let us consider a local section $\frac{\delta}{\delta t}$ of $H_0\mathcal{T}^k M_I$ that projects on $\frac{\partial}{\partial t}$ and the local sections $\left\{ \frac{\delta}{\delta x^i} \right\}_{i=\overline{1,m}}$ of $H_1\mathcal{T}^k M_I \subset T\mathcal{T}^k M_I$ that project, element by element, on the local sections $\left\{ \frac{\partial}{\partial x^i} \right\}_{i=\overline{1,m}}$. These sections have the local forms:

$$\begin{aligned} \frac{\delta}{\delta t} &= \frac{\partial}{\partial t} - N_0^j \frac{\partial}{\partial x^j} - N_1^j \frac{\partial}{\partial y^{(1)j}} - \dots - N_k^j \frac{\partial}{\partial y^{(k)j}}, \\ \frac{\delta}{\delta x^i} &= \frac{\partial}{\partial x^i} - N_1^j \frac{\partial}{\partial y^{(1)j}} - \dots - N_k^j \frac{\partial}{\partial y^{(k)j}}. \end{aligned}$$

Taking account into the dimensions, the set of local sections $\left\{ \frac{\delta}{\delta t} \right\} \cup \left\{ \frac{\delta}{\delta x^i} \right\}_{i=\overline{1,m}}$ is a local base of the sections of $H\theta^{(r)}$. We say that this base is *adapted* if

- $\frac{\delta}{\delta t} = \frac{\delta}{\delta t'}$, i.e. $\frac{\delta}{\delta t}$ comes from a vector field on $\mathcal{T}^k M_I$ and
- on the intersection of two local domains on $E^{(r)}$, the system of vectors $\left\{ \frac{\delta}{\delta x^i} \right\}_{i=\overline{1,m}}$ change according to the rule

$$\frac{\delta}{\delta x^i} = \frac{\partial x^{i'}}{\partial x^i} \frac{\delta}{\delta x^{i'}}.$$

Theorem 3.1. *Every k - t -semispray defines at least one non-linear connection on $\mathcal{T}^k M_I$ that allows local adapted bases of sections.*

A regular Lagrangian on $\mathcal{T}^k M_I$ is a map $L: \mathcal{T}^k M_I \rightarrow \mathbb{R}$ such that its vertical Hessian $\left(\frac{\partial^2 L}{\partial y^{(k)i} \partial y^{(k)j}} \right)$ is non-singular. The next result shows that a regular Lagrangian gives rise to a k - t -semispray.

Proposition 3.4. *If L is a regular Lagrangian ω of order k , then there is a k - t -semispray defined by L .*

Notice that the k - t -semispray is given by the local functions

$$S^i = \frac{1}{k+1} h^{ij} \left(\Gamma^{(k)} \left(\frac{\partial L}{\partial y^{(k)j}} \right) - \frac{\partial L}{\partial y^{(k-1)j}} - \frac{\partial^2 L}{\partial t \partial y^{(k)j}} \right).$$

Finally, the above constructions give a new approach of the time dependent case.

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University of Craiova

Department of Applied Mathematics

13, *Al. I. Cuza st., Craiova, 200585, Romania*

E-mail address: marcelacpopescu@yahoo.com

University of Craiova

Department of Applied Mathematics

13, *Al. I. Cuza st., Craiova, 200585, Romania*

E-mail address: paul_p_popescu@yahoo.com