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FACTORABLE GENERALIZED HAUSDORFF MATRICES

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Invited paper to celebrate Professor Constantin Udrişte, on the occasion of his seventies

ABSTRACT. We determine necessary and sufficient conditions for a conservative generalized Hausdorff matrix to be factorable.

1. INTRODUCTION

Hausdorff [3] defined the matrices which now bear his name, and established many properties of them.

Somewhat less well known are his generalized Hausdorff matrices [4]. They are defined as follows. Let $\{\lambda_n\}$ be a strictly increasing sequence of real numbers satisfying

$$0 \leq \lambda_0 < \lambda_1 < \cdots < \lambda_n < \cdots \tag{1.1}$$

with $\lambda_n \to \infty$, but slowly enough so that

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n} = \infty. \tag{1.2}$$

We shall call a sequence $\{\lambda_n\}$ which satisfies (1.1) and (1.2) admissible.

These generalized Hausdorff matrices are defined by

$$h_{nk}(\lambda;\mu) = \lambda_{k+1}\cdots\lambda_n[\mu_k,\ldots,\mu_n], \quad 0 \leq k \leq n, \tag{1.3}$$

where [ ] is the divided difference defined by

$$[\mu_k,\mu_{k+1}] = \frac{\mu_k - \mu_{k+1}}{\lambda_{k+1}-\lambda_k},$$

and

$$[\mu_k,\ldots,\mu_n] = \frac{[\mu_k,\ldots,\mu_{n-1}] - [\mu_{k+1},\ldots,\mu_n]}{\lambda_n - \lambda_k},$$

with the understanding that the product $\lambda_{k+1}\cdots\lambda_n = 1$ when $k = n$.

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The choice $\lambda_n = n$ yields the ordinary Hausdorff matrices. Since we shall be discussing only the generalized Hausdorff matrices defined by (1.3) we shall omit writing $(\lambda;\mu)$.

Hausdorff considered only admissible sequences $\{\lambda_n\}$ for which $\lambda_0 = 0$. In this case, each row sum is μ_0. (See, e.g., [4] or [10])

Jakimovski [6] considered admissible sequences $\{\lambda_n\}$ for which $\lambda_0 > 0$. In this case the limit of the row sums is μ_0.

It is appropriate, therefore, to call the matrices defined by (1.3) the H-J generalized Hausdorff matrices.

An infinite matrix A is said to be conservative if it maps every convergent sequence into a convergent sequence, not necessarily with the same limit. Necessary and sufficient conditions for a matrix A to be conservative are the well-known Silverman-Toeplitz conditions:

(i) $\lim_n a_k = \alpha_k$ exists for each k,

(ii) $\lim_n \sum_k a_{nk} = t$ exists, and

(iii) $\|A\| := \sup_n \sum_k |a_{nk}| < \infty$.

An H-J matrix is conservative if and only if there exists a function $\chi(t) \in BV[0,1]$ such that

$$\int_0^1 |d\chi(t)| < \infty. \tag{1.4}$$

Moreover, this integral is the norm of the matrix. In addition

$$\mu_n = \int_0^1 t^{\lambda_n} d\chi(t), \quad n = 0, 1, \ldots,$$

where the integral is a Riemann-Siteltjes one.

We shall call $\{\mu_n\}$ the moment generating sequence for H and $\chi(t)$ the mass function for H.

It is also the case that, if (1.4) is satisfied, then each column limit is zero, except possibly for the first one, and for that column, $\lim_n h_{n0}$ exists.

For each admissible sequence $\{\lambda_n\}$, the corresponding set of conservative H-J matrices forms an integral domain of operators.

A lower triangular matrix is said to be factorable if each entry a_{nk} can be written in the form $a_n b_k$, where a_n depends only on n and b_k depends only on k.

A triangle is a lower triangular matrix with no zeros on the main diagonal.

In this paper we determine those H-J matrices which are also factorable.

2. MAIN RESULTS

We begin with a lemma, which is of interest in its own right.

Lemma 2.1. *Let A be a triangle. Then A is factorable if and only if its inverse is bidiagonal.*

Proof. Let A be a factorable triangle. Then a direct calculation verifies that $a^{-1}_{nn} = \dfrac{1}{a_n b_n}$, $a^{-1}_{n,n-1} = -\dfrac{1}{a_{n-1}b_n}$, and $a^{-1}_{nk} = 0$ for $0 \le k < n-1$. Thus A^{-1} is bidiagonal.

Now let A be a triangle with a bidiagonal inverse; i.e., $a^{-1}_{nn} = \alpha_n$, $a^{-1}_{n,n-1} = \beta_n$ and $a^{-1}_{nk} = 0$ for $0 \le k < n-1$.

