

MOND-WEIR DUALITIES WITH LAGRANGIANS FOR
MULTIOBJECTIVE FRACTIONAL AND NON-FRACTIONAL
VARIATIONAL PROBLEMS

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*Invited paper to celebrate Professor Constantin Udriște,
on the occasion of his seventies*

ABSTRACT. Based on the normal efficiency conditions for a multiobjective non-fractional variational problem, we consider fractional duals of generalized Mond-Weir type and fractional Mond-Weir-Zalmai type. Under some assumptions of (ρ, b) -quasiinvexity, weak, direct and converse duality theorems are stated in various variants.

INTRODUCTION

The first result on the necessity of optimal solutions of scalar variational problems was established by Valentine [19] in 1937. The papers of Mond and Hanson [10], Mond, Chandra and Husain [11], Mond and Husain [12], Preda [16] developed the duality of the scalar variational problems involving convex and generalized convex functions. Mititelu [3], Mititelu and Stancu-Minasian [9], Mukherjee and Purnachandra [13], established weak efficiency conditions and developed different types of dualities for multiobjective variational problems under various types of generalized convex functions. Kim and Kim [2] used the efficiency property of the nondifferentiable multiobjective variational problems in duality theory.

In this paper, we use the notion of normal efficient solution introduced by Mititelu [5] and establish certain new results of Mond-Weir duality type and of Zalmai duality type for multiobjective fractional and non-fractional variational problems using (ρ, b) -quasiinvexity assumptions.

For related but different results obtained by other authors on this topic, we address the reader to: [1] by R. Jagannathan, [14] and [15] by Ariana Pitea, C. Udriște and Șt. Mititelu, [18] by I. M. Stancu-Minasian and Șt. Mititelu.

1. NOTATIONS AND PROBLEMS STATEMENT

In \mathbb{R}^n , the n -dimensional Euclidean space, consider the vectors $v = (v_1, \dots, v_n)$ and $w = (w_1, \dots, w_n)$. We recall that the relations $v = w$, $v < w$, $v \leq w$, $v \leq w$ are

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defined as follows:

$$\begin{aligned} v = w &\Leftrightarrow v_i = w_i, & i = \overline{1, n}; & & v < w &\Leftrightarrow v_i < w_i, & i = \overline{1, n}; \\ v \leq w &\Leftrightarrow v_i \leq w_i, & i = \overline{1, n}; & & v \leq w &\Leftrightarrow v \leq w \text{ and } v \neq w. \end{aligned}$$

Let $I = [a, b]$ be a real interval and $f = (f_1, \dots, f_p): I \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^p$, $k = (k_1, \dots, k_p): I \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^p$, $g = (g_1, \dots, g_m): I \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^m$, $h = (h_1, \dots, h_q): I \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^q$ be two times differentiable functions.

Consider a vector-valued function $f(t, x, \dot{x})$, where $t \in I$, $x: I \rightarrow \mathbb{R}^n$ and $\dot{x} = \frac{dx}{dt}$. Denote by f_x and $f_{\dot{x}}$ the $p \times n$ matrices of first-order partial derivatives with respect to x and \dot{x} respectively, that is $f_x = (f_{1x}, f_{2x}, \dots, f_{px})'$ and $f_{\dot{x}} = (f_{1\dot{x}}, f_{2\dot{x}}, \dots, f_{p\dot{x}})'$, with

$$f_{ix} = \left(\frac{\partial f_i}{\partial x_1}, \dots, \frac{\partial f_i}{\partial x_n} \right) \quad \text{and} \quad f_{i\dot{x}} = \left(\frac{\partial f_i}{\partial \dot{x}_1}, \dots, \frac{\partial f_i}{\partial \dot{x}_n} \right), \quad i = 1, 2, \dots, p.$$

By analogy, k_x, g_x, h_x and $k_{\dot{x}}, g_{\dot{x}}, h_{\dot{x}}$ denote the $p \times n, n \times n, q \times n$ matrices of the first order partial derivatives of k, g and h respectively, with respect to x and \dot{x} .

Let X denote the space of piecewise smooth (continuously differentiable) functions x with the norm $\|x\| = \|x\|_\infty + \|Dx\|_\infty$, where the differential operator $D = \frac{d}{dt}$ is given by $u = Dx \Leftrightarrow x(t) = x(a) + \int_a^t u(s)ds$, excepting the discontinuities, where $x(a)$ is a given boundary value.

Consider the multiobjective fractional variational problem

$$(MFP) \left\{ \begin{array}{l} \text{Minimize} \left(\frac{\int_a^b f_1(t, x(t), \dot{x}(t))dt}{\int_a^b k_1(t, x(t), \dot{x}(t))dt}, \dots, \frac{\int_a^b f_p(t, x(t), \dot{x}(t))dt}{\int_a^b k_p(t, x(t), \dot{x}(t))dt} \right) \\ \text{subject to} \\ x(a) = a_0, \quad x(b) = b_0, \\ g(t, x(t), \dot{x}(t)) \leq 0, \quad h(t, x(t), \dot{x}(t)) = 0, \quad \forall t \in I. \end{array} \right.$$

Assume that $\int_a^b k_i(t, x(t), \dot{x}(t))dt > 0$ for all $i = 1, 2, \dots, p$. Let

$$\mathcal{D} = \{x \in X \mid x(a) = a_0, x(b) = b_0, g(t, x(t), \dot{x}(t)) \leq 0, h(t, x(t), \dot{x}(t)) = 0, \forall t \in I\}$$

be the set of all feasible solutions of problem (MFP).

Consider also the following multiobjective variational problem

$$(MP) \left\{ \begin{array}{l} \min \int_a^b f(t, x(t), \dot{x}(t))dt = \left(\int_a^b f_1(t, x(t), \dot{x}(t))dt, \dots, \int_a^b f_p(t, x(t), \dot{x}(t))dt \right) \\ \text{subject to} \\ x(a) = a_0, \quad x(b) = b_0, \\ g(t, x(t), \dot{x}(t)) \leq 0, \quad h(t, x(t), \dot{x}(t)) = 0, \quad t \in I. \end{array} \right.$$

and remark that the domain of (MP) is also \mathcal{D} .

In this paper, we define a duality of Mond-Weir-Zalmai type for problem (MFP) through weak, direct and converse duality theorems. For problem (MP) is developed a duality of generalized Mond-Weir type, also giving theorems of the same type. We give various variants of these results using assumptions of (ρ, b) -quasiinvexity.

2. PRELIMINARIES ON OPTIMALITY AND EFFICIENCY FOR VARIATIONAL PROBLEMS

In this section we recall some definitions and auxiliary results that will be needed later in our discussion of efficiency conditions and Mond-Weir duality for problem (MFP).

