

## INVEX SETS AND PREINVEX FUNCTIONS

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ABSTRACT. We study necessary and sufficient conditions for the invexity of the sets. Some algebraic operations for the invex sets are given and there are presented the definitions of classes of all types of preinvexity. We conclude that the domains of the preinvex and generalized preinvex functions are invex sets. It is proved that a function is preinvex if its epigraph is an invex set. Operations with preinvex and preincave functions are defined. Definitions of the nonsmooth invex, pseudoinvex and quasiinvex functions on open sets are given. For real functions, the equivalences  $invex \equiv preinvex$ ,  $pseudoinvex \equiv prepseudoinvex$  and  $quasiinvex \equiv prequasiinvex$  are mentioned.

### 1. INTRODUCTION

In 1981 Hanson, [8], introduced the invex, pseudoinvex and quasiinvex functions in the differentiable framework. A differentiable function is invex if and only if every stationary point is a global minimum, [2, 6]. For a quasiinvex function, every local minimum point is a global one. Due to these properties, the (generalized) invex functions are used in optimization theory. Recently Mititelu, [14, 15], proves that a function is pseudoinvex if every stationary point is a global minimum point. In 1986 Craven, [5], defined the Lipschitz nonsmooth invex function. All types of nonsmooth invex and generalized invex functions at a point, grouped on ( $\rho$ -) classes, were introduced by Mititelu and Giorgi, [7], using the upper Dini directional derivative and by Mititelu and Stancu-Minasian, [15], using the Clarke directional derivative. The notion of preinvex function was introduced in 1988 by Weir and Mond, [24], and the quasiinvex function was introduced in 1991 by Pini, [19], and easily rectified by Mohan and Neogy, [18], in 1995. The other types of generalized preinvex functions were introduced by Mititelu, [10], in 1977. Mititelu, [10, 15], showed that for a (strictly) preinvex, (strictly) prepseudoinvex or (strictly) precvasiinvex function any (strictly) local minimum point is a (strict) global minimum point. Due to these properties the preinvex functions are used in optimization (for illustrations, see [16] by Mititelu, [20], [21], by Pitea, Udriște and Mititelu and [22] by Pitea). The notion of invex set was introduced by Mititelu, [9], in 1994. In this paper we establish new properties of the invex sets and of the preinvex functions on invex sets. There are given the

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definitions of the invex, pseudoinvex and quasiinvex functions on open invex sets and concerning the real functions, the equivalences  $invex \equiv preinvex$ ,  $pseudoinvex \equiv prepseudoinvex$  and  $quasiinvex \equiv prequasiinvex$  are mentioned.

## 2. INVEX SETS

Let  $X$  be a nonempty set in  $\mathbb{R}^n$  and a mapping  $\eta: X \times X \rightarrow \mathbb{R}^n$ . Pini (1991), [18], named  $X$  an  $\eta$ -invex set if for any,  $x, u \in X$  the segment  $[u, u + \eta(x, u)]$  is contained by  $X$ .

A weaker definition of the invexity of a set is the following:

**Definition 2.1** (Mititelu (1994), [10]). 1) The nonempty set  $X \subseteq \mathbb{R}^n$  is said to be *invex* at the point  $u \in X$  if there exists a vector function  $\eta: X \times X \rightarrow \mathbb{R}^n$  such that

$$\forall x \in X, \forall \lambda \in [0, 1] \Rightarrow u + \lambda\eta(x, u) \in X. \quad (2.1)$$

2) The set  $X$  is *invex* if it is invex at each of its points.

The case  $\eta = 0$  is excluded, when  $X$  is a trivial invex set. If  $X$  is invex with respect to  $\eta$  then,  $X$  is called  $\eta$ -invex.

In general, a set cannot be  $\eta$ -invex (with  $\eta$  previous given), but it can be a  $\eta_1$ -invex set, where  $\eta \neq \eta_1$  are mappings from  $X \times X$  to  $\mathbb{R}^n$ .

