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REGULAR PERMUTATIONS OF QUASIGROUPS

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Invited paper to celebrate Professor Constantin Udrişte,
on the occasion of his seventies

Abstract. The right or left nuclei of loops are subgroups measuring their near-associativity. V. D. Belousov generalized these to proper quasigroups by introduction the notions of right and left regular permutations. The aim of this paper is to investigate extensions of quasigroups such that the orbits of the groups of right or left regular permutations are contained in the kernels of the homomorphism associated with the extension.

1. INTRODUCTION

If a quasigroup has non-empty right or left nucleus, then it has right or left unit element, respectively. V. D. Belousov introduced the notions of right and left regular permutations which can be used for the measure of the near-associativity of quasigroups having neither right nor left unit element. In the case if the quasigroup has right or left unit element, then the right or left regular permutations coincide with right or left translations by elements of the right or left nuclei, respectively. Hence the notions of regular permutations can be considered as natural generalizations of the notions of nuclei.

For the investigation of groups of right or left regular permutations we use the methods of extension theory. Schreier-type loop extensions of groups by loops were investigated by P. T. Nagy and K. Strambach in [3]. Their methods were applied to combinatorial structures by K. Strambach and I. Stuhl in [6]. In [4] P. T. Nagy and I. Stuhl investigated nuclear extensions of quasigroups having left or right unit element. In this paper we investigate quasigroup extensions having empty nuclei and describe their groups of left or right regular permutations. We give conditions under which the orbits of the groups of right or left regular permutations are contained in the kernels of the homomorphism associated with the extension. This construction alerts us quasigroups with prescribed groups of right or left regular permutations of different sizes.

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2. PRELIMINARIES

A *quasigroup* $(Q,\cdot)$ is a set Q with a binary operation $(x,y)\mapsto x\cdot y$, where the equations $a\cdot y=b$ and $x\cdot a=b$ have precisely one solution in Q which we denote by $y=a\backslash b$ and $x=b/a$. The element $e_{a_l}=a/a$ ($e_{a_r}=a\backslash a$) is the *left (right) local unit element* of the element a. If the left (right) local unit elements coincide for all elements of $(Q,\cdot)$, then $e_l=e_{a_l}$ ($e_r=e_{a_r}$) is called the *left (right) unit* of $(Q,\cdot)$. If a quasigroup $(Q,\cdot)$ has left and right unit elements, then they coincide $e=e_l=e_r$ and $(Q,\cdot)$ is called a *loop*. For any $a\in Q$ the maps $\lambda_a\colon y\mapsto a\cdot y$ and $\rho_a\colon y\mapsto y\cdot a$ are the *left* and the *right translations*, respectively. The left translations of $(Q,\cdot)$ generate the group $G_l(Q)$, the right translations of $(Q,\cdot)$ generate the group $G_r(Q)$.

The *left nucleus* $N_l(Q)$ and the *right nucleus* $N_r(Q)$ of a quasigroup $(Q,\cdot)$ are the subgroups defined by

$$N_l\ (Q)=\{n\colon nx\cdot y=n\cdot xy,\ x,y\in Q\},$$
$$N_r\ (Q)=\{n\colon x\cdot yn=xy\cdot n,\ x,y\in Q\},$$

respectively.

A quasigroup $(Q,\cdot)$ has the *left inverse property*, the *right inverse property* or the *cross inverse* property, if for any $x\in Q$ there exists an element x^{-1} such that ${\lambda_x}^{-1}=\lambda_{x^{-1}}$, ${\rho_x}^{-1}=\rho_{x^{-1}}$ or ${\lambda_x}^{-1}=\rho_{x^{-1}}$, respectively. A quasigroup $(Q,\cdot)$ is called *idempotent*, if all its elements are idempotent.

