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A GENERALIZED RICCATI EQUATION METHOD FOR NONLINEAR PDES

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Invited paper to celebrate Professor Constantin Udrişte, on the occasion of his seventies

Abstract. In this paper, a generalized Riccati equation method for obtaining exact solutions of nonlinear partial differential equationds is proposed and applied to the generalized Pochhammer-Chree equation and the Klein-Gordon equation. As a result, new exact travelling wave solutions are obtained. It is shown that the generalized Riccati equation method is direct, effective and can be used for many other nonlinear partial differential equations.

1. INTRODUCTION

Nonlinear partial differential equations (PDEs) are widely used to describe complex phenomena in various fields of science from physics to biology, chemistry, mechanics, etc. As mathematical models of the phenomena, the investigation of exact solutions of nonlinear PDEs will help one to understand these phenomena better. In the past several decades, many effective methods for obtaining exact solutions of nonlinear PDEs have been presented, such as the inverse scattering method [1], Hirota's bilinear method [2], Bäcklund transformation [3], homogenous balance method [4], tanh-function method [5], Jacobi elliptic function expansion method [6], Exp-function method [7], F-expansion method [8], and auxiliary equation method [9].

The last three methods mentioned belong to a class of method called subsidiary ordinary differential equation method (sub-ODE method for short). The key ideas of the sub-ODE method are that the travelling wave solutions of the complicated nonlinear PDE can be expressed as a polynomial, the variable of which is one of the solutions of simple and solvable ODE that called the sub-ODE, and the degree of the polynomial can be determined by balancing the highest order partial derivative with the highest order nonlinear terms in the considered nonlinear PDE. The sub-ODEs which were often used are the Riccati equation, Jacobi elliptic equation, projective

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Riccati equations, etc. With the development of computer science, recently, sub-ODEs with nonlinear terms of high order have attracted much attention [8-12]. This is due to the availability of symbolic computation systems like Mathematica or Maple which enable us to perform the complex and tedious computation on computers.

The present paper is motivated by the desire to introduce a generalized Riccati equation including an arbitrary positive power [12]

$$F'(\xi) = AF^{\eta}(\xi) + \frac{\varepsilon}{1-\eta}F^{2-\eta}(\xi), \tag{1.1}$$

and its some special solutions, then employ Eq. (1.1) as the sub-ODE to obtain new formal exact solutions of nonlinear PDEs. Here A, η and ε are parameters, the prime denotes $\frac{\mathrm{d}}{\mathrm{d}\xi}$. It is obvious to see that Eq. (1.1) includes the well-known Riccati equation

$$F'(\xi) = AF^2(\xi) - \varepsilon. \tag{1.2}$$

as a special case, under this case $\eta = 2$. Eq. (1.2) is the sub-ODE used often in the most existing tanh-function method and its improvements.

The rest of this paper is organized as follows. In Section 2, we introduce some special solutions of Eq. (1.1). In Section 3, we use Eq. (1.1) and its special solutions to solve the generalized Pochhammer-Chree equation and the Klein-Gordon equation. In Section 4, some conclusions are given.

2. SPECIAL SOLUTIONS

In order to find some solutions of Eq. (1.1) conveniently, we set

$$F(\xi) = G^{\frac{1}{\eta-1}}(\xi), \tag{2.1}$$

then Eq. (1.1) becomes

$$G'(\xi) = A(\eta-1)G^2(\xi) - \varepsilon. \tag{2.2}$$

With the aid of Eqs. (2.1) and (2.2), we can easily find some special solutions of Eq. (1.1), which are listed as follows.