**Example 2.1.** The set  $M = (0, \infty)$ :

a) is invex with respect to the function  $\eta: M \times M \rightarrow \mathbb{R}$ ,  $\eta(x, u) = x + u$ ;

b) is not invex at  $u = 1$  with respect to the function  $\eta(x, u) = -xu - 25$ .

Indeed:

a)  $\forall u > 0, \forall \lambda \in [0, 1] \Rightarrow u + \lambda\eta(x, u) = u + \lambda(x + u) > 0, \forall x \in M$ .

b)  $\forall \lambda \in [0, 1] \Rightarrow 1 + \lambda(-xu - 25) > 0 \Rightarrow 1 - 25\lambda > \lambda xu > 0$ .

Relation  $1 - 25\lambda > 0$  is false for  $\lambda \in \left[\frac{1}{25}, 1\right]$ .

This example confirms that Definition 2.1 of invexity of  $X$  is weaker than the definition of  $\eta$ -invexity.

**Remark 2.1.** From geometrical point of view, if  $X$  is invex at  $u$  then, for all  $x \in X$ , the segment-line  $[u, u + \eta(x, u)]$  is included in  $X$ , incident to  $u$ . The invex sets generalize the convex sets that are obtained for  $\eta(x, u) = x - u$ . Any invex set is a  $\eta$ -invex set, but a set cannot be  $\eta$ -invex, with  $\eta$  given apriori.

**Example 2.2** ([10]). The set  $A = \left\{ (x, y) \in \mathbb{R}^2 \mid x^{\frac{2}{3}} + y^{\frac{2}{3}} \leq a^{\frac{2}{3}} \right\}$  (the closed inner of an astroide) is invex.

Let  $(u, v) \in A$ . We have to find a mapping  $\eta = (\eta_1, \eta_2): A \times A \rightarrow A$  such that

$$\forall (x, y) \in A, \forall \lambda \in [0, 1] \Rightarrow (u, v) + \lambda(\eta_1(x, y, u, v), \eta_2(x, y, u, v)) \in A.$$

Usually, such kind of mapping is not unique. Indeed, we could choose

$$\eta(x, y, u, v) = \left( -\frac{u}{2}, -\frac{v}{2} \right) \quad \text{or} \quad \eta(x, y, u, v) = \left( \frac{-u}{1+x^2+y^2}, \frac{-v}{1+x^2+y^2} \right).$$

**2.1. Invexity conditions of the sets.** From condition (2.1) we infer that  $\eta(x, u)$  is a feasible direction to  $X$  at the point  $u \in X$ . Then

$$\eta(x, u) \in F_X(u) = \{v \in \mathbb{R}^n \mid \forall \lambda \in [0, 1] : u + \lambda v \in X\},$$

where  $F_X(u)$  is the cone of feasible directions to  $X$  at  $u$ .

Therefore,

$$\eta(x, u) \in F_X(u) \quad \Rightarrow \quad u + \lambda \eta(x, u) \in X, \quad \forall \lambda \in [0, 1]. \quad (2.2)$$

**Example 2.3.** The graph  $G_f$  of the function  $f: [-1, 1] \rightarrow \mathbb{R}$ ,  $f(x) = x^3$  is not invex at no one of its points. Indeed, the tangent direction to  $G_f$  at the point  $(u, u^3)$ , that is  $(1, 3u^2)$ , does not verify the relation  $(u, u^3) + \lambda(1, 3u^2) \in G_f$  because the relation  $(u + \lambda)^3 = u^3 + 3\lambda u^2$ ,  $\forall \lambda \in [0, 1]$  is false. We have  $F_{G_f}(u, u^3) = \{(0, 0)\}$ . But  $\eta = 0$  is excluded. The convex set  $G_f$  is not invex because  $F_{G_f}(u, u^3) = \{0\}$ . We remark that  $\text{ri } G_f = \emptyset$ . If  $\text{ri } X \neq \emptyset$  and  $F_X(u) = \{0\}$ , then  $u$  is an extremal point of  $X$ .