Then $a_{nn} = \dfrac{1}{\alpha_n}$, and, solving

$$a_{n,n-1}a^{-1}_{n-1,n-1} + a_{nn}a^{-1}_{n,n-1} = 0$$

for $a_{n,n-1}$ yields

$$a_{n,n-1} = -\frac{\beta_{n-1}}{\alpha_{n-1}\alpha_n}.$$

By finite induction it then follows that

$$a_{nk} = (-1)^{n-k}\,\frac{\prod_{j=k+1}^{n} \beta_{j-1}}{\prod_{j=k}^{n} \alpha_j} = (-1)^n\,\frac{\prod_{j=0}^{n-1} \beta_j}{\prod_{j=0}^{n} \alpha_j}\;(-1)^k\,\frac{\prod_{j=0}^{k-1} \alpha_j}{\prod_{j=0}^{k-1} \beta_j},$$

and A is factorable. □

Theorem 2.1. *Let H be a conservative* H-J *matrix with $\lambda_0 = 0$. Then H is factorable if and only if*

$$\mu_n = \frac{a}{\lambda_n + a}, \quad \textit{where} \quad a = \frac{\mu_1\lambda_1}{(1-\mu_1)}, \tag{2.1}$$

or $\mu_0 = 1$, $\mu_n = 0$ for all $n > 0$.

Proof. Since $\lambda_0 = 0$, H has row sums μ_0 and hence, any factorable H must be a weighted mean matrix. A weighted mean matrix is a lower triangular matrix with entries $\dfrac{p_k}{P_n}$, where $p_0 > 0$, $p_k \ge 0$ for $k > 0$, and $P_n\colon = \sum_{k=0}^{n} p_k$. Moreover, $\mu_0 = \dfrac{p_0}{P_0} = 1$.

If H is not a triangle, then at least one diagonal entry of H must be zero; i.e., at least one $\mu_N = 0$. Since H is also a weighted mean matrix, $\mu_0 = 1 \ne 0$.

Suppose that $\mu_0 = 1$, $\mu_n = 0$ for all $n > 0$. Then, for $0 < k \le n$,

$$h_{nk} = \lambda_{k+1}\cdots\lambda_n[\mu_k, \ldots, \mu_n] = 0.$$

For $k = 0$, since $\lambda_0 = 0$,

$$\begin{aligned} h_{n0} &= \lambda_1 \cdots \lambda_n[\mu_0, \dots \mu_n] = \frac{\lambda_1 \cdots \lambda_n}{\lambda_n - \lambda_0}\{[\mu_0, \dots, \mu_{n-1}] - [\mu_1, \dots, \mu_n]\} \\ &= \lambda_1 \cdots \lambda_{n-1}[\mu_0, \dots, \mu_{n-1}] \\ &= \frac{\lambda_1 \cdots \lambda_{n-1}}{\lambda_{n-1} - \lambda_0}\{[\mu_0, \dots, \lambda_{n-2}] - [\mu_1, \dots, \lambda_{n-1}]\} \\ &= \lambda_1 \cdots \lambda_{n-2}[\mu_0, \dots, \lambda_{n-2}] = \cdots \\ &= \mu_0 = 1. \end{aligned}$$

H is clearly conservative, and it is factorable with $p_0 > 0$, $p_n = 0$ for all $n > 0$.

Suppose that $\mu_N = 0$ for some $N > 1$. Since H is also a weighted mean matrix, it follows that $p_N = 0$, and hence column N is zero.

From (1.3) it follows that

$$(\lambda_{N+1} - \lambda_k)h_{N+1,k} = \lambda_{N+1}h_{nk} - \lambda_{k+1}h_{N+1,k+1}. \tag{2.2}$$

Since column N is zero, it follows from (2.2) that column $N+1$, and hence that every column $k > N$ is zero.

Also note that, for $k = N - 1$, $N > 1$,

$$\frac{h_{N+1,N-1}}{h_{N,N-1}} = \frac{\lambda_{N+1}}{\lambda_{N+1} - \lambda_{N-1}} > 1.$$

Therefore $\lim h_{n,k} \neq 0$. Recall that a conservative Hausdorff matrix has all zero column limits except possibly column zero. Thus H is not conservative.

The fact that no conservative ordinary Hausdorff matrix can have only a finite number N of nonzero columns, for any $N > 1$, appears in [9].

Now assume that H is a triangle. Then, by Lemma 2.1, since it is also factorable, its inverse must be bidiagonal.