2.1. If $\varepsilon = -1$, then Eq. (1.1) has the following hyperbolic function solutions:

$$F(\xi) = \left\{\sqrt{-\frac{1}{A(\eta-1)}}\tanh\left(\sqrt{-A(\eta-1)}\xi\right)\right\}^{\frac{1}{\eta-1}}, \tag{2.3}$$

$$F(\xi) = \left\{\sqrt{-\frac{1}{A(\eta-1)}}\coth\left(\sqrt{-A(\eta-1)}\xi\right)\right\}^{\frac{1}{\eta-1}}, \tag{2.4}$$

$$F(\xi) = \left\{\sqrt{-\frac{1}{A(\eta-1)}}\left[\tanh\left(2\sqrt{-A(\eta-1)}\xi\right) \pm \mathrm{isech}\left(2\sqrt{-A(\eta-1)}\xi\right)\right]\right\}^{\frac{1}{\eta-1}}, \tag{2.5}$$

$$F(\xi)=\left\{\sqrt{-\frac{1}{A(\eta-1)}}\left[\frac{\sqrt{2}\tanh\left(2\sqrt{-A(\eta-1)}\xi\right)\pm i\operatorname{sech}\left(2\sqrt{-A(\eta-1)}\xi\right)}{\sqrt{2}-\operatorname{sech}\left(2\sqrt{-A(\eta-1)}\xi\right)}\right]\right\}^{\frac{1}{\eta-1}}, \tag{2.6}$$

2.2. If $\varepsilon = 1$, then Eq. (1.1) has the following trigonometric function solutions:

$$F(\xi)=\left\{-\sqrt{-\frac{1}{A(\eta-1)}}\tan\left(\sqrt{-A(\eta-1)}\xi\right)\right\}^{\frac{1}{\eta-1}}, \tag{2.7}$$

$$F(\xi)=\left\{\sqrt{-\frac{1}{A(\eta-1)}}\cot\left(\sqrt{-A(\eta-1)}\xi\right)\right\}^{\frac{1}{\eta-1}}, \tag{2.8}$$

$$F(\xi)=\left\{-\sqrt{-\frac{1}{A(\eta-1)}}\left[\frac{2\tan\left(2\sqrt{-A(\eta-1)}\xi\right)\pm i\sec\left(2\sqrt{-A(\eta-1)}\xi\right)}{\sqrt{2}+\sqrt{5}\sec\left(2\sqrt{-A(\eta-1)}\xi\right)}\right]\right\}^{\frac{1}{\eta-1}}, \tag{2.9}$$

2.3. If $\varepsilon = 0$, then Eq. (1.1) has the following rational solution:

$$F(\xi)=\left\{\frac{1}{-A(\eta-1)\xi+p}\right\}^{\frac{1}{\eta-1}}, \tag{2.10}$$

where p is an arbitrary constant.

Remark 2.1. Solutions (2.3)-(2.9) can be obtained by using the Exp-function method [7]. To the best of our knowledge, solutions (2.6) and (2.9) are new and they have not been reported in literature.

3. APPLICATIONS

First, let us consider the so-called Pochhammer-Chree equation

$$u_{tt}-u_{ttxx}-u_{xx}-\frac{1}{p}\left(u^{p}\right)_{xx}=0, \tag{3.1}$$

which models the propagation of longitudinal deformation waves in an elastic rod, where $p = 3$ or $p = 5$ reflects two possible constitutive choices for the material. Bogolusky [13] and Clarkson et al. [14] gave some solitary wave solutions of Eq. (3.1) with $p = 2, 3, 5$ and studied the interaction of two solitary waves numerically. Moreover, Clarkson et al. showed that this equation and even certain generalizations do not pass the Painlevé test, and hence are probably not completely integrable. In this paper, we consider the generalized Pochhammer-Chree equation [12]

$$u_{tt}-u_{xxtt}-\left(au+bu^{3}+cu^{5}\right)=0, \tag{3.2}$$

where a, b and c are real parameters.

In what follows, we show the detailed steps of the generalized Riccati equation method to construct exact travelling wave solutions of Eq. (3.2).

