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ON THE JACKSON NETWORKS

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ABSTRACT. In this note, we consider a network with Poisson distributed flows and exponentially distributed durations of services.The main contribution is a new proof of a classical formula due to Jackson, regrading the probability of a stationary network states.

1. OPEN NETWORKS OF FLOWS

There are many theoretical models for waiting and service lines, with or without losses, with different service disciplines, cyclic service lines etc. In this respect, one crucial step was the proliferation of broadly connected networks - communication networks, Internet, industrial processing networks.

A *nework* is an attributed graph $\mathcal{R} = (N, A, \Phi, \delta)$ with $N =$ the set of nodes, $A =$ the edges set, $\Phi =$ the data flow (messages, packets, commands, information etc), transmitteders the edges and $\delta =$ transmission discipline and facility process for servers on edges and nodes. The main challenge for any network is to define a random process that specifies the full operation and allows the estimation of some standard measures including comutation moments as well as behavior on equilibrium, stationary and ergodic. The networks are *open* if the source for data flow Φ is internal and external or they can be *closed* if the source for Φ is only internal.

As opposed to standard waiting lines, the network in Figure 1 below contains *the flow decomposition* (e.g. node n_1 with probability p), *flow composition* (node n_3), *flow sequences* (sequence $n_4 \to n_5 \to n_6$) etc.

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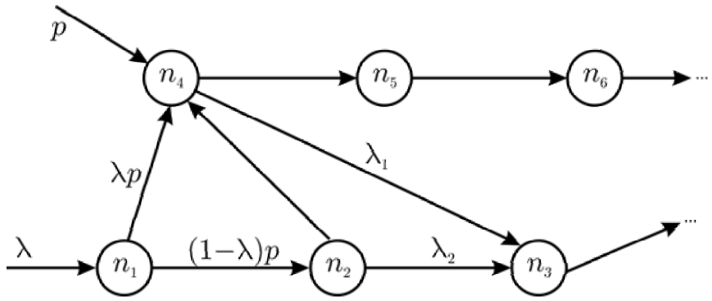

Figure 1. A typical network

The analysis of this network is complex; some progress was done by using Poisson flows with exponentially distributed serving durations. This is the basis for the Jackson networks that will be considered in the rest of the paper.

A broad diversity of *protocols* is available for the network access. A protocol is a set of standard logical conventions for meaningful communication. All protocols have specific requirements, standardized form and specifications for all communication between network entities, communication code, service protocols etc.

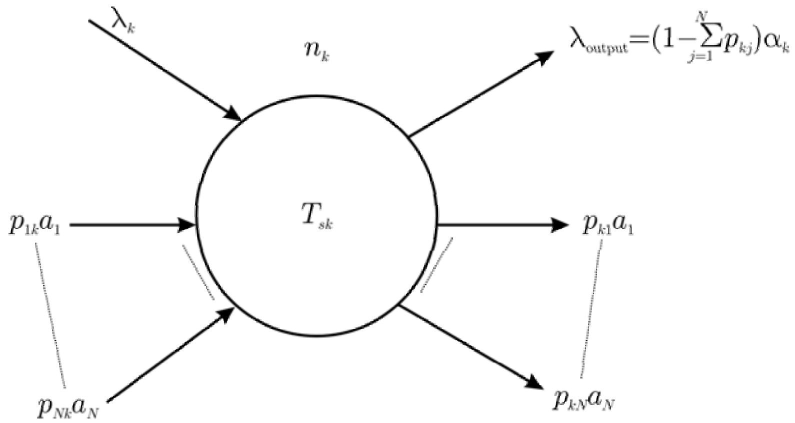

Figure 2. A typical Poisson node in a network

Given an open network with N nodes $n_1, n_2, \ldots, n_N$, with exponentially distributed durations of service; n_k contains s_k servers, each node has serving time T_{sk} $(1 \le k \le N)$. Messages (including packets, clients) are flowing from exterior to each node n_k in Poisson flux with the average rate λ_k messages/second; messages can also originate from other internal nodes, they are processed in n_k and they flow instantaneously to node j $(1 \le j \le N)$ with probability p_{kj}, or leave the network with probability $1 - \sum_{j=1}^{N} p_{kj}$. Suppose that the service discipline is *First In First Out* (FIFO). We denote by α_k the total traffic rate for the node n_k, which includes the internal and external data flow (Figure 2).

The flow in this node can be represent as a linear system of balance equations with unknowns α_k:

$$\alpha_k = \lambda_k + \sum_{j=1}^{N} p_{kj}\alpha_k, \quad 1 \leq k \leq N. \tag{1.1}$$

In matriceal form, the equivalent equation is

$$A = \Lambda + A \cdot P \quad \text{or} \quad A \cdot (I_N - P) = \Lambda, \tag{1.1$'$}$$

where $A = (\alpha_1, \ldots, \alpha_N)$, $\Lambda = (\lambda_1, \ldots, \lambda_N)$ are vectors and $P = (p_{ij})$, $1 \leq i, j \leq N$ the square matrix $N \times N$ with the above mentioned probabilities.