**Theorem 2.1** ([13]). *If the nonempty set  $X$  is  $\eta$ -invex at  $u$  then  $F_X(u) \supset \{0\}$ .*

**Theorem 2.2** ([13]). *Let  $X$  be a nonempty set and  $u \in X$ . If  $\text{ri } F_X(u) \neq \emptyset$  then  $X$  is invex set at  $u$ .*

**Remark 2.2.** If  $X$  is  $\eta$ -invex at  $u$  then, according to relation (2.1) the mapping  $\eta$  has the form  $\eta: X \times X \rightarrow F_X(u) \setminus \{0\}$ . If  $X$  is invex at each  $u$  (of  $X$ ) then it remains  $\eta: X \times X \rightarrow \mathbb{R}^n \setminus \{0\}$ .

Let the set  $A$  be open in  $\mathbb{R}^n$ . We recall that the upper Dini directional derivative of  $f: A \rightarrow \mathbb{R}$  at  $u \in A$ , in the direction  $v \in \mathbb{R}$  is defined by, [7],

$$f'_+(u; v) = \limsup_{\lambda \downarrow 0} \frac{f(u + \lambda v) - f(u)}{\lambda}.$$

**Theorem 2.3.** *Let  $A$  be an open set in  $\mathbb{R}^n$ ,  $u \in A$  and the functions  $f: A \rightarrow \mathbb{R}^n$  and  $\eta: A \times A \rightarrow \mathbb{R}^n \setminus \{0\}$ . Suppose that  $f'_+(u; \eta(x, u))$  is finite for all  $x \in A$ . Then  $A$  is an invex set at  $u$  with respect to  $\eta$ . If  $u$  is arbitrary in  $A$ , then  $A$  is an invex set.*

*Proof.* Since the function  $f'_+(u; \cdot)$  is positively defined, then for any  $t > 0$ , we have

$$f'_+(u; t\eta(x, u)) = \limsup_{\lambda \downarrow 0} \frac{f(u + \lambda t\eta(x, u)) - f(u)}{\lambda}.$$

For any  $x \in A$  and  $t > 0$  we have  $u + \lambda t\eta(x, u) \in A$ . It results  $u + \lambda t\eta(x, u) \in A$ ,  $\forall \lambda t \geq 0$ . We denote  $\mu = \lambda t$  and we obtain  $u + \mu\eta(x, u) \in A$ ,  $\forall \mu \in [0, 1]$ , that is  $A$  is an  $\eta$ -invex set at  $u$ . But  $u$  is arbitrary, then  $A$  is an invex set.  $\square$

## 2.2. Algebraic properties of the invex sets.

**Theorem 2.4** ([13]). *If nonempty set  $X$  is invex with respect to the vector function  $\eta: X \times X \rightarrow \mathbb{R}^n \setminus \{0\}$ , then it admits the representation*

$$X = \bigcup_{u \in X} \bigcup_{x \in X} \{u + \lambda \eta(x, u) \in X \mid \forall \lambda \in [0, 1]\}.$$

**Theorem 2.5.** Consider in  $\mathbb{R}^n$  the open sets  $A_1, \dots, A_m$ , such that  $\bigcap_{i=1}^m A_i \neq \emptyset$ . If  $u \in \bigcap_{i=1}^m A_i$  and the sets  $A_1, \dots, A_m$  are invex at  $u$ , then the set  $\bigcap_{i=1}^m A_i$  is invex at  $u$ .