Definition 2.1. A feasible solution $x^0 \in \mathcal{D}$ is an *efficient solution* of problem (MP) if there is no $x \in \mathcal{D}$, $x \neq x^0$, such that

$$\int_a^b f(t, x(t), \dot{x}(t))dt \leq \int_a^b f(t, x^0(t), \dot{x}^0(t))dt.$$

For problem (MP) we quote the following result of efficiency

Theorem 2.1 (Necessary efficiency conditions for problem (MP)). ([6], [9]) *Let $x^0 \in \mathcal{D}$ be an efficient solution of problem (MP). Then there exist a vector $\lambda^0 \in \mathbb{R}^p$ and piecewise smooth functions $\mu^0: I \rightarrow \mathbb{R}^m$ and $\nu^0: I \rightarrow \mathbb{R}^q$ which satisfy the following conditions*

$$(MV) \left\{ \begin{array}{l} \lambda^0' f_x(t, x^0(t), \dot{x}^0(t)) + \mu^0(t)' g_x(t, x^0(t), \dot{x}^0(t)) + \nu^0(t)' h_x(t, x^0(t), \dot{x}^0(t)) \\ = \frac{d}{dt} [\lambda^0' f_{\dot{x}}(t, x^0(t), \dot{x}^0(t)) + \mu^0(t)' g_{\dot{x}}(t, x^0(t), \dot{x}^0(t)) \\ + \nu^0(t)' h_{\dot{x}}(t, x^0(t), \dot{x}^0(t))] \\ \mu^0(t)' g(t, x^0(t), \dot{x}^0(t)) = 0, \quad \mu_i(t) \geq 0, \quad \forall t \in I, \\ \lambda^0 \geq 0, \quad e' \lambda^0 = 1, \quad e = (1, \dots, 1)' \in \mathbb{R}^p \text{ if } \lambda^0 \geq 0. \end{array} \right.$$

Definition 2.2. [5] The point $x^0 \in \mathcal{D}$ is a *normal efficient solution* of problem (MP) if the vector λ^0 from the previous theorem satisfies the inequality $\lambda^0 \geq 0$.

We present now the efficiency necessary conditions for problem (MFP).

For each $i = \overline{1, p}$, we consider the functionals

$$F_i, K_i: X \rightarrow \mathbb{R}, \quad F_i(x) = \int_a^b f_i(t, x(t), \dot{x}(t))dt, \quad K_i(x) = \int_a^b k_i(t, x(t), \dot{x}(t))dt.$$

Mititelu established the following necessary efficiency conditions for (MFP):

Theorem 2.2 (Necessary efficiency conditions for problem (MFP)). ([6], [9]). *Let x^0 be a normal efficient solution of problem (MFP). Then there exist $\lambda^0 \in \mathbb{R}^n$ and*

piecewise smooth functions $\mu^0: I \rightarrow \mathbb{R}^q$ and $\nu^0: I \rightarrow \mathbb{R}^n$ that satisfy the conditions

$$(\text{MFV}) \left\{ \begin{array}{l} \sum_{i=1}^p \lambda_i^0 [K_i(x^0) f_{ix}(t, x^0(t), \dot{x}^0(t)) - F_i(x^0) k_{ix}(t, x^0(t), \dot{x}^0(t))] \\ + \mu^0(t)' g_x(t, x^0(t), \dot{x}^0(t)) + \nu^0(t)' h_x(t, x^0(t), \dot{x}^0(t)) \\ = \frac{d}{dt} \left\{ \sum_{i=1}^p \lambda_i^0 [K_i(x^0) f_{i\dot{x}}(t, x^0(t), \dot{x}^0(t)) - F_i(x^0) k_{i\dot{x}}(t, x^0(t), \dot{x}^0(t))] \right. \\ \left. + \mu^0(t)' g_{\dot{x}}(t, x^0(t), \dot{x}^0(t)) + \nu^0(t)' h_{\dot{x}}(t, x^0(t), \dot{x}^0(t)) \right\} \\ \mu^0(t)' g(t, x^0(t), \dot{x}^0(t)) = 0, \quad \mu^0(t) \geq 0, \quad \forall t \in I, \\ \lambda^0 \geq 0, \quad e' \lambda^0 = 1. \end{array} \right.$$

Let $\rho \in \mathbb{R}^n$ and $b: X \times X \rightarrow [0, \infty)$. Denote $H(x) = \int_a^b h(t, x(t), \dot{x}(t)) dt$.

Definition 2.3. ([6]) The function H is called (*strictly*) (ρ, b) -*quasinvex* at the point x^0 if there exist vector functions $\eta: I \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^q$, with $\eta(t, x(t), \dot{x}(t)) = 0$ for $x(t) = x^0(t)$, and $\theta: X \times X \rightarrow \mathbb{R}^n$ such that for any x ($x \neq x^0$),

$$H(x) \leq H(x^0) \Rightarrow b(x, x^0) \int_a^b [\eta' h_x(t, x^0(t), \dot{x}^0(t)) + (D\eta)' h_{\dot{x}}(t, x^0(t), \dot{x}^0(t))] dt \\ (<) \leq -\rho b(x, x^0) \|\theta(x, x^0)\|^2.$$

Definition 2.4. ([3]) The function H is *monotonic* (ρ, b) -*quasinvex* at the point x^0 if there exist vector functions $\eta: I \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^q$ with $\eta(t, x(t), \dot{x}(t)) = 0$ for $x(t) = x^0(t)$, and $\theta: X \times X \rightarrow \mathbb{R}^n$ such that for any x ($x \neq x^0$),

$$H(x) = H(x^0) \Rightarrow b(x, x^0) \int_a^b [\eta' h_x(t, x^0(t), \dot{x}^0(t)) + (D\eta)' h_{\dot{x}}(t, x^0(t), \dot{x}^0(t))] dt \\ = -\rho b(x, x^0) \|\theta(x, x^0)\|^2.$$

3. GENERALIZED MOND-WEIR DUALITY FOR (MP)

Let $\{J_0, J_1, \dots, J_r\}$ be a partition of the set $J = \{1, \dots, m\}$ that is $J_\alpha \subseteq J$, $J_\alpha \cap J_\beta = \emptyset$ if $\alpha \neq \beta$, $\bigcup_{\alpha=0}^r J_\alpha = J$. Let also $\{S_0, S_1, \dots, S_r\}$ be a partition of the set $S = \{1, \dots, q\}$. Consider a function $y \in X$ and the piecewise non-smooth functions $\mu: I \rightarrow \mathbb{R}^m$ and $\nu: I \rightarrow \mathbb{R}^q$. The Mond-Weir Lagrangian associated to problem (MP) is

$$L_e(t, y(t), \dot{y}(t)) = f(t, y(t), \dot{y}(t)) \\ + [\mu_{J_0}(t)' g_{J_0}(t, y(t), \dot{y}(t)) + \nu_{S_0}(t)' h_{S_0}(t, y(t), \dot{y}(t))] e,$$

where $L_e = (L_1, \dots, L_p)$ and

$$L_i(t, y(t), \dot{y}(t)) = f_i(t, y(t), \dot{y}(t)) + \mu_{J_0}(t)'g_{J_0}(t, y(t), \dot{y}(t)) + \nu_{S_0}(t)'h_{S_0}(t, y(t), \dot{y}(t)),$$

for $i = \overline{1, p}$.

The generalized multiobjective Mond-Weir dual problem associated to problem (MP) is the following variational problem:

$$(MWD) \left\{ \begin{array}{l} \text{Maximize (Pareto)} \int_a^b L_e(t, y(t), \dot{y}(t)) dt \\ = \left(\int_a^b L_1(t, y(t), \dot{y}(t)), \dots, \int_a^b L_p(t, y(t), \dot{y}(t)) dt \right) \\ \text{subject to} \\ y(a) = y_0, \quad y(b) = b_0, \\ \lambda' f_x(t, y(t), \dot{y}(t)) + \mu(t)'g_x(t, y(t), \dot{y}(t)) + \nu(t)'h_x(t, y(t), \dot{y}(t)) \\ = \frac{d}{dt} \{ \lambda' f_{\dot{x}}(t, y(t), \dot{y}(t)) + \mu(t)'g_{\dot{x}}(t, y(t), \dot{y}(t)) + \nu(t)'h_{\dot{x}}(t, y(t), \dot{y}(t)) \} \\ \mu_{J_\alpha}(t)'g_{J_\alpha}(t, y(t), \dot{y}(t)) + \nu_{S_\alpha}(t)'h_{S_\alpha}(t, y(t), \dot{y}(t)) \geq 0, \quad \alpha = \overline{1, r}, \quad \forall t \in I, \\ \lambda \geq 0, \quad e' \lambda = 1. \end{array} \right.$$

We denote by $\pi(x)$ the value of problem (MP) at $x \in \mathcal{D}$ and by $\delta(y, \lambda, \mu, \nu)$ the value of dual (MWD) at $(y, \lambda, \mu, \nu) \in \Delta$, where Δ is the domain of (MWD).

