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A DOUBLE INEQUALITY INVOLVING $\Gamma(r)^{\frac{1}{r}}$ VIA THE MULTIPLICATION FORMULA OF GAUSS

CRISTINEL MORTICI

Abstract. We prove a new double inequality involving $\Gamma(r)^{1/r}$, using the multiplication formula of Gauss, as a Riemann sum.

1. INTRODUCTION

For a function $f\colon [a,b] \to \mathbb{R}$, we put

$$A_n = \frac{b-a}{n}\sum_{i=1}^{n} f\left(x_i^{(n)}\right) \quad \text{and} \quad B_n = \frac{b-a}{n}\sum_{i=0}^{n-1} f\left(x_i^{(n)}\right), \tag{1.1}$$

where

$$x_i^{(n)} = a + i\frac{b-a}{n} \qquad (i = 0,1,2,\dots,n;\ n = 1,2,3,\dots).$$

It is elegantly proven in [12], Cor. 2.2, that if f is an increasing convex or concave function on $[a,b]$, then

$$A_{n+1} \le A_n \quad \text{and} \quad B_n \le B_{n+1} \quad (n = 1,2,3,\dots). \tag{1.2}$$

In fact, this is a generalization of the following inequality

$$\frac{n}{n+1} \le \left(\frac{\frac{1}{n}\sum\limits_{i=1}^{n} i^r}{\frac{1}{n+1}\sum\limits_{i=1}^{n+1} i^r}\right)^{1/r}, \quad r > 0.$$

due to Alzer [2]. In a word, the Riemann sums of the function x^r is a decreasing sequence. Such inequalities were studied by many authors, see [6], [11], [13].

The aim of this paper is to indicate a concrete application of the inequalities (1.2) to the gamma function and the idea is to apply (1.2) to a function involving

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the gamma function, using the multiplication formula of Gauss, *e.g.,* [1], [3], [5], [14]:

$$\Gamma(na)=\frac{n^{na-\frac{1}{2}}}{\sqrt{2\pi}^{n-1}}\Gamma(a)\Gamma\left(a+\frac{1}{n}\right)\cdots\Gamma\left(a+\frac{n-1}{n}\right). \tag{1.3}$$

A new double inequality involving $(\Gamma(r))^{\frac{1}{r}}$ is established. It is well-known that such inequalities are of great interest, since, for example, they produce bounds for the permanents of $(0,1)$-matrices, see [7]. This technique that uses the multiplication formula (1.3) and Riemann sums was also developed by Mortici [8] to derive the Stirling's formula and to establish other interesting properties about the gamma function. For related advances in numerical analysis , see [9], [10].

2. THE RESULTS

The gamma function is the natural extension from the nonnegative integers to $\mathbb{R}\smallsetminus\{-1,-2,\ldots\}$ (or to $\mathbb{C}\smallsetminus\{-1,-2,\ldots\}$) of the factorial function,

$$\Gamma(x+1)=\int_0^\infty t^x e^{-t}dt=\lim_{n\to\infty}\frac{n^x n!}{(x+1)(x+2)\cdots(x+n)}.$$

By various manipulations, other formulas can be obtained:

$$\Gamma(x+1)=\int_0^1\left(\ln\frac{1}{t}\right)^x dt=\lim_{n\to\infty}\frac{n^x}{\prod\limits_{p=1}^{n}\left(1+\frac{x}{p}\right)}=\prod_{n=1}^{\infty}\frac{\left(1+\frac{1}{n}\right)^x}{\left(1+\frac{x}{n}\right)},$$

or

$$\Gamma(x+1)=e^{-\gamma x}\prod_{n=1}^{\infty}\frac{e^{\frac{x}{n}}}{\left(1+\frac{x}{n}\right)},$$

in terms of the Euler-Mascheroni constant

$$\gamma=\lim_{n\to\infty}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}-\ln n\right)=0.577215664901532860606512\ldots\ .$$

In fact, this definition is natural, if we take into account the famous Bohr-Mollerup-Artin theorem, that states that the gamma function is the unique extension of $n!$, via the log-convex property, see [3], [4].