*Proof.* The set  $\bigcap_{i=1}^m A_i$  is open. Then  $\text{ri} \bigcap_{i=1}^m A_i = \bigcap_{i=1}^m A_i \neq \emptyset$  and the cone of the feasible directions to the set  $\bigcap_{i=1}^m A_i$  at the point  $u$  is nonempty. Then, according to Theorem 2.1, the set  $\bigcap_{i=1}^m A_i$  is invex at  $u$  (with respect to any of the mappings  $\eta_i: A_i \times A_i \rightarrow A_i$ , where  $A_i$  is invex with respect to  $\eta_i$ , for each  $i$ ).  $\square$

**Theorem 2.6.** Consider in  $\mathbb{R}^n$  the nonempty and disjoint sets  $X_1, \dots, X_m$  that are invex with respect to the mappings  $\eta_i: X_i \times X_i \rightarrow \mathbb{R}^n$ ,  $i = \overline{1, m}$ , respectively. Then the set  $\bigcup_{i=1}^m X_i$  is invex at the point  $u \in \bigcup_{i=1}^m X_i$ , with respect to

$$\eta: \left( \bigcup_{i=1}^m X_i \right) \times \left( \bigcup_{i=1}^m X_i \right) \rightarrow \mathbb{R}^n, \quad \eta(x, u) = \begin{cases} \eta_i(x, u), & (x, u) \in X_i \times X_i \\ 0, & (x, u) \notin X_i \times X_i. \end{cases}$$

*Proof.* There are two cases.

CASE 1.  $x, u \in X_i$ . We have  $(x, u) \in X_i \times X_i$  and it follows

$$u + \lambda \eta(x, u) = u + \lambda \eta_i(x, u) \in X_i \subset \bigcup_{i=1}^m X_i.$$

CASE 2.  $u \in X_i, x \in X_j$ , where  $i \neq j$ . Then  $(x, u) \notin X_i \times X_i$  and obtain

$$u + \lambda \eta(x, u) = u + \lambda 0 = u \in X_i \subset \bigcup_{i=1}^m X_i.$$

$\square$

**Example 2.4** (Theorem 2.6 for  $m = 2$ ). Let the nonempty sets  $X_1, X_2 \subset \mathbb{R}^n$ , invexes with respect to the mappings  $\eta_i: X_i \times X_i \rightarrow \mathbb{R}^n$ ,  $i = 1, 2$ . Then the set  $X_1 \cup X_2$  is invex with respect to  $\eta: (X_1 \cup X_2) \times (X_1 \cup X_2) \rightarrow \mathbb{R}^n$ , defined by

$$\eta(x, u) = \begin{cases} \eta_1(x, u), & x, u \in X_1 \\ \eta_2(x, u), & x, u \in X_2 \\ 0, & x \in X_1, u \in X_2 \\ 0, & x \in X_2, u \in X_1. \end{cases}$$

**Example 2.5** (Mohan and Noagy, [18]). The set  $X = [-7, -2] \cup [2, 10]$  is invex with respect to the function  $\eta: X \times X \rightarrow \mathbb{R}$ , defined by

$$\eta(x, u) = \begin{cases} x - u, & x, u \in [-7, -2] \\ x - u, & x, u \in [2, 10] \\ -7 - u, & x \in [-7, -2], u \in [2, 10] \\ 2 - u, & x \in [2, 10], u \in [-7, -2]. \end{cases}$$

**Example 2.6** (A generalization of Example 2.5). Consider in  $\mathbb{R}^n$  the nonempty convex sets  $C_1$  and  $C_2$  and the fixed points  $a \in C_1$  and  $b \in C_2$ . Then the set  $C_1 \cup C_2$  is invex with respect to  $\eta: (C_1 \cup C_2) \times (C_1 \cup C_2) \rightarrow \mathbb{R}^n$ , defined by

$$\eta(x, u) = \begin{cases} x - u, & x, u \in C_1 \\ x - u, & x, u \in C_2 \\ a - u, & x \in C_1, u \in C_2 \\ b - u, & x \in C_2, u \in C_1. \end{cases}$$