2. JACKSON NETWORKS

Jackson indicated in [1] different cases when $\lim_{n\to\infty} P^n = 0$.

Lemma 2.1. *The equation* (1.1′) *has a unique solution* A *for any given* λ.

Proof. If α is an eigen value of P, with the eigen vector x $(x \neq 0)$, then $Px = \alpha x$, hence $P^n x = \alpha^n x$. Therefore $\lim_{x\to\infty} \alpha^n = 0$, hence $|\alpha| < 1$. Then the matrix $I_N - P$ has its eigen values different from O. As a consequence, the matrix $I_N - P$ is nonsingular and finally, $A = \lambda(I_N - P)^{-1}$. □

Thus, the equations of balance (1.1) have a unique solution for any traffic vector Λ, which represents the network throughput. Jackson proved that in such a network, every node is an independent multi-server with the Poisson entry (α_k) and much more, any node can be independently analysed. Then, the results can be extended to global network by using standard statistical methods. A similar extension process can be applied for congestions in nodes. The rigorous formulation of the Jackson theorem was the following: "Denote at any moment t, the network state as begin the random vector. $C(t) = (c_1(t), \ldots, c_n(t))$ where $c_i(t) =$ the number of messages in node i at time t (including those in service). The family $\{C(t)\}$, $t \geq 0$ form a stationary Markov process and any network line is an *multi-server independent* $(M/M/s)$. Furthermore, if $\lim_{n\to\infty} P^n = 0$ then the probability $P(C)$ of stationary network in the state $C = (c_1, c_2, \ldots, c_N)$ has the product form:

$$P(C) = \psi_1(c_1) \cdot \psi_2(c_2) \cdots \psi_N(c_N), \tag{$*$}$$

where $\psi_k(c_k) =$ the probability of c_k messages on a line $M/M/s_k$ with s_k the number of servers in node k $(1 \leq k \leq N)$."

In the simplified case of one server in each node ($s_k = 1$, with serving rate μ_k), $\psi_k(c_k) = \rho_k^{c_k}(1 - \rho_k)$ and $\rho_k = \alpha_k/\mu_k$, for $1 \leq k \leq N$.

Example 2.1. A feedback system containing one server with exponentially distributed serving duration and r traffic rate, $0 < r < 1$, is shown in Figure 3.

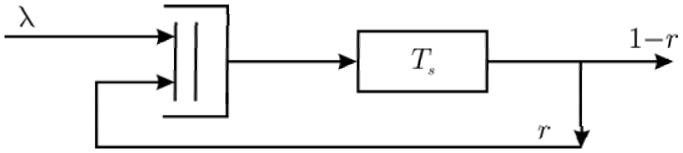

Figure 3. A feedback system with one server

The rate of the system α is given by the balance equation $\alpha = \lambda + r\alpha$. So $\alpha = \dfrac{\lambda}{1-r}$ and the server utilization coefficient is $\rho = \alpha \cdot T_s = \dfrac{\lambda \cdot T_s}{1-r}$ and $T_q = \dfrac{T_s}{1-\rho} = \dfrac{(1-r)\,T_s}{1-r-\lambda T_s}$.

Note. Many real cases differ from the simplified ideal case from Figure 3, however a similar approach can be applied by extrapolation. One recommends that the average traffic rate is computed by solving linear system even in the case of a non-Poisson traffic. Also, the server utilization coefficient in node k is given by $\rho_k = \dfrac{\alpha_k \cdot T_{sk}}{s_k}$. Starting from these parameters, we can extract the utilization coefficients for different system parts and estimate congestion. The Jackson model is also valid for finite number of simultaneous lines $M/M/s$ in cascade.

There are different generalizations of $(*)$, called probability distributions of states, in *product form.*

3. A NEW PROOF FOR JACKSON FORMULA $(*)$

In order to simplify, consider an open network with N nodes (or quens) $n_1, n_2, \dots,$ n_N, one source S and a destination D (Figure 4). λ is Poisson arrival rate at the source S.

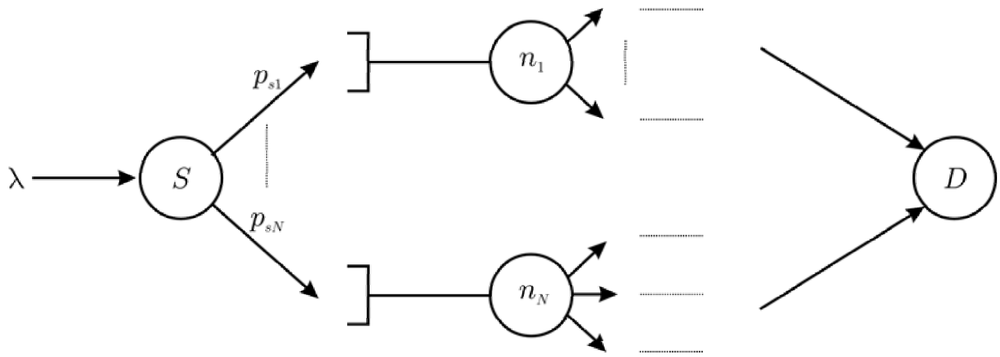

Figure 4. Simple Jackson network

μ_k is the serving rate at each node n_k, $1 \le k \le N$ and p_{kj} is the probability that a message (packet, client etc.) starting from the node k to reach the node j.