Making the travelling wave transformation

$$u(x,t) = u(\xi), \quad \xi = k(x + \lambda t), \tag{3.3}$$

and integrating twice, here k and λ are constants to be determined later, then Eq. (3.2) becomes an ordinary differential equation for $u(\xi)$ in the form:

$$k^2\lambda^2 u'' + (a - \lambda^2)u + bu^3 + cu^5 = 0, \tag{3.4}$$

where $u'' = \mathrm{d}^2u/\mathrm{d}\xi^2$ and the integration constants are chosen as zero.

Supposing $u(\xi)$ can be expressed as

$$u(\xi) = a_0 + \sum_{i=1}^{M} \left\{a_i F^i(\xi) + b_i F^{-i}(\xi)\right\}, \tag{3.5}$$

where $F(\xi)$ satisfies Eq. (1.1), while a_0, a_i and b_i $(i = 1, 2, \ldots, M)$ are undermined constants. Balancing u'' and u^5 in Eq. (3.4), we get $M + 2(\eta - 1) = 5M$, namely $2M = \eta - 1$, we here choose $\eta = 3$ and $M = 1$.

Substituting (3.5) given the value of $M = 1$ along with Eq. (1.1) into (3.4), collecting all terms with the same power of $F^j(\xi)$ $(j = 0, \pm1, \pm2, \ldots, \pm5)$ together, then setting each coefficient of $F^j(\xi)$ to zero, we obtain a set of algebraic equations for A, a_0, a_1 and b_1 as follows:

$F^0(\xi):\ aa_0 + a_0^3 b + a_0^5 c + 6a_0 ba_1 b_1 + 20a_0^3 ca_1 b_1 + 30a_0 ca_1^2 b_1^2 = 0,$

$F^1(\xi):\ aa_1 - 3a_0^2 ba_1 + 5a_0^4 ca_1 - Ak^2\varepsilon\lambda^2 a_1 + 3ba_1^2 b_1 + 30a_0^2 ca_1^2 b_1 + 10ca_1^3 b_1^2 = 0,$

$F^2(\xi):\ 3a_0 ba_1^2 + 10a_0^3 ca_1^2 + 20a_0 ca_1^3 b_1 = 0,$

$F^3(\xi):\ ba_1^3 + 10a_0^2 ca_1^3 - A^2k^2\lambda^2 b_1 + 5ca_1^4 b_1 = 0,$

$F^4(\xi):\ 5a_0 ca_1^4 = 0,$

$F^5(\xi):\ 3A^2k^2\lambda^2 a_1 + ca_1^5 = 0,$

$F^{-1}(\xi):\ ab_1 + 3a_0^2 bb_1 + 5a_0^4 cb_1 - \lambda^2 b_1 - Ak^2\varepsilon\lambda^2 b_1 + 3ba_1 b_1^2 + 30a_0^2 ca_1 b_1^2 + 10ca_1^2 b_1^3 = 0,$

$F^2(\xi):\ 3a_0 bb_1^2 + 10a_0^3 cb_1^2 + 20a_0 ca_1 b_1^3 = 0,$

$F^{-3}(\xi):\ -\frac{1}{4}k^2\varepsilon^2 a_1 + bb_1^3 + 10a_0^2 cb_1^3 + 5ca_1 b_1^4 = 0,$

$F^{-4}(\xi):\ 5a_0 cb_1^4 = 0,$

$F^{-5}(\xi):\ \frac{3}{4}k^2\varepsilon^2 b_1 + cb_1^5 = 0.$

Solving the set of algebraic equations, we have

$$a_0 = 0, \quad a_1 = \pm\frac{(k^2\lambda^4\varepsilon^2 - ak^2\lambda^2\varepsilon^2)^{\frac{3}{4}}}{\sqrt{2b}k^2\lambda^2\varepsilon^2}, \tag{3.6}$$

$$b_1 = \pm\frac{\sqrt{2}(k^2\lambda^4\varepsilon^2 - ak^2\lambda^2\varepsilon^2)^{\frac{1}{4}}}{\sqrt{b}}, \quad A = \frac{\lambda^2 - a}{8k^2\lambda^2\varepsilon}, \quad c = \frac{3b^2}{16(a - \lambda^2)}, \tag{3.7}$$