**Remark 2.3.** In general, the invex sets are not connected.

**Theorem 2.7.** Let the sets  $X_1 \subseteq \mathbb{R}^{n_1}, \dots, X_m \subseteq \mathbb{R}^{n_m}$  be invexes at the points  $u_i \in X_i$ ,  $i = \overline{1, m}$ , respectively with respect to the mappings  $\eta_i: X_i \times X_i \rightarrow \mathbb{R}^{n_i}$ ,  $i = \overline{1, m}$ . Then the cartesian product set  $\prod_{i=1}^m X_i$  is invex at the point  $u = (u_1, \dots, u_m) \in \prod_{i=1}^m X_i$  with respect to

$$\eta = (\eta_1, \dots, \eta_m): \left( \prod_{i=1}^m X_i \right) \times \left( \prod_{i=1}^m X_i \right) \rightarrow \mathbb{R}^{n_1} \times \dots \times \mathbb{R}^{n_m},$$

given by the formula  $\eta(x; u) = (\eta_1(x_1, u_1), \dots, \eta_m(x_m, u_m))$ .

For  $m = 1$ ,  $n_1 = 1$  and  $u$  arbitrary, the result of Mohan and Neogy, [15] is obtained.

*Proof.* Indeed, we have

$$\begin{aligned} u + \lambda \eta(x, u) &\in (u_1, \dots, u_m) + \lambda(\eta_1(x_1, u_1), \dots, \eta_m(x_m, u_m)) \\ &= (u_1 + \lambda \eta_1(x_1, u_1), \dots, u_m + \lambda \eta_m(x_m, u_m)) \in X_1 \times \dots \times X_m = \prod_{i=1}^m X_i \end{aligned}$$

for any  $\lambda \in [0, 1]$ .  $\square$

**Example 2.7** (Mohan and Neogy, [18]). The set  $X_1 = [-7, -2] \cup [2, 10]$  is invex with respect to the function  $\eta_1 = \eta$  given in Example 2.5. The set  $X_2 = [-5, -2] \cup [2, 7]$  is invex with respect to the function  $\eta_2: X_2 \times X_2 \rightarrow \mathbb{R}$ , defined by

$$\eta_2(x, u) = \begin{cases} x - u, & x, u \in [2, 7] \\ x - u, & x, u \in [-5, -2] \\ -5 - u, & x \in [2, 7], u \in [-5, -2] \\ 2 - u, & x \in [-5, -2], u \in [2, 7]. \end{cases}$$

Then the set  $X_1 \times X_2$  is invex with respect to  $\eta: (X_1 \times X_2) \times (X_1 \times X_2) \rightarrow \mathbb{R} \times \mathbb{R}$ , defined by  $\eta(x, u, y, v) = (\eta_1(x, u), \eta_2(y, v))$ .

For any sets  $X$  and  $Y$  in  $\mathbb{R}^n$  is defined their *sum* by the set

$$X + Y = \{x + y \mid \forall x \in X, \forall y \in Y\}.$$

**Theorem 2.8** ([13]). *Consider in  $\mathbb{R}^n$  the sets  $X$  and  $Y$ . If the set  $X$  is invex at  $u \in X$  with respect to  $\eta: X \times X \rightarrow \mathbb{R}^n$ , and the set  $Y$  is invex at  $v \in Y$  with respect to  $\mu: Y \times Y \rightarrow \mathbb{R}^n$ , then the set  $X + Y$  is invex at the point  $u + v \in X + Y$  with respect to  $\sigma: (X + Y) \times (X + Y) \rightarrow \mathbb{R}^n$ , given by*

$$(O_1) \quad \forall x \in X, \forall y \in Y : \sigma(x + y, u + v) = \eta(x, u) + \mu(y, v).$$