Then

$$p_{kd} = 1 - \sum_{j=1}^{N} p_{kj} \quad \text{for } 1 \le k \le N \ (d \equiv \text{ destination}).$$

With α_k the total traffic rate at node k, we obtain the balance equations:

$$\alpha_k = p_{sk} \cdot \lambda + \sum_{j=1}^{N} p_{kj} \cdot \alpha_j, \quad 1 \le k \le N \ (s \equiv \text{ source}).$$

In a stationary regime the goal is to determine probability $P(C)$ of a state $C = (c_1, c_2, \dots, c_N)$, where $c_k =$ the number of messages in n_k.

Starting with the N-dimensional vector $E_k = (0, \dots, \overset{k}{1}, \dots, 0)$, we have

$$C \pm E_k = (c_1, c_2, \dots, c_k \pm 1, \dots, c_N)$$

and $C + E_j - E_k = (c_1, \dots, c_j + 1, \dots, c_k - 1, \dots, c_N)$, therefore

$$\left(\lambda + \sum_{k=1}^{N} \mu_k\right) \cdot P(C) = \lambda \cdot \sum_{k=1}^{N} p_{sk} \cdot P(C - E_k) + \sum_{k=1}^{N} p_{kd} \cdot \mu_k \cdot P(C + E_k) + \sum_{k=1}^{N} \sum_{j=1}^{N} p_{jk} \cdot \mu_j \cdot P(C + E_j - E_k). \tag{3.1}$$

The left side of the equation represents the rate of output from state C which includes arrival rate λ and the departure rate μ_k from node n_k; the right hand side encompasses the total number of lines to reach the state C (for example starting with $C - E_k$, the state C can be reached with node n_k in state $c_k - 1$, after arriving at node n_k; there are two options: external message with rate λp_{sk} or arrival from node n_j with rate $p_{jk}\mu_j$, in latter case node j needs to be in state $c_j + 1$. Also a message can depart from node n_k to destination d, represented by last term on right hand side). In order to solve equation (3.1), we need to explicit p_{sk} and we are left to prove that equation is satisfied if:

$$\alpha_k \cdot P(C - E_k) = \mu_k \cdot P(C) \quad \text{or} \quad \alpha_j \cdot P(C - E_k) = \mu_j \cdot P(C + E_j - E_k). \tag{3.2}$$

After some simple computations, it will follow that

$$\lambda \cdot P(C) = \sum_{k=1}^{N} p_{kd} \cdot \mu_k \cdot P(C + E_k)$$

and using $\alpha_k \cdot P(C) = \mu_k \cdot P(C + E_k)$, we get exactly the flow conservation equations for the flow from the source S to destination D. In fact we have also proved that (3.2) can be used to prove (3.1).

From (3.2) we have $P(C) = (\alpha_k/\mu_k) \cdot P(C - E_k)$ and by looping c_k times, we obtain

$$P(C) = (\alpha_k/\mu_k)^{c_k} \cdot P(C - c_k E_k) = (\alpha_k/\mu_k)^{c_k} \cdot P(c_1, c_2, \dots, \overset{k}{0}, \dots, c_N).$$

Formula below is built by applying the same process for all other nodes, as follows $P(C) = \prod_{k=1}^{N} (\alpha_k/\mu_k)^{c_k} \cdot P(0,0,\ldots,0)$, where the probability $P(0,0,\ldots,0)$ means that no message exists in the network with the normalization condition $\sum_{C} P(C) = 1$. Assuming a finite number of states C,

$$\sum_{C} \prod_{k=1}^{N} (\alpha_k/\mu_k)^{c_k} = \prod_{k=1}^{N} \sum_{c_k=0}^{\infty} (\alpha_k/\mu_k)^{c_k}.$$

We denote $\rho_k = \alpha_k/\mu_k$, then $\sum_{c_k=0}^{\infty} \rho_k^{c_k} = \frac{1}{1-\rho_k}$ (if $\alpha_k < \mu_k$ any k), therefore we obtain $P(0,0,\ldots,0) = \frac{1}{(1-\rho_1)\cdots(1-\rho_k)}$.

So we have proved *Jackson formula* $(*)$ in a slightly simplified, but significant case. Namely,

$$P(C) = \prod_{k=1}^{N} \rho_k^{c_k}(1-\rho_k).$$

This formula provides the global probability for an open network in product form and proves that all N nodes can be considered as an one single server queue $M/M/1$.

4. CONCLUSIONS

The study of the different interconnected networks requires not only the graph theory, but also study of waiting lines (queues), random processes etc. A special kind of networks it the Jackson networks, where the probability of any stationary state has a particular from, called "product-form". In this note, one presents a new proof for the Jackson formula, in a simplified but generic case.

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