A bijection $\lambda\colon Q\to Q$ is a *left-regular permutation* or *right-regular permutation* of $(Q,\cdot)$, if for all $x,y\in Q$ one has $\lambda(xy)=\lambda(x)\cdot y$ or $\rho(xy)=x\cdot\rho(y)$, respectively. If λ is a left-regular permutation then $\lambda=\lambda_{\lambda(x)}\lambda_x^{-1}$ for all $x\in Q$. Similarly, if ρ is a right-regular permutation then $\rho=\rho_{\rho(x)}\rho_x^{-1}$ for all $x\in Q$. Hence the left-regular permutations form a subgroup $\Lambda(Q)$ of the group $G_l(Q)$, and the right-regular permutations form a subgroup $\mathrm{R}(Q)$ of the group $G_r(Q)$. If $\Lambda(Q)$ or $\mathrm{R}(Q)$ consists only the identity map of Q, then we say that the group of left-regular permutations or right-regular permutations is *trivial*.

In a quasigroup $(Q,\cdot)$ with left unit element a left translation λ_n, $n\in Q$, is left-regular if and only if $n\in N_l(Q)$ and hence $\Lambda(Q)$ is isomorphic to the left nucleus $N_l(Q)$. Analogously, in a quasigroup $(Q,\cdot)$ with right unit element a right translation ρ_n, $n\in Q$, is right-regular if and only if $n\in N_r(Q)$ and hence $\mathrm{R}(Q)$ is isomorphic to the right nucleus $N_r(Q)$ (cf. [1] p. 22-25 or [5] Chap. III, Sec. 3, p. 71-73).

3. RIGHT-REGULAR PERMUTATIONS OF F-EXTENSIONS

Let $(K,\cdot)$, $(Q,\cdot)$ be quasigroups and let $f\colon K\times K\to Q$ be a function. The operation

$$(a,\alpha)\circ(b,\beta)\colon=(ab,f(a,b)\cdot\alpha\beta)\,,\quad a,b\in K,\ \alpha,\beta\in Q \tag{3.1}$$

defines a quasigroup on the set $K\times Q=\{(a,\alpha),a\in K,\alpha\in Q\}$ which will be called the f-extension of $(Q,\cdot)$ by $(K,\cdot)$ and will be denoted by $(\mathfrak{Q}_f,\circ)$. f-extensions of loops and quasigroups are investigated in [4], [6] and [3].

Let $\varphi\colon \mathfrak{Q}_f \to K\colon (a,\alpha) \mapsto a$ be the canonical homomorphism of the f-extension $((\mathfrak{Q}_f,\circ)$. The kernel θ of $\varphi\colon \mathfrak{Q}_f \to K : (a,\alpha) \mapsto a$ is a normal equivalence relation on $\mathfrak{Q}_f$ given by

$$(a,\alpha)\theta(b,\beta) \Longleftrightarrow \varphi(a,\alpha) = \varphi(b,\beta).$$

The equivalence classes have the shape $\{(a,\alpha);\alpha \in Q\}$ for $a \in K$. We call θ *the equivalence relation of the extension* and the equivalence classes of θ the *equivalence classes of the extension.* If the quasigroup K is idempotent, then these equivalence classes are normal subquasigroups of $(\mathfrak{Q}_f,\circ)$, which are called *kernel quasigroups* of the extension.

Lemma 3.1. *A bijection $\rho\colon (x,\xi) \mapsto (\rho_1(x,\xi),\rho_2(x,\xi)) : K \times Q \to K \times Q$ is a right-regular permutation of an f-extension $(\mathfrak{Q}_f,\circ)$ if and only if*

(i) *ρ_1 is constant on the equivalence classes of the extension and induces a right-regular permutation $\rho_1\colon K \to K$ of the quasigroup $(K,\cdot)$,*

(ii) *ρ_2 satisfies*

$$\rho_2(xy, f(x,y)\cdot\xi\eta) = f(x,\rho_1(y,\eta))\cdot\xi\rho_2(y,\eta)$$

for all $x,y \in K,\ \xi,\eta \in Q$.