*Proof.* Since  $X$  is invex at  $u$  with respect to  $\eta$ , and  $Y$  is invex at  $v$  with respect to  $\mu$  the implications hold:

$$\begin{aligned} \forall x \in X, \forall \lambda \in [0, 1] &\Rightarrow u + \lambda\eta(x, u) \in X, \\ \forall y \in Y, \forall \lambda \in [0, 1] &\Rightarrow v + \lambda\mu(y, v) \in Y, \end{aligned}$$

respectively. Summing side by side these two relations it results the implication

$$\begin{aligned} \forall x + y \in X + Y, \forall \lambda \in [0, 1] &\Rightarrow \\ \Rightarrow u + v + \lambda[\eta(x, u) + \mu(y, v)] &= u + v + \lambda\sigma(x + y, u + v) \in X + Y. \end{aligned}$$

Therefore, the set  $X + Y$  is invex at  $u + v$  with respect to the mapping  $\sigma = \eta + \mu$ .  $\square$

**Theorem 2.9.** *If  $X \subseteq \mathbb{R}^n$  is invex at  $u \in X$  with respect to the mapping  $\eta: X \times X \rightarrow \mathbb{R}^n$  and  $\eta(x, u)$  is homogeneous with respect to the variable  $u$  for any  $x$ , then the set  $\alpha X$  is invex with respect to the same  $\eta$ .*

*Proof.* If  $X$  is invex at  $u$  with respect to  $\eta$ , then

$$\forall x \in X, \forall \lambda \in [0, 1] \Rightarrow u + \lambda\eta(x, u) \in X.$$

For  $\alpha \in \mathbb{R}$ , it results

$$(O_2) \quad \forall \alpha x \in \alpha X, \forall \lambda \in [0, 1] \Rightarrow \alpha u + \lambda\eta(x, \alpha u) \in \alpha X.$$

Therefore, the set  $\alpha X$  is invex at  $\alpha u$ .  $\square$

**Example 2.8.** The set  $[0, \infty)$  is invex with respect to the function  $\eta$ , given by  $\eta(x, u) = u(x^2 + 1)$ , where  $\eta(x, \lambda u) = \lambda\eta(x, u)$ . Then the set  $\alpha[0, \infty) = \{\alpha x \mid \forall x \geq 0\}$  is invex with respect to  $\eta$ .

Denote by  $H(\mathbb{R}^n)$  the family of subsets of  $\mathbb{R}^n$  which are invex with respect to the mapping  $\eta(x, u)$ , supposed homogeneous with respect to the variable  $u$ . We have

**Corollary 2.1.** *The family  $H(\mathbb{R}^n)$  is a vector subspace of  $\mathbb{R}^n$  with respect to the operations  $(O_1)$  and  $(O_2)$ .*

### 3. PREINVEX FUNCTIONS AND GENERALIZATIONS

**3.1. Classes and types of preinvex functions. New invex sets.** Let  $X \subseteq \mathbb{R}^n$  be a nonempty set,  $f: X \rightarrow \mathbb{R}$  a function and  $\rho$  a real number.

**Definition 3.1** ([11]). **A** (Preinvexities). The function  $f$  is said to be  $\rho$ -preinvex ( $\rho$  pI) at the point  $u \in X$  if there exist the vector functions  $\eta, \theta: X \times X \rightarrow \mathbb{R}^n$  ( $\eta \neq 0$ ) such that

$$\forall x \in X, \forall \lambda \in [0, 1]: f(u + \lambda\eta(x, u)) \leq \lambda f(x) + (1 - \lambda)f(u) - \lambda\rho\|\theta(x, u)\|^2.$$

If  $\rho > 0$ ,  $\rho = 0$ ,  $\rho < 0$  then  $f$  is called *strongly preinvex*, *preinvex*, *weakly invex* at  $u$ , respectively.

If  $\forall x \in X, x \neq u, \forall \lambda \in (0, 1): f(u + \lambda\eta(x, u)) < \lambda f(x) + (1 - \lambda)f(u)$ , then  $f$  is called *strictly preinvex* at  $u$ .