• CASE STUDY 1

Theorem 3.1 (Weak duality). *Let $x \in \mathcal{D}$ and $(y, \lambda, \mu, \nu) \in \Delta$ be feasible points of problems (MP) and (MWD). Assume that the following conditions are satisfied:*

a) *For each $i = \overline{1, p}$, the integral $\int_a^b L_i(t, x(t), \mu(t), \nu(t)) dt$ is (ρ'_i, b) -quasiinvex at the point y with respect to η and θ .*

b) *For each $\alpha = \overline{1, r}$, $\int_a^b [\mu_{J_\alpha}(t)'g_{J_\alpha}(t, x(t), \dot{x}(t)) + \nu_{K_\alpha}(t)'h_{K_\alpha}(t, x(t), \dot{x}(t))] dt$ is (ρ''_α, b) -quasiinvex at the point y with respect to η and θ .*

c) *At least one of the integrals from a)-b) is strictly (ρ, b) -quasiinvex at the point y with respect to η and θ .*

d) $\sum_{i=1}^p \lambda_i \rho'_i + \sum_{\alpha=1}^r \rho''_\alpha \geq 0$.

Then the inequality $\pi(x) \leq \delta(y, \lambda, \mu, \nu)$ is false.

Proof. By reductio ad absurdum, suppose there exist $x \in \mathcal{D}$ and $(y, \lambda, \mu, \nu) \in \Delta$ such that $\pi(x) \leq \delta(y, \lambda, \mu, \nu)$. Hence, for all $i = \overline{1, p}$, we have

$$\int_a^b f(t, x(t), \dot{x}(t)) dt \leq \int_a^b L_e(t, y(t), \dot{y}(t)) dt,$$

or componentwise

$$\int_a^b f_i(t, x(t), \dot{x}(t)) dt \leq \int_a^b [f_i(t, y(t), \dot{y}(t)) + \mu_{J_0}(t)' g_{J_0}(t, y(t), \dot{y}(t)) + \nu_{S_0}(t)' h_{S_0}(t, y(t), \dot{y}(t))] dt. \quad (3.1)$$

But $\mu_{J_0}(t)' g_{J_0}(t, x(t), \dot{x}(t)) \leq 0$ and $\nu_{S_0}(t)' h_{S_0}(t, x(t), \dot{x}(t)) = 0$, $\forall t \in I$, and from (3.1), we obtain

$$\int_a^b L_i(t, x(t), \dot{x}(t)) dt \leq \int_a^b L_i(t, y(t), \dot{y}(t)) dt, \quad i = \overline{1, p}. \quad (3.2)$$

According to hypothesis a), relation (3.2) implies

$$b(x, y) \int_a^b [\eta L_{ix}(t, y(t), \dot{y}(t)) + (D\eta)' L_{i\dot{x}}(t, y(t), \dot{y}(t))] dt \leq -\rho'_i b(x, y) \|\theta(x, y)\|^2. \quad (3.3)$$

Consider $\lambda = (\lambda_1, \dots, \lambda_p) \geq 0$ such that $e'\lambda = 1$. Multiplying (3.3) by λ_i and summing over $i = \overline{1, p}$, we obtain

$$\begin{aligned} b(x, y) \int_a^b [\eta(\lambda' L_x)(t, y(t), \dot{y}(t)) + (D\eta)'(\lambda' L_{\dot{x}})(t, y(t), \dot{y}(t))] dt \\ \leq -\left(\sum_{i=1}^p \lambda_i \rho'_i\right) b(x, y) \|\theta(x, y)\|^2. \end{aligned} \quad (3.4)$$

From the constraints of \mathcal{D} and Δ , we have

$$\begin{aligned} \int_a^b [\mu_{J_\alpha}(t)' g_{J_\alpha}(t, x(t), \dot{x}(t)) + \nu_{S_\alpha}(t)' h_{S_\alpha}(t, x(t), \dot{x}(t))] dt \\ \leq \int_a^b [\mu_{J_\alpha}(t)' g_{J_\alpha}(t, y(t), \dot{y}(t)) + \nu_{S_\alpha}(t)' h_{S_\alpha}(t, y(t), \dot{y}(t))] dt \end{aligned}$$

and according to b), we obtain

$$\begin{aligned} b(x, y) \int_a^b [\eta(\mu_{J_\alpha}(t)' (g_{J_\alpha})_y(t, y(t), \dot{y}(t)) + \nu_{S_\alpha}(t)' (h_{S_\alpha})_y(t, y(t), \dot{y}(t)))] dt \\ + b(x, y) \int_a^b (D\eta)' [\mu_{J_\alpha}(t)' (g_{J_\alpha})_{\dot{x}}(t, y(t), \dot{y}(t)) + \nu_{S_\alpha}(t)' (h_{S_\alpha})_{\dot{x}}(t, y(t), \dot{y}(t))] dt \\ \leq -\rho''_\alpha b(x, y) \|\theta(x, y)\|. \end{aligned} \quad (3.5)$$

Summing (3.4) and (3.5) side by side, we obtain

$$\begin{aligned} b(x, y) \int_a^b [\eta[\lambda' f_x(t, y(t), \dot{y}(t)) + \mu(t)' g_x(t, y(t), \dot{y}(t)) + \nu(t)' h_x(t, y(t), \dot{y}(t))] \\ + (D\eta)' [\lambda' f_{\dot{x}}(t, y(t), \dot{y}(t)) + \mu(t)' g_{\dot{x}}(t, y(t), \dot{y}(t)) + \nu(t)' h_{\dot{x}}(t, y(t), \dot{y}(t))]] dt \\ \leq \left(\sum_{i=1}^p \lambda_i \rho'_i + \sum_{\alpha=1}^r \rho''_\alpha\right) b(x, y) \|\theta(x, y)\|^2. \end{aligned} \quad (3.6)$$

Taking into account the hypothesis c) at inequality (3.6) it follows $b(x, y) > 0$ and consequently (3.6) becomes

$$\begin{aligned} & \int_a^b \left[\eta [\lambda' f_x(t, y(t), \dot{y}(t)) + \mu(t)' g_x(t, y(t), \dot{y}(t)) + \nu(t)' h_x(t, y(t), \dot{y}(t))] \right. \\ & \left. + (D\eta)' [\lambda' f_{\dot{x}}(t, y(t), \dot{y}(t)) + \mu(t)' g_{\dot{x}}(t, y(t), \dot{y}(t)) + \nu(t)' h_{\dot{x}}(t, y(t), \dot{y}(t))] \right] dt \\ & < \left(\sum_{i=1}^p \lambda_i \rho'_i + \sum_{\alpha=1}^r \rho''_{\alpha} \right) b(x, y) \|\theta(x, y)\|^2. \end{aligned} \tag{3.7}$$

Denoting $V = \lambda' f + \mu(t)' g + \nu(t)' h$ relation (3.7) becomes

$$\int_a^b [\eta' V_x + (D\eta)' V_{\dot{x}}] dt < -\|\theta(x, y)\|^2 \left(\sum_{i=1}^p \lambda_i \rho'_i + \sum_{\alpha=1}^r \rho''_{\alpha} \right). \tag{3.8}$$