or

$$a_0 = 0, \quad a_1 = \pm\frac{2(k^2\lambda^4\varepsilon^2 - ak^2\lambda^2\varepsilon^2)^{\frac{3}{4}}}{5^{\frac{3}{4}}\sqrt{b}k^2\lambda^2\varepsilon^2}, \tag{3.8}$$

$$b_1 = \pm\frac{2(k^2\lambda^4\varepsilon^2 - ak^2\lambda^2\varepsilon^2)^{\frac{1}{4}}}{5^{\frac{1}{4}}\sqrt{b}}, \quad A = \frac{a-\lambda^2}{10k^2\lambda^2\varepsilon}, \quad c = \frac{15b^2}{64(a-\lambda^2)}, \tag{3.9}$$

where k and λ are arbitrary constants.

Now we use the results obtained above to obtain a series of exact travelling wave solutions of Eq. (1.1). When $\varepsilon = -1$, from Eqs. (2.3), (3.3), (3.5)-(3.7) we obtain the hyperbolic function solution of Eq. (3.2):

$$u_{1\pm}(x,t) = \pm\frac{\sqrt{\lambda^2-a}}{\sqrt{b}}\left\{\tanh\left(\frac{k}{2}\sqrt{\frac{\lambda^2-a}{k^2\lambda^2}}(x+ct)\right)\right\}^{\frac{1}{2}}$$
$$\pm\frac{\sqrt{\lambda^2-a}}{\sqrt{b}}\left\{\coth\left(\frac{k}{2}\sqrt{\frac{\lambda^2-a}{k^2\lambda^2}}(x+ct)\right)\right\}^{\frac{1}{2}}. \tag{3.10}$$

When $\varepsilon = -1$, from Eqs. (2.5), (3.3), (3.5)-(3.7) we obtain the hyperbolic function solution of Eq. (3.2):

$$u_{2\pm}(x,t) = \pm\frac{\sqrt{\lambda^2-a}}{\sqrt{b}}\left\{\tanh\left(k\sqrt{\frac{\lambda^2-a}{k^2\lambda^2}}(x+ct)\right) \pm \text{isech}\left(k\sqrt{\frac{\lambda^2-a}{k^2\lambda^2}}(x+ct)\right)\right\}^{\frac{1}{2}}$$
$$\pm\frac{\sqrt{\lambda^2-a}}{\sqrt{b}}\left\{\tanh\left(k\sqrt{\frac{\lambda^2-a}{k^2\lambda^2}}(x+ct)\right) \mp \text{isech}\left(k\sqrt{\frac{\lambda^2-a}{k^2\lambda^2}}(x+ct)\right)\right\}^{\frac{1}{2}}. \tag{3.11}$$

When $\varepsilon = -1$, from Eqs. (2.6), (3.3), (3.5)-(3.7) we obtain the hyperbolic function solution of Eq. (3.2):

$$u_{3\pm}(x,t) = \pm\frac{\sqrt{\lambda^2-a}}{\sqrt{b}}\left\{\frac{\sqrt{2}\tanh\left(k\sqrt{\frac{\lambda^2-a}{k^2\lambda^2}}(x+\lambda t)\right) \pm \text{isech}\left(k\sqrt{\frac{\lambda^2-a}{k^2\lambda^2}}(x+\lambda t)\right)}{\sqrt{2}-\text{sech}\left(k\sqrt{\frac{\lambda^2-a}{k^2\lambda^2}}(x+ct)\right)}\right\}^{\frac{1}{2}}$$
$$\pm\frac{\sqrt{\lambda^2-a}}{\sqrt{b}}\left\{\frac{\sqrt{2}-\text{sech}\left(k\sqrt{\frac{\lambda^2-a}{k^2\lambda^2}}(x+\lambda t)\right)}{\sqrt{2}\tanh\left(k\sqrt{\frac{\lambda^2-a}{k^2\lambda^2}}(x+ct)\right) \pm \text{isech}\left(k\sqrt{\frac{\lambda^2-a}{k^2\lambda^2}}(x+\lambda t)\right)}\right\}^{\frac{1}{2}}. \tag{3.12}$$