Jakimovski and Tietze [7] have shown that every H-J matrix has the decomposition

$$H = T^{-1}\mu T,$$

where μ is a diagonal matrix with diagonal entries $\{\mu_n\}$ and T is a triangle with entries

$$t_{nk} = (-1)^k \frac{\prod_{\nu=k+1}^{n} \lambda_\nu}{\prod_{\nu=k}^{n-1}(\lambda_n - \lambda_\nu)}.$$

Therefore $H^{-1} = T^{-1}\mu^{-1}T$, which implies that H^{-1} has entries defined by (1.3) with $\{\mu_n\}$ replace by $\left\{\frac{1}{\mu_n}\right\}$.

Since H^{-1} is bidiagonal, $h_{nk}^{-1} = 0$ for $0 \le k < n-1$. In particular, with $\xi_n = \dfrac{1}{\mu_n}$,

$$0 = h_{n,n-2}^{-1} = \lambda_{n-1}\lambda_n[\xi_{n-2}, \xi_{n-1}, \xi_n],$$

which implies that $[\xi_{n-2}, \xi_{n-1}] = [\xi_{n-1}, \xi_n]$, or that

$$\frac{\xi_{n-2} - \xi_{n-1}}{\lambda_{n-1} - \lambda_{n-2}} = \frac{\xi_{n-1} - \xi_n}{\lambda_n - \lambda_{n-1}};$$

i.e.,

$$\begin{aligned} \xi_n - \xi_{n-1} &= \frac{(\lambda_n - \lambda_{n-1})}{\lambda_{n-1} - \lambda_{n-2}}(\xi_{n-1} - \xi_{n-2}) \\ &= \cdots \\ &= \frac{(\lambda_n - \lambda_{n-1})}{(\lambda_1 - \lambda_0)}(\xi_1 - \xi_0). \end{aligned} \tag{2.3}$$

Thus

$$\sum_{k=0}^{n-1}(\xi_{k+1} - \xi_k) = \frac{(\xi_1 - \xi_0)}{(\lambda_1 - \lambda_0)}\sum_{k=0}^{n-1}(\lambda_{k+1} - \lambda_k), \tag{2.4}$$

or, since $\lambda_0 = 0$,

$$\xi_n = \frac{(\xi_1 - \xi_0)\lambda_n}{\lambda_1} + \xi_0,$$

or

$$\mu_n = \frac{\lambda_1}{\lambda_n\left(\dfrac{1}{\mu_1} - \dfrac{1}{\mu_0}\right) + \dfrac{\lambda_1}{\mu_0}} = \frac{\dfrac{\mu_0\mu_1\lambda_1}{\mu_0 - \mu_1}}{\lambda_n + \dfrac{\mu_1\lambda_1}{\mu_0 - \mu_1}}.$$

Set

$$a = \frac{\lambda_1\mu_1}{(\mu_0 - \mu_1)}.$$

Then μ_n takes the form of (2.1), since $\mu_0 = 1$.

Note that, from (2.3), if $\mu_1 = 1$, then $\mu_n = 1$ for all n, and $H = I$. But the identity matrix cannot be obtained by any weighted mean matrix. Also note that

$$1 - \mu_1 = 1 - \frac{p_1}{P_1} = \frac{p_0}{P_1} > 0,$$

and a is positive.

Conversely, if H has the form (2.1), then

$$\begin{aligned} [\mu_k, \mu_{k+1}] &= \frac{\mu_k - \mu_{k+1}}{\lambda_{k+1} - \lambda_k} = \frac{1}{\lambda_{k+1} - \lambda_k}\Big(\frac{a}{\lambda_k + a} - \frac{a}{\lambda_{k+1} + a}\Big) \\ &= \frac{a}{(\lambda_k + a)(\lambda_{k+1} + a)}, \end{aligned}$$

and, by induction,

$$[\mu_k, \ldots, \mu_n] = \frac{a}{\prod_{i=k}^{n}(\lambda_i + a)}.$$

Thus

$$h_{nk} = \frac{\lambda_{k+1}\cdots\lambda_n a}{\prod_{i=k}^{n}(\lambda_i + a)} = \frac{\prod_{i=1}^{n}\lambda_i}{\prod_{i=1}^{n}(\lambda_i + a)} \; \frac{\prod_{i=0}^{k-1}(\lambda_i + a)}{\prod_{i=1}^{k}\lambda_i},$$

since $\lambda_0 = 0$, and H is a weighted mean matrix (which is factorable) with p_0 arbitrary and $p_k = \prod_{i=0}^{k-1}(\lambda_i + 1)/\prod_{i=1}^{k}\lambda_i$ for $k > 0$. □

An incomplete form of Theorem 2.1 appears as Theorem 6 in [8].