Proof. If $(x,\xi) \mapsto (\rho_1(x,\xi),\rho_2(x,\xi))$ is a right-regular permutation, then

$$\begin{aligned}\rho\left((x,\xi)\circ(y,\eta)\right) &= (\rho_1(xy,f(x,y)\cdot\xi\eta),\rho_2(xy,f(x,y)\cdot\xi\eta)) = (x,\xi)\circ\rho(y,\eta)\\ &= (x\rho_1(y,\eta), f(x,\rho_1(y,\eta))\cdot\xi\rho_2(y,\eta)).\end{aligned}$$

It follows that that the condition (ii) is satisfied and $\rho_1(xy,f(x,y)\cdot\xi\eta) = x\rho_1(y,\eta)$ for all $x,y \in K,\ \xi,\eta \in Q$. Consequently $\rho_1(xy,f(x,y)\cdot\xi\eta)$ is independent on ξ, hence ρ_1 is constant on the equivalence classes of the extension and the induced map $\rho_1 : K \to K$ satisfies $\rho_1(xy) = x\rho_1(y)$. ☐

3.1. ***f*-extensions by quasigroups $(K,\cdot)$ with trivial $\mathbf{R}(K)$.** The following assertion follows from the previous theorem:

Remark 3.1. If the group of right-regular permutations of a quasigroup $(K,\cdot)$ is trivial, then the orbits of the group of right-regular permutations of the f-extension $(\mathfrak{Q}_f,\circ)$ are contained in the congruence classes of the extension.

This result motivates the investigation of extensions by quasigroups which have trivial group of right-regular permutations. Quasigroups with trivial right-regular permutation groups form a wide class. For example, idempotent quasigroups have this property, since if a quasigroup $(K,\cdot)$ is idempotent and a map $\phi\colon K \to K$ satisfies $x\phi(x) = \phi(x^2)$ for any $x \in K$, then $x\phi(x) = \phi(x)^2$ and hence $\phi(x) = x$. Many constructions of idempotent quasigroups are given in [2], Sections 9 and 10 by the study of the core of Bol loops.

Theorem 3.1. *Assume that the group of right-regular permutations of the quasigroup $(K,\cdot)$ is trivial and $(Q,\cdot)$ is a loop. A map $\rho\colon \mathfrak{Q}_f \to \mathfrak{Q}_f$ is a right-regular permutation of the f-extension $(\mathfrak{Q}_f,\circ)$ if and only if it has the shape $\rho = (id,\rho_\nu)$, where $\nu \in N_r(Q)$.*

Proof. Using Lemma 3.1 we prove that a map $\rho_2\colon K\times Q\to Q$ satisfying the identity

$$\rho_2(xy, f(x,y)\cdot\xi\eta) = f(x,y)\cdot\xi\rho_2(y,\eta)$$

is a right translation of $(Q,\cdot)$ by an element $\nu\in N_r(Q)$. Putting $\xi=\varepsilon$ into this identity, where ε is the unit element of $(Q,\cdot)$, we get

$$\rho_2(xy, f(x,y)\eta) = f(x,y)\rho_2(y,\eta). \tag{3.2}$$

It follows, that

$$f(x,y)\cdot\xi\rho_2(y,\eta) = \rho_2(xy, f(x,y)\cdot\xi\eta) = f(x,y)\cdot\rho_2(y,\xi\eta),$$

and hence $\rho_2(y,\xi\eta)=\xi\rho_2(y,\eta)$ for any $y\in K$. Applying this to equation (3.2) we obtain

$$f(x,y)\rho_2(xy,\eta) = \rho_2(xy, f(x,y)\eta) = f(x,y)\rho_2(y,\eta).$$

Hence the map $(x,\xi)\mapsto\rho_2(x,\xi)$ does not depend on $x\in K$ and $\xi\mapsto\rho_2(\xi)$ is a right-regular permutation of $(Q,\cdot)$. Consequently ρ_2 is a right translation ρ_ν of $(Q,\cdot)$, where $\nu\in N_r(Q)$. Conversely, it is clear, that for any $\nu\in N_r(Q)$ the map $\rho=(id,\rho_\nu)$ is a right regular permutation of $(\mathfrak{Q}_f,\circ)$. □