Integrating by parts, the foregoing inequality becomes

$$\eta' V_{\dot{x}} \Big|_a^b + \int_a^b \eta' \left[V_x - \frac{d}{dt} V_{\dot{x}} \right] dt < -\|\theta(x, y)\|^2 \left(\sum_{i=1}^p \lambda_i \rho'_i + \sum_{\alpha=1}^r \rho''_{\alpha} \right). \tag{3.9}$$

But $\eta(a, y(a), \dot{y}(a)) = \eta(b, y(b), \dot{y}(b)) = 0$ and taking into account the first constraint of problem (MWD), relation (3.9) reduces to

$$0 < -\|\theta(x, y)\|^2 \left(\sum_{i=1}^p \lambda_i \rho'_i + \sum_{\alpha=1}^r \rho''_{\alpha} \right).$$

According to hypothesis d), this inequality becomes $0 < 0$, that is false. □

Theorem 3.2 (Direct duality). *Let x^0 be a normal efficient solution of the primal (MP) and suppose satisfied the hypotheses of Theorem 3.1 at the point x^0 . Then there are $\lambda^0 \in \mathbb{R}^p$ and the piecewise smooth functions $\mu^0: I \rightarrow \mathbb{R}^m$ and $\nu^0: I \rightarrow \mathbb{R}^q$ such that $(x^0, \lambda^0, \mu^0, \nu^0)$ is an efficient solution of the dual (MWD). Moreover, we have $\pi(x^0) = \delta(x^0, \lambda^0, \mu^0, \nu^0)$.*

Proof. Since the point x^0 is a normal efficient solution of problem (MP), according to Theorem 2.1 there are a vector $\lambda^0 \in \mathbb{R}^n$ and the piecewise non-smooth functions $\mu^0: I \rightarrow \mathbb{R}^m$ and $\nu^0: I \rightarrow \mathbb{R}^q$ which satisfy relations (MV). The vector relation $\mu^0(t)' g(t, x^0(t), \dot{x}^0(t)) = 0$ from (MV) is equivalent to $\mu_j^0(t)' g_j(t, x^0(t), \dot{x}^0(t)) = 0$, $j = \overline{1, m}$. It follows $\mu_{J_{\alpha}}(t)' g_{J_{\alpha}}(t, x^0(t), \dot{x}^0(t)) = 0$ for $\alpha = \overline{1, r}$. Also, we remark that $\nu_{S_{\alpha}}^0(t)' h_{S_{\alpha}}(t, x^0(t), \dot{x}^0(t)) = 0$, for $\alpha = \overline{1, r}$. Therefore, we obtain

$$\mu_{J_{\alpha}}(t)' g_{J_{\alpha}}(t, x^0(t), \dot{x}^0(t)) + \nu_{S_{\alpha}}^0(t)' h_{S_{\alpha}}(t, x^0(t), \dot{x}^0(t)) = 0 \text{ for } \alpha = \overline{1, r}$$

and then, we get $(x^0, \lambda^0, \mu^0, \nu^0) \in \Delta$. Moreover,

$$\mu_{J_0}(t)' g_{J_0}(t, x^0(t), \dot{x}^0(t)) + \nu_{S_0}^0(t)' h_{S_0}(t, x^0(t), \dot{x}^0(t)) = 0$$

and consequently, $\pi(x^0) = \delta(x^0, \lambda^0, \mu^0, \nu^0) [= f(t, x^0(t), \dot{x}^0(t))]$. □

Theorem 3.3 (Converse duality). *Let $(x^0, \lambda^0, \mu^0, \nu^0)$ be an efficient solution of the dual (MWD). Suppose that the following conditions are satisfied:*

i) $x^0 \in \mathcal{D}$.

a) *For each $i = \overline{1, p}$, the functional $\int_a^b f_i(t, x(t), \dot{x}(t))dt$ is (ρ'_i, b) -quasiinvex at the point x^0 with respect to η and θ .*

b) *For each $\alpha = \overline{1, r}$, $\int_a^b [\mu_{J_\alpha}^0(t)'g_{J_\alpha}(t, x(t), \dot{x}(t)) + \nu_{S_\alpha}^0(t)'h_{S_\alpha}(t, x(t), \dot{x}(t))]dt$ is (ρ''_α, b) -quasiinvex at the point x^0 with respect to η and θ .*

c) *At least one of the integrals from a)-b) is strictly (ρ, b) -quasiinvex at the point x^0 with respect to η and θ .*

d) $\sum_{i=1}^p \lambda_i^0 \rho'_i + \sum_{\alpha=1}^r \rho''_\alpha \geq 0$.

Then x^0 is an efficient solution of (MP). Moreover, $\pi(x^0) = \delta(x^0, \lambda^0, \mu^0, \nu^0)$.

Proof. On the contrary, suppose that the point x^0 is not an efficient solution of problem (MP) and we will find a contradiction. Then, there exists $x \in \mathcal{D}$ such that $(f_1, \dots, f_p) \leq (L_1, \dots, L_p)$. Repeating the proof of Theorem 3.1, with $(x^0, \lambda^0, \mu^0, \nu^0)$ instead of (y, λ, μ, ν) , we obtain

$$0 < -\|\theta(x^0, y)\|^2 \left(\sum_{i=1}^p \lambda_i^0 \rho'_i + \sum_{\alpha=1}^r \rho''_\alpha \right),$$

that is $0 < 0$. It means that the above made supposition is false. Therefore x^0 is an efficient solution of problem (MP), and we obtain $\pi(x^0) = \delta(x^0, \lambda^0, \mu^0, \nu^0)$. \square

• CASE STUDY 2

Theorem 3.4 (Weak duality). *Let $x \in \mathcal{D}$ and $(y, \lambda, \mu, \nu) \in \Delta$ be feasible points of problems (MP) and (MWD). Assume satisfied the following conditions:*

a) *For each $i = \overline{1, p}$, the functional $\int_a^b L_i(t, x(t), \dot{x}(t))dt$ is (ρ'_i, b) -quasiinvex at the point y with respect to η and θ .*

b) *For each $\alpha = \overline{1, r}$, $\int_a^b \mu_{J_\alpha}(t)'g_{J_\alpha}(t, x(t), \dot{x}(t))dt$ is (ρ''_α, b) -quasiinvex at the point y with respect to η and θ .*

c) *For each $\alpha = \overline{1, r}$, the functional $\int_a^b \nu_{K_\alpha}(t)'h_{K_\alpha}(t, x(t), \dot{x}(t))dt$ is monotonic (ρ'''_α, b) -quasiinvex at the point y with respect to η and θ .*

d) *At least one of the integrals from a)-c) is strictly (ρ, b) -quasiinvex at the point y with respect to η and θ .*

e) $\sum_{i=1}^p \lambda_i \rho'_i + \sum_{\alpha=1}^r (\rho''_\alpha + \rho'''_\alpha) \geq 0$.

Then the inequality $\pi(x) \leq \delta(y, \lambda, \mu, \nu)$ is false.