In what follows, we denote by $\phi=1.461\,632\,145\ldots$ the unique positive root of the equation $\psi(x)=0$, where

$$\psi(x)=\frac{d}{dx}\ln\Gamma(x)=\frac{\Gamma'(x)}{\Gamma(x)}$$

is the psi, or the digamma function. We also have the integral representation:

$$\psi(x) = -\gamma + \int_0^\infty \frac{e^{-t} - e^{-tx}}{1 - e^{-t}} dt.$$

As a direct consequence, the gamma function Γ is strictly increasing on $[\phi, \infty)$. The derivative ψ' is also called the trigamma function. As

$$\psi'(x) = \int_0^\infty \frac{te^{-tx}}{1 - e^{-t}} dt,$$

it results that $\psi' > 0$. For details, see, for example, [1], [15].

Now we are in position to give the following main result:

Theorem 2.1. For every counting number n and for every real $a \in [\phi, \infty)$, the following double inequality holds true:

$$\left(\frac{n}{2\pi}\right)^{\frac{1}{2n(n+1)}} \left(1 + \frac{1}{n}\right)^{a - \frac{1}{2(n+1)}} \leq$$

$$\leq \frac{\sqrt[n+1]{\Gamma((n+1)a)}}{\sqrt[n]{\Gamma(na)}} \leq \left(\frac{na}{2\pi}\right)^{\frac{1}{2n(n+1)}} \left(1 + \frac{1}{n}\right)^{a - \frac{1}{2(n+1)}}. \tag{2.1}$$

Proof. Let us consider the function $f : [0,1] \to \mathbb{R}$ by

$$f(x) = \ln \Gamma(a + x) \ , \quad a \in [\phi, \infty).$$

As $f'(x) = \psi(a + x) > 0$, for every $a + x > \phi = 1.461\,632\,145\ldots$, it results that f is strictly increasing on $[0,1]$. Furthermore, $f''(x) = \psi'(a + x) > 0$, and thus f is convex. Now we can apply the inequalities (1.1) for the function f, where

$$A_n = \frac{1}{n} \sum_{i=1}^{n} \ln \Gamma\left(a + \frac{i}{n}\right) = \frac{1}{n} \ln \sqrt[n]{\prod_{i=1}^{n} \Gamma\left(a + \frac{i}{n}\right)} \tag{2.2}$$

and similarly,

$$B_n = \frac{1}{n} \sum_{i=0}^{n-1} \ln \Gamma\left(a + \frac{i}{n}\right) = \frac{1}{n} \ln \sqrt[n]{\prod_{i=0}^{n-1} \Gamma\left(a + \frac{i}{n}\right)}. \tag{2.3}$$

Using the multiplication formula of Gauss (1.3), (2.2)-(2.3) can be written as

$$A_n = \frac{1}{n} \ln \frac{\sqrt{2\pi}^{n-1} \Gamma(na)}{an^{na - \frac{1}{2}}} = \ln \frac{\sqrt{2\pi}^{\frac{n-1}{n}} \sqrt[n]{\Gamma(na)}}{\sqrt[n]{a} n^{a - \frac{1}{2n}}},$$

respectively

$$B_n = \frac{1}{n} \ln \frac{\sqrt{2\pi}^{n-1} \Gamma(na)}{n^{na - \frac{1}{2}}} = \frac{\sqrt{2\pi}^{\frac{n-1}{n}} \sqrt[n]{\Gamma(na)}}{n^{a - \frac{1}{2n}}}.$$

Hence $A_n \geq A_{n+1}$ becomes equivalent with

$$\frac{\sqrt{2\pi}^{\frac{n-1}{n}}\ \sqrt[n]{\Gamma\left(na\right)}}{\sqrt[n]{a}n^{a-\frac{1}{2n}}} \geq \frac{\sqrt{2\pi}^{\frac{n}{n+1}}\ \sqrt[n+1]{\Gamma\left(\left(n+1\right)a\right)}}{\sqrt[n+1]{a}\left(n+1\right)^{a-\frac{1}{2(n+1)}}} \tag{2.4}$$