When $\varepsilon = 1$, from Eqs. (2.8), (3.3), (3.5), (3.8) and (3.9) we obtain the trigonometric function solution of Eq. (3.2):

$$u_{4\pm}(x,t) = \pm\frac{2\sqrt{\lambda^2-a}}{\sqrt{5b}}\left\{\cot\left(k\sqrt{\frac{\lambda^2-a}{5k^2\lambda^2}}(x+\lambda t)\right)\right\}^{\frac{1}{2}}$$

$$\pm\frac{2\sqrt{\lambda^2-a}}{\sqrt{5b}}\left\{\tan\left(k\sqrt{\frac{\lambda^2-a}{5k^2\lambda^2}}(x+\lambda t)\right)\right\}^{\frac{1}{2}}. \tag{3.13}$$

When $\varepsilon = 0$, from the set of algebraic equations obtained in the process of computation we derive $a_1 = b_1 = 0$, therefore, Eq. (3.2) has not rational solution.

Remark 3.1. To the best of our knowledge, solutions (3.11) and (3.12) are new and they have not been reported in literature.

We next consider the nonlinear Klein-Gordon equation

$$u_{tt} - u_{xx} + g(u) = 0, \tag{3.14}$$

which is used to model many different nonlinear phenomena [15], including the propagation of dislocations in crystals and the behavior of elementary particles and the propagation of fluxons in Josephson junctions. The function $g(u)$ is chosen as many forms such as $\sin u$, $\sinh u$, e^u, $e^u + e^{-2u}$ and $e^{-u} + e^{-2u}$ which characterize the sin-Gordon equation, sinh-Gordon equation, Liouville equation, Dodd-Bullough-Mikhailov equation and the Tziteica-Dodd-Bullough equation, respectively. Also, $g(u)$ appears as a polynomial such as $g(u) = au + bu^3$ and $g(u) = au + bu^3 + cu^5$. In what follows, we consider the nonlinear Klein-Gordon [12, 16]

$$u_{tt} - \lambda^2 u_{xx} + \alpha u - \beta u^3 + \gamma u^5 = 0, \tag{3.15}$$

where α, β, γ and λ are nonzero constants.

Making the travelling wave transformation

$$u(x,t) = u(\xi), \quad \xi = x + ct, \tag{3.16}$$

where c is a constant to be determined later, then Eq. (3.15) becomes an ordinary differential equation for $u(\xi)$ in the form:

$$(c^2 - \lambda^2)u'' + \alpha u - \beta u^3 + \gamma u^5 = 0, \tag{3.17}$$

where $u'' = \mathrm{d}^2u/\mathrm{d}\xi^2$.

Supposing $u(\xi)$ can be expressed as

$$u(\xi) = a_0 + \sum_{i=1}^{M}\left\{a_i F^i(\xi) + b_i F^{-i}(\xi)\right\}, \tag{3.18}$$

where $F(\xi)$ satisfies Eq. (1.1), while a_0, a_i and b_i $(i = 1, 2, \dots, M)$ are undermined constants. Balancing u'' and u^5 in Eq. (3.17), we get $M + 2(\eta - 1) = 5M$, namely $2M = \eta - 1$, we may choose $\eta = 3$ and $M = 1$.