Theorem 3.5 (Direct duality). *Let x^0 be a normal efficient solution of the primal (MP) and suppose satisfied the hypotheses of Theorem 3.4 at the point x^0 . Then there are $\lambda^0 \in \mathbb{R}^p$ and the piecewise smooth functions $\mu^0: I \rightarrow \mathbb{R}^m$ and $\nu^0: I \rightarrow \mathbb{R}^q$ such that $(x^0, \lambda^0, \mu^0, \nu^0)$ is an efficient solution of the dual (MWD) and moreover, $\pi(x^0) = \delta(x^0, \lambda^0, \mu^0, \nu^0)$.*

Theorem 3.6 (Converse duality). *Let $(x^0, \lambda^0, \mu^0, \nu^0)$ be an efficient solution of the dual (MD) and suppose satisfied the following conditions:*

- i) $x^0 \in \mathcal{D}$.
- ii) *The hypotheses a)-e) of Theorem 3.4 hold for $(y, \lambda, \mu, \nu) = (x^0, \lambda^0, \mu^0, \nu^0)$. Then x^0 is an efficient solution of (MP). Moreover, $\pi(x^0) = \delta(x^0, \lambda^0, \mu^0, \nu^0)$.*

• CASE STUDY 3

Theorem 3.7 (Weak duality). *Let $x \in \mathcal{D}$ and $(y, \lambda, \mu, \nu) \in \Delta$ be feasible points of problems (MP) and (MWD). Assume satisfied the following conditions:*

- a) *The functional $\int_a^b \lambda' f_x(t, x(t), \dot{x}(t)) dt$ is (ρ, b) -quasiinvex at the point y with respect to η and θ .*
- b) *For each $\alpha = \overline{1, r}$, $\int_a^b \mu_{J_\alpha}(t)' g_{J_\alpha}(t, x(t), \dot{x}(t)) dt$ is (ρ'_α, b) -quasiinvex at the point y with respect to η and θ .*
- c) *For each $\alpha = \overline{1, r}$, the functional $\int_a^b \nu_{K_\alpha}(t)' h_{K_\alpha}(t, x(t), \dot{x}(t)) dt$ is monotonic (ρ''_α, b) -quasiinvex at the point y with respect to η and θ .*
- d) *At least one of the integrals from a)-c) is strictly (ρ, b) -quasiinvex at the point y with respect to η and θ .*
- e) $\rho + \sum_{\alpha=1}^r (\rho'_\alpha + \rho''_\alpha) \geq 0, \quad \rho \in \mathbb{R}^p.$

Then the inequality $\pi(x) \leq \delta(t, \lambda, \mu, \nu)$ is false.

Theorem 3.8 (Direct duality). *Let x^0 be a normal efficient solution of the primal (MP) and suppose satisfied the hypotheses of Theorem 3.7 at the point x^0 . Then there are $\lambda^0 \in \mathbb{R}^p$ and the piecewise smooth functions $\mu^0: I \rightarrow \mathbb{R}^m$ and $\nu^0: I \rightarrow \mathbb{R}^q$ such that $(x^0, \lambda^0, \mu^0, \nu^0)$ is an efficient solution of the dual (MWD) and moreover, $\pi(x^0) = \delta(x^0, \lambda^0, \mu^0, \nu^0)$.*

Theorem 3.9 (Converse duality). *Let $(x^0, \lambda^0, \mu^0, \nu^0)$ be an efficient solution of the dual (MWD) and suppose satisfied the following conditions:*

- i) $x^0 \in \mathcal{D}$.
- ii) *The hypotheses a)-e) of Theorem 3.7 hold for $(y, \lambda, \mu, \nu) = (x^0, \lambda^0, \mu^0, \nu^0)$. Then x^0 is an efficient solution of (MP) and moreover, $\pi(x^0) = \delta(x^0, \lambda^0, \mu^0, \nu^0)$.*

• CASE STUDY 4

Theorem 3.10 (Weak duality). *Let $x \in \mathcal{D}$ and $(y, \lambda, \mu, \nu) \in \Delta$ be feasible points of problems (MP) and (MWD). Assume satisfied the following conditions:*

a) *The functional $\int_a^b \lambda' f_x(t, x(t), \dot{x}(t)) dt$ is (ρ, b) -quasiinvex at the point y with respect to η and θ .*

b) *For each $\alpha = \overline{1, r}$, $\int_a^b [\mu_{J_\alpha}(t)' g_{J_\alpha}(t, x(t), \dot{x}(t)) + \nu_{K_\alpha}(t)' h_{K_\alpha}(t, x(t), \dot{x}(t))] dt$ is (ρ'_α, b) -quasiinvex at the point y with respect to η and θ .*

c) *At least one of the integrals from a)-b) is strictly (ρ, b) -quasiinvex at the point y with respect to η and θ .*

$$\text{d) } \rho + \sum_{\alpha=1}^r \rho'_\alpha \geq 0, \quad \rho \in \mathbb{R}^p.$$

Then the relation $\pi(x) \leq \delta(t, \lambda, \mu, \nu)$ is false.

Theorem 3.11 (Direct duality). *Let x^0 be a normal efficient solution of primal (MP) and suppose satisfied the hypotheses of Theorem 3.10 at the point x^0 . Then there are $\lambda^0 \in \mathbb{R}^p$ and the piecewise smooth functions $\mu^0: I \rightarrow \mathbb{R}^m$ and $\nu^0: I \rightarrow \mathbb{R}^q$ such that $(x^0, \lambda^0, \mu^0, \nu^0)$ is an efficient solution of the dual (MWD) and moreover, $\pi(x^0) = \delta(x^0, \lambda^0, \mu^0, \nu^0)$.*

Theorem 3.12 (Converse duality). *Let $(x^0, \lambda^0, \mu^0, \nu^0)$ be an efficient solution of dual (MWD) and suppose satisfied the following conditions:*

i) $x^0 \in \mathcal{D}$.

ii) *The hypotheses a)-d) of Theorem 3.10 hold for $(y, \lambda, \mu, \nu) = (x^0, \lambda^0, \mu^0, \nu^0)$.*

Then the point x^0 is an efficient solution of (MP), and $\pi(x^0) = \delta(x^0, \lambda^0, \mu^0, \nu^0)$.

4. FRACTIONAL DUAL MOND-WEIR-ZALMAI WITH LAGRANGIANS FOR PROBLEM (MFP)

Let $\{J_0, J_1, \dots, J_r\}$ be a partition of $\{1, \dots, m\}$ and $\{S_0, S_1, \dots, S_r\}$ a partition of $\{1, \dots, q\}$. We denote

$$F_i(y) = \int_a^b f_i(t, y(t), \dot{y}(t)) dt, \quad K_i(y) = \int_a^b k_i(t, y(t), \dot{y}(t)) dt,$$

$$\bar{F}_i(t, y(t), \dot{y}(t)) = f_i(t, y(t), \dot{y}(t)) + \mu_{J_0}(t)' g_{J_0}(t, y(t), \dot{y}(t)) + \nu_{S_0}(t)' h_{S_0}(t, y(t), \dot{y}(t)).$$

Consider a function $y \in X$. We associate to problem (MFP) the following multiobjective variational problem:

$$\text{(MFZD)} \left\{ \begin{array}{l}
 \text{Maximize } \left(\frac{\int_a^b \bar{F}_1(t, y(t), \dot{y}(t)) dt}{\int_a^b k_1(t, y(t), \dot{y}(t)) dt}, \dots, \frac{\int_a^b \bar{F}_p(t, y(t), \dot{y}(t)) dt}{\int_a^b k_p(t, y(t), \dot{y}(t)) dt} \right) \\
 \text{subject to } y(a) = y_0, \quad y(b) = b_0, \\
 \sum_{i=1}^p \lambda_i [K_i(y) f_{ix}(t, y(t), \dot{y}(t)) - F_i(y) k_{ix}(t, y(t), \dot{y}(t))] \\
 \quad + \mu(t)' g_x(t, y(t), \dot{y}(t)) + \nu(t)' h_x(t, y(t), \dot{y}(t)) = \\
 = \frac{d}{dt} \left\{ \sum_{i=1}^p \lambda_i [K_i(y) f_{i\dot{x}}(t, y(t), \dot{y}(t)) - F_i(y) k_{i\dot{x}}(t, y(t), \dot{y}(t))] \right. \\
 \quad \left. + \mu(t)' g_{\dot{x}}(t, y(t), \dot{y}(t)) + \nu(t)' h_{\dot{x}}(t, y(t), \dot{y}(t)) \right\} \\
 \mu_{J_\alpha}(t)' g_{J_\alpha}(t, y(t), \dot{y}(t)) + \nu_{S_\alpha}(t) h_{S_\alpha}(t, y(t), \dot{y}(t)) \geq 0, \quad \alpha = \overline{1, r}, \quad \forall t \in I \\
 \lambda \geq 0, \quad e' \lambda = 1
 \end{array} \right.$$

We denote by $\pi(x)$ the value of problem (MFP) at the point $x \in \mathcal{D}$ and by $\delta(y, \lambda, \mu, \nu)$ the value of dual (MFZD) at the point $(y, \lambda, \mu, \nu) \in \Delta$, where Δ is the domain of (MFZD).