while $B_n \leq B_{n+1}$ is

$$\frac{\sqrt{2\pi}^{\frac{n-1}{n}}\ \sqrt[n]{\Gamma\left(na\right)}}{n^{a-\frac{1}{2n}}} \leq \frac{\sqrt{2\pi}^{\frac{n}{n+1}}\ \sqrt[n+1]{\Gamma\left(\left(n+1\right)a\right)}}{\left(n+1\right)^{a-\frac{1}{2(n+1)}}} \tag{2.5}$$

and (2.1) follows straightfoward from (2.4)-(2.5). $\square$

Now we can see that the ratio $\sqrt[2n(n+1)]{a}$ of the extreme sides of the double inequality (2.1) tends rapidly to 1, as n becomes large. This means accurate approximations of the quantity $\dfrac{\sqrt[n+1]{\Gamma\left(\left(n+1\right)a\right)}}{\sqrt[n]{\Gamma\left(na\right)}}$.

We illustrate the performance of (2.1) by the following numerical example, where we denote:

$$\alpha_n = \left(\frac{n}{2\pi}\right)^{\frac{1}{2n(n+1)}}\left(1+\frac{1}{n}\right)^{\phi-\frac{1}{2(n+1)}};$$

$$\rho_n = \frac{\sqrt[n+1]{\Gamma\left(\left(n+1\right)\phi\right)}}{\sqrt[n]{\Gamma\left(n\phi\right)}};$$

$$\beta_n = \left(\frac{n\phi}{2\pi}\right)^{\frac{1}{2n(n+1)}}\left(1+\frac{1}{n}\right)^{\phi-\frac{1}{2(n+1)}},$$

with $\phi = 1.461\,632\,145\ldots$.

n	α_n	ρ_n	β_n
5	1.280 801 772	1.288 032 753	1.288 929 670
10	1.146 929 591	1.148 796 386	1.148 910 032
25	1.059 327 547	1.059 629 586	1.059 636 878
50	1.029 585 927	1.029 661 642	1.029 662 554
100	1.014 739 023	1.014 757 976	1.014 758 090
500	1.002 931 375	1.002 932 134	1.002 932 135
1000	1.001 464 006	1.001 464 195	1.001 464 195
12500	1.000 116 954	1.000 116 955	1.000 116 955

As we can see from this table, the approximation $\rho_n \approx \beta_n$ is better than $\rho_n \approx \alpha_n$. By raising $\rho_n \approx \beta_n$ to the $(1/\phi)$-th power, we obtain

$$\frac{\Gamma\left((n+1)\phi\right)^{\frac{1}{(n+1)\phi}}}{\Gamma(n\phi)^{\frac{1}{n\phi}}} \approx \left(\frac{n\phi}{2\pi}\right)^{\frac{1}{2\phi n(n+1)}} \left(1+\frac{1}{n}\right)^{1-\frac{1}{2\phi(n+1)}}.$$

With the notations $\rho(r) = \Gamma(r)^{\frac{1}{r}}$ and $n\phi = r$, we obtain, for every $r \geq \phi$, the following accurate estimation:

$$\frac{\Gamma(r+\phi)^{\frac{1}{r+\phi}}}{\Gamma(r)^{\frac{1}{r}}} \approx \left(\frac{r}{2\pi}\right)^{\frac{\phi}{2r(r+\phi)}} \left(1+\frac{\phi}{r}\right)^{1-\frac{1}{2(r+\phi)}}.$$

Note that ϕ can be replaced by any step-value $a \geq \phi$.

Some of numerical computations are given in the next table. Even for $n = 850$, the error is smaller than 10^{-10}.

n	$\frac{\Gamma(r+\phi)^{\frac{1}{r+\phi}}}{\Gamma(r)^{\frac{1}{r}}}$	$\left(\frac{r}{2\pi}\right)^{\frac{\phi}{2r(r+\phi)}} \left(1+\frac{\phi}{r}\right)^{1-\frac{1}{2(r+\phi)}}$
10	1.142 516 249	1.142 743 486
25	1.058 928 638	1.058 943 803
100	1.014 745 773	1.014 746 015
250	1.005 877 927	1.005 877 942
500	1.002 933 138	1.002 933 140
850	1.001 723 520	1.001 723 520

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Valahia University
Faculty of Sciences and Arts
Department of Mathematics
Bd. Unirii 18*,* 130082*, Târgovişte, Romania*
E-mail address: `cmortici@valahia.ro`