Substituting (3.18) given the value of $M = 1$ along with Eq. (1.1) into (3.17), collecting all terms with the same power of $F^j(\xi)$ $(j = 0, \pm1, \pm2, \dots, \pm5)$ together, then setting each coefficient of $F^j(\xi)$ to zero, we obtain a set of algebraic equations for A, a_0, a_1 and b_1 as follows:

$F^0(\xi):\ a_0\alpha - a_0^3\beta + a_0^5\gamma - 6a_0\beta a_1b_1 + 20a_0^3\gamma a_1b_1 + 30a_0\gamma a_1^2b_1^2 = 0,$

$F^1(\xi):\ a_1\alpha - 3a_0^2\beta a_1 + 5a_0^4\gamma a_1 - Ac^2\varepsilon a_1 + A\varepsilon\lambda^2 a_1 - 3\beta a_1^2b_1 + 30a_0^2\gamma a_1^2b_1 + 10\gamma a_1^3b_1^2 = 0,$

$F^2(\xi):\ -3a_0\beta a_1^2 + 10a_0^3\gamma a_1^2 + 20a_0\gamma a_1^3b_1 = 0,$

$F^3(\xi):\ -\beta a_1^3 + 10a_0^2\gamma a_1^3 - A^2c^2b_1 + A^2\lambda^2b_1 + 5\gamma a_1^4b_1 = 0,$

$F^4(\xi):\ 5a_0\gamma a_1^4 = 0,$

$F^5(\xi):\ 3A^2c^2a_1 - 3A^2\lambda^2a_1 + \gamma a_1^5 = 0,$

$F^{-1}(\xi):\ \alpha b_1 - 3a_0^2\beta b_1 + 5a_0^4\gamma b_1 - Ac^2\varepsilon b_1 + A\varepsilon\lambda^2b_1 - 3\beta a_1b_1^2 + 30a_0^2\gamma a_1b_1^2 + 10\gamma a_1^2b_1^3 = 0,$

$F^2(\xi):\ -3a_0\beta b_1^2 + 10a_0^3\gamma b_1^2 + 20a_0\gamma a_1b_1^3 = 0,$

$F^{-3}(\xi):\ -\frac{1}{4}c^2\varepsilon^2a_1 + \frac{1}{4}\varepsilon^2\lambda^2a_1 - \beta b_1^3 + 10a_0^2\gamma b_1^3 + 5\gamma a_1b_1^4 = 0,$

$F^{-4}(\xi):\ 5a_0\gamma b_1^4 = 0,$

$F^{-5}(\xi):\ \frac{3}{4}c^2\varepsilon^2b_1 - \frac{3}{4}\varepsilon^2\lambda^2b_1 + \gamma b_1^5 = 0.$

Solving the set of algebraic equations, we get

$$a_0 = 0, \quad a_1 = \pm\frac{(\alpha\lambda^2\varepsilon^2 - \alpha c^2\varepsilon^2)^{\frac{3}{4}}}{\sqrt{2\beta}(c^2-\lambda^2)}, \quad b_1 = \mp\frac{\sqrt{2}(\alpha\lambda^2\varepsilon^2 - \alpha c^2\varepsilon^2)^{\frac{1}{4}}}{\sqrt{\beta}}, \tag{3.19}$$

$$A = \frac{\alpha}{8\varepsilon(\lambda^2-c^2)}, \quad \gamma = \frac{3\beta^2}{16\alpha}, \tag{3.20}$$

or

$$a_0 = 0, \quad a_1 = \pm\frac{2(\alpha\lambda^2\varepsilon^2 - \alpha c^2\varepsilon^2)^{\frac{3}{4}}}{5^{\frac{3}{4}}(c^2-\lambda^2)}, \quad b_1 = \mp\frac{2(\alpha\lambda^2\varepsilon^2 - \alpha c^2\varepsilon^2)^{\frac{1}{4}}}{5^{\frac{1}{4}}\sqrt{\beta}}, \tag{3.21}$$

$$A = \frac{\alpha}{10\varepsilon(c^2-\lambda^2)}, \quad \gamma = \frac{15\beta^2}{64\alpha}, \tag{3.22}$$

where c is an arbitrary constant.