• CASE STUDY 1

Theorem 4.1 (Weak duality). *Let $x \in \mathcal{D}$ and $(y, \lambda, \mu, \nu) \in \Delta$ be feasible points of problems (MFP) and (MFZD). Assume that the following conditions are satisfied:*

a) $K_i(x) [\mu_{J_0}(t)' g_{J_0}(t, y(t), \dot{y}(t)) + \nu_{S_0}(t)' h_{S_0}(t, y(t), \dot{y}(t))] \leq 0, \forall x \in \mathcal{D}, \forall t \in I$, for $i = \overline{1, p}$.

b) For each $i = \overline{1, p}$, the integral $\int_a^b [K_i(y) f_i(t, x(t), \dot{x}(t)) - F_i(y) k_i(t, x(t), \dot{x}(t))] dt$ is (ρ'_i, b) -quasiinvex at the point y with respect to η and θ .

c) For each $\alpha = \overline{1, r}$, $\int_a^b [\mu_{J_\alpha}(t)' g_{J_\alpha}(t, y(t), \dot{y}(t)) + \nu_{K_\alpha}(t)' h_{K_\alpha}(t, y(t), \dot{y}(t))] dt$ is (ρ''_i, b) -quasiinvex at the point y with respect to η and θ .

d) At least one of the integrals from a)-b) is strictly (ρ, b) -quasiinvex at the point y with respect to η and θ .

e) $\sum_{i=1}^p \lambda_i \rho'_i + \sum_{\alpha=1}^r \rho''_\alpha \geq 0$.

Then the inequality $\pi(x) \leq \delta(y, \lambda, \mu, \nu)$ is false.

Proof. By reductio ad absurdum, we have

$$\left(\frac{\int_a^b f_1(t, x(t), \dot{x}(t)) dt}{\int_a^b k_1(t, x(t), \dot{x}(t)) dt}, \dots, \frac{\int_a^b f_p(t, x(t), \dot{x}(t)) dt}{\int_a^b k_p(t, x(t), \dot{x}(t)) dt} \right) \\ \leq \left(\frac{\int_a^b \bar{F}_1(t, x(t), \dot{x}(t)) dt}{\int_a^b k_1(t, x(t), \dot{x}(t)) dt}, \dots, \frac{\int_a^b \bar{F}_p(t, x(t), \dot{x}(t)) dt}{\int_a^b k_p(t, x(t), \dot{x}(t)) dt} \right)$$

or componentwise,

$$\frac{\int_a^b f_i(t, x(t), \dot{x}(t)) dt}{\int_a^b k_i(t, x(t), \dot{x}(t)) dt} \leq \frac{\int_a^b \bar{F}_i(t, y(t), \dot{y}(t)) dt}{\int_a^b k_i(t, y(t), \dot{y}(t)) dt} = \\ = \frac{\int_a^b [f_i(t, y(t), \dot{y}(t)) + \mu_{J_0}(t)g_{J_0}(t, y(t), \dot{y}(t)) + \nu_{S_0}(t)h_{S_0}(t, y(t), \dot{y}(t))] dt}{\int_a^b k_i(t, y(t), \dot{y}(t)) dt}$$

for $i = \overline{1, p}$. From this inequality it follows

$$\int_a^b K_i(y) f_i(t, x(t), \dot{x}(t)) dt \leq \int_a^b F_i(y) k_i(t, x(t), \dot{x}(t)) dt \\ + \int_a^b K_i(x) [\mu_{J_0}(t)g_{J_0}(t, y(t), \dot{y}(t)) + \nu_{S_0}(t)h_{S_0}(t, y(t), \dot{y}(t))] dt$$

and taking into account the hypothesis a) we get

$$\int_a^b K_i(y) f_i(t, x(t), \dot{x}(t)) dt \leq \int_a^b F_i(y) k_i(t, x(t), \dot{x}(t)) dt.$$

We have

$$\int_a^b [K_i(y) f_i(t, x(t), \dot{x}(t)) - F_i(y) k_i(t, x(t), \dot{x}(t))] dt \leq 0 \\ = \int_a^b [K_i(y) f_i(t, y(t), \dot{y}(t)) - F_i(y) k_i(t, y(t), \dot{y}(t))] dt \quad (4.1)$$

or $F_i(x)K_i(y) - K_i(x)F_i(y) \leq 0$.

According to b), relation (4.1) implies

$$\begin{aligned}
 b(x, y) \int_a^b & \left\{ \eta' [K_i(y) f_{ix}(t, y(t), \dot{y}(t)) - F_i(y) k_{ix}(t, y(t), \dot{y}(t))] \right. \\
 & \left. + (D\eta)' [K_i(y) f_{i\dot{x}}(t, y(t), \dot{y}(t)) - F_i(y) k_{i\dot{x}}(t, y(t), \dot{y}(t))] \right\} dt \\
 & \leq -\rho'_i b(x, y) \|\theta(x, y)\|.
 \end{aligned} \tag{4.2}$$

Multiplying (4.1) and (4.2) by λ_i , $i = \overline{1, p}$ ($\lambda \geq 0$) and summing over $i = \overline{1, p}$, we obtain

$$\begin{aligned}
 \sum_{i=1}^p \lambda_i [F_i(x) K_i(y) - K_i(x) F_i(y)] & \leq 0 \Rightarrow \\
 b(x, y) \int_a^b & \left\{ \eta' \sum_{i=1}^p \lambda_i [K_i(y) f_{ix}(t, y(t), \dot{y}(t)) - F_i(y) k_{ix}(t, y(t), \dot{y}(t))] \right. \\
 & \left. + (D\eta)' \sum_{i=1}^p \lambda_i [K_i(y) f_{i\dot{x}}(t, y(t), \dot{y}(t)) - F_i(y) k_{i\dot{x}}(t, y(t), \dot{y}(t))] \right\} dt \\
 & \leq -b(x, y) \|\theta(x, y)\|^2 \sum_{i=1}^p \lambda_i \rho'_i.
 \end{aligned} \tag{4.3}$$