Corollary 3.1. *The group of the right-regular permutations of the quasigroup $(\mathfrak{Q}_f,\circ)$ is isomorphic with the right nucleus of $(Q,\cdot)$.*

Corollary 3.2. *If the right nucleus $N_r(Q)$ is a normal subgroup, then the equivalence relation induced by the orbits of the right-regular permutation group $R(\mathfrak{Q}_f)$ is a normal congruence.*

Corollary 3.3. *If $(K,\cdot)$ is an idempotent quasigroup and the right nucleus of $(Q,\cdot)$ is a normal subgroup, then the orbits of the right-regular permutation group are normal subquasigroups of the kernel quasigroups of the extension.*

4. LEFT-REGULAR PERMUTATIONS OF F-EXTENSIONS

Similarly to the right regular case we have

Lemma 4.1. *A bijection $\lambda\colon(x,\xi)\mapsto(\lambda_1(x,\xi),\lambda_2(x,\xi)) : K\times Q\to K\times Q$ is a left-regular permutation of an f-extension $(\mathfrak{Q}_f,\circ)$ if and only if*

(i) *λ_1 is constant on the equivalence classes of the extension and induces a left-regular permutation $\lambda_1\colon K\to K$ of the quasigroup $(K,\cdot)$,*

(ii) *λ_2 satisfies*

$$\lambda_2(xy, f(x,y)\cdot\xi\eta) = f(\lambda_1(x,\xi),y)\cdot\lambda_2(x,\xi)\eta$$

for all $x,y\in K,\ \xi,\eta\in Q$.

If the group of left-regular permutations of the quasigroup $(K,\cdot)$ is trivial, then the orbits of the group of left-regular permutations are contained in the congruence classes of the extension.

Theorem 4.1. *Assume that the group of left-regular permutations of the quasigroup* $(K,\cdot)$ *is trivial and* $(Q,\cdot)$ *is a loop. A map* $\lambda\colon K\times Q\to K\times Q$ *is a left-regular permutation of the* f*-extension* $(\mathfrak{Q}_f,\circ)$ *if and only if it has the shape* $(x,\xi)\mapsto(x,\nu(x)\xi)$*, where* ν *is a mapping* $\nu\colon K\to N_l(Q)$ *satisfying*

$$\nu(xy)f(x,y)=f(x,y)\nu(x) \quad \text{and} \quad \lambda_{f(x,y)\nu(x)}=\lambda_{f(x,y)}\lambda_{\nu(x)} \tag{4.3}$$

for any $x,y\in K$.

Proof. According to Lemma 4.1 the component λ_2 of a left-regular permutation $\lambda\colon\mathfrak{Q}_f\to\mathfrak{Q}_f$ satisfies

$$\lambda_2(xy,f(x,y)\cdot\xi\eta)=f(x,y)\cdot\lambda_2(x,\xi)\eta. \tag{4.4}$$

Putting $\eta=\varepsilon$ we get

$$\lambda_2(xy,f(x,y)\xi)=f(x,y)\lambda_2(x,\xi).$$

Applying this to the previous identity we have

$$f(x,y)\lambda_2(x,\xi\eta)=\lambda_2(xy,f(x,y)\cdot\xi\eta)=f(x,y)\cdot\lambda_2(x,\xi)\eta.$$

Consequently for any $x\in K$ the map $\xi\mapsto\lambda_2(x,\xi)$ is a left-regular permutation of the loop $(Q,\cdot)$, and hence it is a left multiplication by an element $\nu(x)\in N_l(Q)$. Now, equation (4.4) gives

$$\nu(xy)\left(f(x,y)\cdot\xi\eta\right)=f(x,y)\left(\nu(x)\xi\cdot\eta\right).$$

Since $\nu(x)\in N_l(Q)$ we obtain the equivalent identity

$$\nu(xy)\cdot f(x,y)\xi=f(x,y)\cdot\nu(x)\xi. \tag{4.5}$$

With $\xi=\varepsilon$ we obtain the first part of condition (4.3). Using $\nu(xy)\in N_l(Q)$ equation (4.5) implies the second part of (4.3).