When $\varepsilon = -1$, from Eqs. (2.3), (3.16), (3.18)-(3.20) we obtain the hyperbolic function solution of Eq. (3.15):

$$u_{1\mp}(x,t) = \mp\frac{\sqrt{\alpha}}{\sqrt{\beta}}\left\{\tanh\left(\frac{1}{2}\sqrt{\frac{\alpha}{\lambda^2-c^2}}(x+ct)\right)\right\}^{\frac{1}{2}}$$
$$\mp\frac{\sqrt{\alpha}}{\sqrt{\beta}}\left\{\coth\left(\frac{1}{2}\sqrt{\frac{\alpha}{\lambda^2-c^2}}(x+ct)\right)\right\}^{\frac{1}{2}}. \tag{3.23}$$

When $\varepsilon = -1$, from Eqs. (2.5), (3.16), (3.18)-(3.20) we obtain the hyperbolic function solution of Eq. (3.15):

$$u_{2\mp}(x,t) = \mp\frac{\sqrt{\alpha}}{\sqrt{\beta}}\left\{\tanh\left(\sqrt{\frac{\alpha}{\lambda^2-c^2}}(x+ct)\right) \pm \text{isech}\left(\sqrt{\frac{\alpha}{\lambda^2-c^2}}(x+ct)\right)\right\}^{\frac{1}{2}}$$

$$\mp\frac{\sqrt{\alpha}}{\sqrt{\beta}}\left\{\tanh\left(\sqrt{\frac{\alpha}{\lambda^2-c^2}}(x+ct)\right) \pm \text{isech}\left(\sqrt{\frac{\alpha}{\lambda^2-c^2}}(x+ct)\right)\right\}^{-\frac{1}{2}}. \qquad (3.24)$$

When $\varepsilon = -1$, from Eqs. (2.6), (3.16), (3.18)-(3.20) we obtain the hyperbolic function solution of Eq. (3.15):

$$u_{2\mp}(x,t) = \mp\frac{\sqrt{\alpha}}{\sqrt{\beta}}\left\{\frac{\sqrt{2}\tanh\left(\sqrt{\frac{\alpha}{\lambda^2-c^2}}(x+ct)\right) \pm \text{isech}\left(\sqrt{\frac{\alpha}{\lambda^2-c^2}}(x+ct)\right)}{\sqrt{2}-\text{sech}\left(\sqrt{\frac{\alpha}{\lambda^2-c^2}}(x+ct)\right)}\right\}^{\frac{1}{2}}$$

$$\mp\frac{\sqrt{\alpha}}{\sqrt{\beta}}\left\{\frac{\sqrt{2}\tanh\left(\sqrt{\frac{\alpha}{\lambda^2-c^2}}(x+ct)\right) \pm \text{isech}\left(\sqrt{\frac{\alpha}{\lambda^2-c^2}}(x+ct)\right)}{\sqrt{2}-\text{sech}\left(\sqrt{\frac{\alpha}{\lambda^2-c^2}}(x+ct)\right)}\right\}^{-\frac{1}{2}}. \qquad (3.25)$$

When $\varepsilon = 1$, from Eqs. (2.8), (3.16), (3.18) (3.21)and (3.22) we obtain the trigonometric function solution of Eq. (3.15):

$$u_{4\mp}(x,t) = \mp\frac{2\sqrt{\alpha}}{\sqrt{5\beta}}\left\{\tan\left(\sqrt{\frac{\alpha}{5(c^2-\lambda^2)}}(x+ct)\right)\right\}^{\frac{1}{2}}$$

$$\mp\frac{2\sqrt{\alpha}}{\sqrt{5\beta}}\left\{\cot\left(\sqrt{\frac{\alpha}{5(c^2-\lambda^2)}}(x+ct)\right)\right\}^{\frac{1}{2}}. \qquad (3.26)$$

When $\varepsilon = 0$, from the set of algebraic equations obtained in the process of computation we derive $a_1 = b_1 = 0$, therefore, Eq. (3.15) has not rational solution.

Remark 3.2. To the best of our knowledge, solutions (3.24) and (3.25) are new and they have not been reported in literature.