According to c), we have

$$\begin{aligned}
 & \int_a^b [\mu_{J_\alpha}(t)' g_{J_\alpha}(t, x(t), \dot{x}(t)) + \nu_{S_\alpha}(t)' h_{S_\alpha}(t, x(t), \dot{x}(t))] dt \\
 & \leq \int_a^b [\mu_{J_\alpha}(t)' g_{J_\alpha}(t, y(t), \dot{y}(t)) + \nu_{S_\alpha}(t)' h_{S_\alpha}(t, y(t), \dot{y}(t))] dt \\
 \Rightarrow b(x, y) \int_a^b & \left\{ \eta' [\mu_{J_\alpha}(t)' (g_{J_\alpha})_x(t, y(t), \dot{y}(t)) + \nu_{S_\alpha}(t)' (h_{S_\alpha})_x(t, y(t), \dot{y}(t))] \right. \\
 & \left. + (D\eta)' [\mu_{J_\alpha}(t)' (g_{J_\alpha})_{\dot{x}}(t, y(t), \dot{y}(t)) + \nu_{S_\alpha}(t)' (h_{S_\alpha})_{\dot{x}}(t, y(t), \dot{y}(t))] \right\} dt \\
 & \leq -\rho''_\alpha b(x, y) \|\theta(x, y)\|^2.
 \end{aligned} \tag{4.4}$$

Summing side by side over $\alpha = \overline{1, r}$ the double implication (4.4), we obtain

$$\begin{aligned}
 & \int_a^b [\mu(t)' g(t, x(t), \dot{x}(t)) + \nu(t)' h(t, x(t), \dot{x}(t))] dt \\
 & - \int_a^b [\mu(t)' g(t, y(t), \dot{y}(t)) + \nu(t)' h(t, y(t), \dot{y}(t))] dt \leq 0 \Rightarrow
 \end{aligned}$$

$$\begin{aligned}
\Rightarrow \quad & b(x, y) \int_a^b \left\{ \eta' [\mu(t)'g_x(t, y(t), \dot{y}(t)) + \nu(t)'h_x(t, y(t), \dot{y}(t))] \right. \\
& \quad \left. + (D\eta)' [\mu(t)'g_{\dot{x}}(t, y(t), \dot{y}(t)) + \nu(t)'h_{\dot{x}}(t, y(t), \dot{y}(t))] \right\} dt \\
& \leq -b(x, y) \|\theta(x, y)\|^2 \sum_{\alpha=1}^r \rho''_{\alpha}. \tag{4.5}
\end{aligned}$$

Summing now side by side the double implications (4.3) and (4.5) and taking into account c) it follows

$$\begin{aligned}
& \sum_{i=1}^p \lambda_i [F_i(x)K_i(y) - K_i(x)F_i(y)] + \int_a^b [\mu(t)'g(t, x(t), \dot{x}(t)) + \nu(t)'h(t, x(t), \dot{x}(t))] dt \\
& \quad - \int_a^b [\mu(t)'g(t, y(t), \dot{y}(t)) + \nu(t)'h(t, y(t), \dot{y}(t))] dt \leq 0 \\
\Rightarrow \quad & b(x, y) \int_a^b \left\{ \eta' \left[\sum_{i=1}^p \lambda_i [K_i(y)f_{ix}(t, y(t), \dot{y}(t)) - F_i(y)k_{ix}(t, y(t), \dot{y}(t))] \right. \right. \\
& \quad \left. \left. + \mu(t)'g_x(t, y(t), \dot{y}(t)) + \nu(t)'h_x(t, y(t), \dot{y}(t))] \right] \right. \\
& \quad \left. + (D\eta)' \left[\sum_{i=1}^p \lambda_i [K_i(y)f_{i\dot{x}}(t, y(t), \dot{y}(t)) - F_i(y)k_{i\dot{x}}(t, y(t), \dot{y}(t))] \right. \right. \\
& \quad \left. \left. + \mu(t)'g_{\dot{x}}(t, y(t), \dot{y}(t)) + \nu(t)'h_{\dot{x}}(t, y(t), \dot{y}(t))] \right] \right\} dt \\
& < -b(x, y) \|\theta(x, y)\|^2 \left(\sum_{i=1}^p \lambda_i \rho'_i + \sum_{\alpha=1}^r \rho''_{\alpha} \right). \tag{4.6}
\end{aligned}$$

By the second implication of (4.6) we get $b(x, y) > 0$. Then, the second implication of (4.6), shortly, becomes

$$\int_a^b [\eta' V_x + (D\eta)' V_{\dot{x}}] dt < -\|\theta(x, y)\|^2 \left(\sum_{i=1}^p \lambda_i \rho'_i + \sum_{\alpha=1}^r \rho''_{\alpha} \right), \tag{4.7}$$

where we denoted

$$\begin{aligned}
V &= \sum_{i=1}^p \lambda_i [K_i(y)f_i(t, y(t), \dot{y}(t)) - F_i(y)k_i(t, y(t), \dot{y}(t))] \\
& \quad + \mu(t)'g(t, y(t), \dot{y}(t)) + \nu(t)'h(t, y(t), \dot{y}(t)).
\end{aligned}$$

Using an integration by parts, the previous inequality becomes

$$\eta' V_{\dot{x}} \Big|_a^b + \int_a^b \eta' \left[V_x - \frac{d}{dt} V_{\dot{x}} \right] dt < -\|\theta(x, y)\|^2 \left(\sum_{i=1}^p \lambda_i \rho'_i + \sum_{\alpha=1}^r \rho''_{\alpha} \right) \tag{4.8}$$

But $\eta(a, y(a), \dot{y}(a)) = \eta(b, y(b), \dot{y}(b)) = 0$ and taking into account the first constraint of problem (MFZD), relation (4.8) reduces to

$$0 < -\|\theta(x, y)\|^2 \left(\sum_{i=1}^p \lambda_i \rho'_i + \sum_{\alpha=1}^r \rho''_{\alpha} \right).$$

Taking into account now the hypothesis d), this inequality becomes $0 < 0$, that is false. Consequently, the above made supposition is false. \square

Theorem 4.2 (Direct duality). *Let x^0 be a normal efficient solution of primal (MFP) and suppose satisfied the hypotheses of Theorem 4.1 at x^0 . Then there are $\lambda^0 \in \mathbb{R}^p$ and the piecewise smooth functions $\mu^0: I \rightarrow \mathbb{R}^m$ and $\nu^0: I \rightarrow \mathbb{R}^r$ such that $(x^0, \lambda^0, \mu^0, \nu^0)$ is an efficient solution of dual (MFZD) and moreover, $\pi(x^0) = \delta(x^0, \lambda^0, \mu^0, \nu^0)$.*

Proof. Because x^0 is a normal efficient solution of problem (MFP), according to Theorem 2.2 there are a vector $\lambda^0 \in \mathbb{R}^n$ and the piecewise non-smooth functions $\mu^0: I \rightarrow \mathbb{R}^m$ and $\nu^0: I \rightarrow \mathbb{R}^q$ which satisfy relations (MFV).

Vector relation $\mu^0(t)'g(t, x^0(t), \dot{x}^0(t)) = 0$ is equivalent to relations

$$\mu_j^0(t)'g_j(t, x^0(t), \dot{x}^0(t)) = 0, \quad j = \overline{1, m},$$

from which it results $\mu_{J_{\alpha}}^0(t)'g_{J_{\alpha}}(t, x^0(t), \dot{x}^0(t)) = 0, \quad \alpha = \overline{0, r}$.