Theorem 2.2. *Let H be a conservative* H-J *triangle with $\lambda_0 > 0$. Then H is factorable if and only if*

$$\mu_n = \frac{\mu_0 b}{\lambda_n - \lambda_0 + b}, \quad \text{where} \quad b = \frac{\mu_1(\lambda_1 - \lambda_0)}{\mu_0 - \mu_1} > \lambda_0.$$

Proof. From (2.4) in the proof of Theorem 2.1, with $\lambda_0 > 0$, one obtains

$$\xi_n = \frac{(\xi_1 - \xi_0)(\lambda_n - \lambda_0)}{\lambda_1 - \lambda_0} + \xi_0,$$

or,

$$\begin{aligned}
\mu_n &= \frac{(\lambda_1 - \lambda_0)}{(1/\mu_1 - 1/\mu_0)(\lambda_n - \lambda_0) + (\lambda_1 - \lambda_0)/\mu_0} \\
&= \frac{\mu_0\mu_1(\lambda_1 - \lambda_0)}{(\mu_0 - \mu_1)(\lambda_n - \lambda_0) + \mu_1(\lambda_1 - \lambda_0)} \\
&= \frac{\mu_0\mu_1(\lambda_1 - \lambda_0)/(\mu_0 - \mu_1)}{\lambda_n - \lambda_0 + \mu_1(\lambda_1 - \lambda_0)/(\mu_0 - \mu_1)} \\
&= \frac{\mu_0 b}{\lambda_n - \lambda_0 + b}, \qquad \text{where} \quad b = \frac{\mu_1(\lambda_1 - \lambda_0)}{(\mu_0 - \mu_1)}.
\end{aligned}$$

The condition $b > \lambda_0$ is needed in order for H to be conservative. □

In Theorem 2.2, one cannot remove the assumption that H is a triangle. For, suppose that H is not a triangle. Then $\mu_n = 0$ for some N; i.e., $a_N b_N = 0$. If $b_N = 0$, then the proof of Theorem 2.1 applies. But, if $a_N = 0$, the question has not been resolved, although the following two examples are of interest.

If $\mu_n = \dfrac{\lambda_n - \lambda_0}{\lambda_n + 1}$, then H is conservative, but not factorable.

If H is an E-J matrix with $\mu_0 = 0$, $\mu_n = \dfrac{1}{n+\alpha+1}$ for $n > 0$ (see the paragraph below for the definition of E-J matrices), then H is factorable, but

$$\begin{aligned}\Delta^n \mu_0 &= \sum_{k=0}^{n} (-1)^k \binom{n}{k} \mu_k = \sum_{k=1}^{n} (-1)^k \binom{n}{k} \mu_k \\ &= \sum_{k=0}^{n} (-1)^k \frac{n}{k} \binom{n-1}{k-1} \mu_k = -n \sum_{j=0}^{n-1} (-1)^j \binom{n-1}{j} \epsilon_j,\end{aligned}$$

where $\epsilon_j = \dfrac{\mu_{j+1}}{j+1}$. The sequence $\{\epsilon_j\}$ is the product of two totally monotone sequences, so it is totally monotone, and all of the forward differences are nonnegative and bounded. Therefore

$$|h_{n0}^{\alpha)}| = \binom{n+\alpha}{n} n \Delta^n \lambda_0 \to \infty, \quad \text{as} \quad n \to \infty,$$

and H is not conservative.

A different generalization of Hausdorff matrices, which was defined independently by Endl [1] and [2] and Jakimovski [6], we shall call the E-J matrices. These matrices have entries

$$h_{nk}^{(\alpha)} = \binom{n+\alpha}{n-k} \Delta^{n-k} \mu_k, \quad \alpha \geq 0,$$

where Δ is the forward difference operator defined by $\Delta \mu_k = \mu_k - \mu_{k+1}$, and $\Delta^{n+1} \mu_k = \Delta(\Delta^n \mu_k)$.

Again the necessary and sufficient condition for an E-J matrix to be conservative is the existence of a function $\chi(t) \in BV[0,1]$ such that (1.4) is finite. For the E-J matrices the diagonal entries take the form

$$\mu_n^{(\alpha)} = \int_0^1 t^{n+\alpha} d\chi(t).$$

The E-J matrices are the special case of the H-J matrices with $\lambda_n = n + \alpha$.

Corollary 2.1. *Let* $H^{(\alpha)}$ *be a conservative* E-J *triangle. Then* $H^{(\alpha)}$ *is a factorable triangle if and only if*

$$\mu_n^{(\alpha)} = \frac{\mu_0 c}{n+c}, \quad \textit{where} \quad c = \frac{\mu_1}{\mu_0 - \mu_1}.$$

Proof. In Theorem 2.2 set $\lambda_n = n + \alpha$. □

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