Remark 3.3. It is obvious that all known solutions in [12] of Eqs. (3.2) and (3.15) can be recovered from the solutions obtained above by setting the corresponding parameters as special cases. It shows that the method developed in this paper is more powerful than Geng and San's method [12]. With the aid of Mathematica, we have checked all the solutions obtained in this paper by putting them back into the original Eqs. (3.2) and (3.15), respectively.

4. CONCLUSIONS

In this paper, some special solutions of a generalized Riccati equation including an arbitrary positive power have been reported. Based on the generalized Riccati equation and its special solutions, new exact travelling wave solutions of the generalized Pochhammer-Chree equation and the Klein-Gordon equation have been obtained. Some of the obtained solutions are unbounded and they develop a singularity at a finite point, i.e., for any fixed $t = t_0$, there always exist x_0 at which these solutions blow up. There is much current interest in the so-called "blow-up" phenomena [17]. It appears that these singular solutions will model the physical phenomena. The paper shows that the generalized Riccati equation method is direct, effective and can be used for many other nonlinear evolution equations in mathematical physics.

REFERENCES

[1] M. J. Ablowitz and P. A. Clarkson: *Soliton, Nonlinear Evolution Equations and Inverse Scattering*, Cambridge University Press, New York, 1991.

[2] I. L. Bogolubsky: *Some examples of inelastic soliton interaction*, Comput. Phys. Commun., **13**(1977), 149-155.

[3] P. A. Clarkson, R. J. Leveque and R. Saxton: *Solitary-wave interaction in elastic rods*, Stud. Appl. Math., **75**(1986), 95-122.

[4] D. B. Duncan: *Symplectic finite difference approximations of the nonlinear Klein-Gordon equation*, SIAM J. Numer. Anal., **34**(1997), 1742-1760.

[5] T. Geng and W. R. Shan: *A new application of Riccati equation to some nonlinear evolution equations*, Phys. Lett. A, **372**(2008), 1626-1630.

[6] J. H. He and X. H. Wu: *Exp-function method for nonlinear wave equations*, Chaos Solitons Fractals, **30**(2006), 700-708.

[7] R. Hirota: *Exact solution of the Korteweg-de Vries equation for multiple collisions of solitons*, Phys. Rev. Lett., **27**(1971), 1192-1194.

[8] W. Malfliet: *Solitary wave solutions of nonlinear wave equations*, Am. J. Phys., **60**(1992), 650-654.

[9] M. R. Miurs: *Bäcklund Transformation*, Springer, Berlin, 1978.

[10] E. J. Parkes, B. R. Duffy and P. C. Abbott, *Jacobi elliptic function rational expansion method with symbolic computation to construct new doubly-periodic solutions of nonlinear evolution equations*, Phys. Lett. A, **295**(2002), 280-286.

[11] Y. Z. Peng: *Exact travelling wave solutions for the Zakharov-Kuznetsov equation*, Appl. Math. Comput., **199**(2008), 397-405.

[12] M. L. Wang: *Exact solution for a compound KdV-Burgers equations*, Phys. Lett. A, **213**(1996), 279-287.

[13] A. M. Wazwaz: *Compactons,solitons and periodic solutions for some forms of nonlinear Klein-Gordon equations*, Chaos Solitons Fractals, **28**(2006), 1005-1013.

[14] E. Yomba: *The modified extended Fan sub-equation method and its application to the (2+1)-dimensional Broer-Kaup-Kupershmidt equation*, Chaos Solitons Fractals, **27**(2006), 187-196.

[15] S. Zhang: *Explicit and exact non-travelling wave solutions for Konoplchenko-Dubrovsky equations*, Z. Naturforsch. A **62**(2007), 689-697.

[16] S. Zhang and T. C. Xia: *A generalized F-expansion method and new exact solutions of Konopelchenko-Dubrovsky equations*, Appl. Math. Comput., **183**(2006), 1190-1200.

[17] S. Zhang and T. C. Xia: *A generalized auxiliary equation method and its application to* (2+1)*-dimensional asymmetric Nizhnik-Novikov-Vesselov equations*, J. Phys. A: Math. Theor., **40**(2007), 227-248.

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