Also $\nu_{S_{\alpha}}^0(t)'h_{S_{\alpha}}(t, x^0, \dot{x}^0) = 0, \alpha = \overline{0, r}$. Then we have

$$\mu_{J_{\alpha}}^0(t)'g_{J_{\alpha}}(t, x^0(t), \dot{x}^0(t)) + \nu_{S_{\alpha}}^0(t)'h_{S_{\alpha}}(t, x^0(t), \dot{x}^0(t)) = 0, \quad \alpha = \overline{0, r},$$

that is $(x^0, \lambda^0, \mu^0, \nu^0) \in \Delta$ and in addition, $\pi(x^0) = \delta(x^0, \lambda^0, \mu^0, \nu^0)$. \square

Theorem 4.3 (Converse duality). *Let $(x^0, \lambda^0, \mu^0, \nu^0)$ be an efficient solution to dual (MFZD) and suppose satisfied the following conditions:*

- i) $x^0 \in \mathcal{D}$.
- a) $K_i(x^0) [\mu_{J_0}(t)'g_{J_0}(t, y(t), \dot{y}(t)) + \nu_{S_0}(t)'h_{S_0}(t, y(t), \dot{y}(t))] \leq 0, \forall x \in \mathcal{D}, \forall t \in I, i = \overline{1, p}$.

b) For each $i = \overline{1, p}, \int_a^b [K_i(x^0)f_i(t, x(t), \dot{x}(t)) - F_i(x^0)k_i(t, x(t), \dot{x}(t))] dt$ is (ρ'_i, b) -quasiinvex at the point $x = x^0$ with respect to η and θ .

c) For each $\alpha = \overline{1, r}, \int_a^b [\mu_{J_{\alpha}}^0(t)'g_{J_{\alpha}}(t, x(t), \dot{x}(t)) + \nu_{S_{\alpha}}^0(t)'h_{S_{\alpha}}(t, x(t), \dot{x}(t))] dt$ is (ρ''_{α}, b) -quasiinvex at the point $x = x^0$ with respect to η and θ .

d) At least one of the integrals from a)-c) is strictly (ρ, b) -quasiinvex at the point $x = x^0$ with respect to η and θ .

e) $\sum_{i=1}^p \lambda_i^0 \rho'_i + \sum_{\alpha=1}^r \rho''_{\alpha} \geq 0$.

Then x^0 is an efficient solution to (MFP) and moreover, $\pi(x^0) = \delta(x^0, \lambda^0, \mu^0, \nu^0)$.

Proof. On the contrary, suppose that the point x^0 is an efficient solution of problem (MFP) and then, we will find a contradiction. Then there exists $x \in \mathcal{D}$ such that for each $i = \overline{1, p}$:

$$\begin{aligned} & \frac{\int_a^b f_i(t, x(t), \dot{x}(t)) dt}{\int_a^b k_i(t, x(t), \dot{x}(t)) dt} \leq \\ & \leq \frac{\int_a^b [f_i(t, x^0(t), \dot{x}^0(t)) + \mu_{J_0}(t)' g_{J_0}(t, x^0(t), \dot{x}^0(t)) + \nu_{S_0}(t)' h_{S_0}(t, x^0(t), \dot{x}^0(t))] dt}{\int_a^b k_i(t, x^0(t), \dot{x}^0(t)) dt}. \end{aligned}$$

Following the proof of Theorem 4.1, with $(x^0, \lambda^0, \mu^0, \nu^0)$ instead of (y, λ, μ, ν) , we obtain

$$0 < -\|\theta(x, x^0)\|^2 \left(\sum_{i=1}^p \lambda_i^0 \rho'_i + \sum_{\alpha=1}^r \rho''_{\alpha} \right),$$

that yields $0 < 0$. It means that the above made supposition is false. Therefore the point x^0 is an efficient solution of problem (MFP) and we have $\pi(\bar{x}) = \delta(x^0, \lambda^0, \mu^0, \nu^0)$. \square

• CASE STUDY 2

Theorem 4.4 (Weak duality). *Let $x \in \mathcal{D}$ and $(y, \lambda, \mu, \nu) \in \Delta$ be feasible points of problems (MFP) and (MFZD). Assume satisfied the following conditions:*

a) $K_i(x) [\mu_{J_0}(t)' g_{J_0}(t, y(t), \dot{y}(t)) + \nu_{S_0}(t)' h_{S_0}(t, y(t), \dot{y}(t))] \leq 0, \forall x \in \mathcal{D}, \forall t \in I, i = \overline{1, p}$.

b) For each $i = \overline{1, p}$, the integral $\int_a^b [K_i(y) f_i(t, x(t), \dot{x}(t)) - F_i(y) k_i(t, x(t), \dot{x}(t))] dt$ is (ρ'_i, b) -quasiinvex at y with respect to η and θ .

c) For each $\alpha = \overline{1, r}$, the functional $\int_a^b \mu_{J_{\alpha}}(t)' g_{J_{\alpha}}(t, x(t), \dot{x}(t)) dt$ is (ρ''_{α}, b) -quasiinvex at y with respect to η and θ .

d) For each $\alpha = \overline{1, r}$, $\int_a^b \nu_{K_{\alpha}}(t)' h_{K_{\alpha}}(t, x(t), \dot{x}(t)) dt$ is monotonic (ρ'''_{α}, b) -quasiinvex at the point y with respect to η and θ .

e) At least one of the integrals from a)-b) is strictly (ρ, b) -quasiinvex at y with respect to η and θ .

$$\text{f) } \sum_{i=1}^p \lambda_i \rho'_i + \sum_{\alpha=1}^r (\rho''_{\alpha} + \rho'''_{\alpha}) \geq 0.$$

Then the inequality $\pi(x) \leq \delta(y, \lambda, \mu, \nu)$ is false.

Theorem 4.5 (Direct duality). *Let x^0 be a normal efficient solution of primal (MFP) and suppose satisfied the hypotheses of Theorem 4.4 at x^0 . Then there*

are $\lambda^0 \in \mathbb{R}^p$ and the piecewise smooth functions $\mu^0: I \rightarrow \mathbb{R}^m$ and $\nu^0: I \rightarrow \mathbb{R}^r$ such that $(x^0, \lambda^0, \mu^0, \nu^0)$ is an efficient solution of dual (MFZD) and moreover, $\pi(x^0) = \delta(x^0, \lambda^0, \mu^0, \nu^0)$.

Theorem 4.6 (Converse duality). *Let $(x^0, \lambda^0, \mu^0, \nu^0)$ be an efficient solution of dual (MFZD) and suppose satisfied the following conditions:*

- i) $x^0 \in \mathcal{D}$.
- a) $K_i(x^0)[\mu_{J_0}(t)'g_{J_0}(t, y(t), \dot{y}(t)) + \nu_{S_0}(t)'h_{S_0}(t, y(t), \dot{y}(t))] \leq 0, \forall t \in I, i = \overline{1, p}$.
- b) For each $i = \overline{1, p}$, $\int_a^b [K_i(x^0)f_i(t, x(t), \dot{x}(t)) - F_i(x^0)k_i(t, x(t), \dot{x}(t))]dt$ is (ρ'_i, b) -quasiinvex at the point $x = x^0$ with respect to η and θ .
- c) For each $\alpha = \overline{1, r}$, the integral $\int_a^b \mu_{J_\alpha}(t)'g_{J_\alpha}(t, x(t), \dot{x}(t))dt$ is (ρ''_α, b) -quasiinvex at the point $x = x_0$ with respect to η and θ .
- d) For each $\alpha = \overline{1, r}$, the integral $\int_a^b \nu_{K_\alpha}(t)'h_{K_\alpha}(t, x(t), \dot{x}(t))dt$ is monotonic (ρ'''_α, b) -quasiinvex at $x = x_0$ with respect to η and θ .
- e) One of the integrals from a)-c) is strictly (ρ, b) -quasiinvex at x^0 with respect to η and θ .
- f) $\sum_{i=1}^p \lambda_i^0 \rho'_i + \sum_{\alpha=1}^r (\rho''_\alpha + \rho'''_\alpha) \geq 0$.

Then x^0 is an efficient solution of (MFP). Moreover, $\pi(x^0) = \delta(x^0, \lambda^0, \mu^0, \nu^0)$.

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