Obviously, a map $(x,\xi)\mapsto(x,\nu(x)\xi)$ satisfying $\nu(x)\in N_l(Q)$ and condition (4.3) for any $x,y\in K$ is a left-regular permutation. □

Let F denote the subloop of $(Q,\cdot)$ the element of which commute with all $f(x,y)$; $x,y\in K$. The previous theorem yields the following

Corollary 4.1. *The map* $\lambda=(id,\lambda_\nu)$ *is a left regular permutation of the* f*-extension* $(\mathfrak{Q}_f,\circ)$ *if* $\nu\in N_l(Q)\cap F$ *and* $\lambda_{f(x,y)\nu}=\lambda_{f(x,y)}\lambda_\nu$ *for any* $x,y\in K$.

Theorem 4.2. *Assume that the group of left-regular permutations of the quasigroup* $(K,\cdot)$ *is trivial,* $(Q,\cdot)$ *is a loop and there exists* $k\in K$ *such that* $f(k,y)=\kappa$ *is constant for any* $y\in K$ *with* $\kappa\in F$*. A map* $\lambda\colon K\times Q\to K\times Q$ *is a left-regular permutation of the* f*-extension* $(\mathfrak{Q}_f,\circ)$ *if and only if* $\lambda=(id,\lambda_\nu)$*, where* $\nu\in N_l(Q)\cap F$ *and* $\lambda_{f(x,y)\nu}=\lambda_{f(x,y)}\lambda_\nu$ *for any* $x,y\in K$.

Proof. We prove the first condition in (4.3) putting $x=k$ and $y=k\backslash t$ in the first condition in (4.3) one has $\nu(t)f(k,k\backslash t)=f(k,k\backslash t)\nu(k)$ for any $t\in K$ and we get that $\nu(t)=\nu(k)$ is constant for all $t\in K$. The element $\nu(k)$ has to commute with any element of F, hence $\nu(k)\in N_l(Q)\cap F$. The second condition in (4.3) imlies $\lambda_{f(x,y)\nu}=\lambda_{f(x,y)}\lambda_\nu$ for any $x,y\in K$. Conversely, any element $\nu\in N_l(Q)\cap F$

satisfying $\lambda_{f(x,y)\nu} = \lambda_{f(x,y)}\lambda_\nu$ for any $x, y \in K$ fulfils the conditions (4.3) and hence $\lambda = (id, \lambda_\nu)$ is a left-regular permutation of the f-extension $(\mathfrak{Q}_f, \circ)$. ☐

Theorem 4.3. *Assume that the group of left-regular permutations of the quasigroup* $(K, \cdot)$ *is trivial and the loop* $(Q, \cdot)$ *has the cross inverse property. A map* $\lambda\colon K \times Q \to K \times Q$ *is a left-regular permutation of the* f*-extension* $(\mathfrak{Q}_f, \circ)$ *if and only if* $\lambda = (id, \lambda_\nu)$*, where* $\nu \in N_l(Q)$ *and* $\lambda_{f(x,y)\nu} = \lambda_{f(x,y)}\lambda_\nu$ *for any* $x, y \in K$*.*

Proof. We multiply from the right the first identity of (4.3) with the cross inverse $f(x,y)^{-1}$ of the element $f(x,y)$. Since $\nu(x) \in N_l(Q)$ for any $x \in K$ we obtain

$$\begin{aligned} \nu(xy) &= \nu(xy) \cdot f(x,y)f(x,y)^{-1} = \nu(xy)f(x,y) \cdot f(x,y)^{-1} \\ &= f(x,y)\nu(x) \cdot f(x,y)^{-1} = \nu(x), \end{aligned}$$

i. e. $\nu : K \to Q$ is a constant function and hence the assertion is proved. ☐

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