**B** (Preincavities). The function  $f$  is said to be  $\rho$ -preincave at  $u$  if the function  $-f$  is  $\rho$ -preinvex at  $u$ . If  $\rho > 0$ ,  $\rho = 0$  or  $\rho < 0$ , then the function  $f$  is called *strongly preincave*, *preincave* or *weakly preincave* at  $u$ . The function  $f$  is called *strictly preincave* at  $u$  if the function  $-f$  is strictly preinvex at  $u$ .

**C**. The function  $f$  is *preinvex* (*preincave*) etc. on  $X$  if it is preinvex (*preincave*) etc. at each point of  $X$ .

**Definition 3.2** ([11]). **A** (Prepseudoinvexities). The function  $f$  is said to be (*strictly*)  $\rho$ -prepseudoinvex ( $\rho$  ppI) at the point  $u \in X$  if there exist the vector functions  $\eta, \theta: X \times X \rightarrow \mathbb{R}^n$  ( $\eta \neq 0$ ) such that

$$\forall x \in A, \forall \lambda \in (0, 1): f(u + \lambda\eta(x, u)) + \rho\lambda\|\theta(x, u)\|^2 \geq 0 \Rightarrow f(x) \geq f(u).$$

If  $\rho > 0$ ,  $\rho = 0$ ,  $\rho < 0$  then  $f$  is called *strongly prepseudoinvex*, *prepseudoinvex*, respectively *weakly prepseudoinvex* at  $u$ .

If  $\forall x \in A, x \neq u, \forall \lambda \in (0, 1): f(u + \lambda\eta(x, u)) \geq 0 \Rightarrow f(x) \geq f(u)$ , then  $f$  is called *strictly prepseudoinvex* function at  $u$ .

**B** (Prepseudoincavities). The function  $f$  is said to be  $\rho$ -prepseudoincave at  $u$  if the function  $-f$  is  $\rho$ -prepseudoinvex at  $u$ . If  $\rho > 0$ ,  $\rho = 0$  or  $\rho < 0$ , then the function  $f$  is called *strongly prepseudoincave*, *prepseudoincave* or *weakly prepseudoincave* at  $u$ . The function  $f$  is called *strictly prepseudoincave* at  $u$  if the function  $-f$  is strictly prepseudoinvex at  $u$ .

**C**. The function  $f$  is *prepseudoinvex* (*prepseudoincave*) etc. on  $X$  if it is prepseudoinvex (*prepseudoincave*) etc. at each point of  $X$ .

**Definition 3.3** ([11]). **A** (Prequasiinvexities). The function  $f$  is said to be (*strictly*)  $\rho$ -prequasiinvex ( $\rho$  ppI) at the point  $u \in X$  if there exist the vector functions  $\eta, \theta: X \times X \rightarrow \mathbb{R}^n$  ( $\eta \neq 0$ ) such that

$$\forall x \in A, \forall \lambda \in [0, 1]: f(x) \leq f(u) \Rightarrow f(u + \lambda\eta(x, u)) + \rho\lambda\|\theta(x, u)\|^2 \leq 0.$$

If  $\rho > 0$ ,  $\rho = 0$ ,  $\rho < 0$  then  $f$  is called *strongly prequasiinvex*, *prequasiinvex*, *weakly prepseudoinvex* at  $u$ , respectively.

If  $\forall x \in A, x \neq u, \forall \lambda \in (0, 1) : f(u + \lambda\eta(x, u)) \geq 0 \Rightarrow f(x) \geq f(u)$ , then  $f$  is called *strictly prequasiinvex* at  $u$ .

**B** (Prepseudoincavities). The function  $f$  is said to be  $\rho$ -*prepseudoincave* at  $u$  if the function  $-f$  is  $\rho$ -prequasiinvex at  $u$ . If  $\rho > 0$ ,  $\rho = 0$  or  $\rho < 0$ , then the function  $f$  is called *strongly prequasiincave*, *prequasiincave* or *weakly prequasiincave* at  $u$ . The function  $f$  is called *strictly prequasiincave* at  $u$  if the function  $-f$  is strictly prequasiinvex at  $u$ .

**C**. The function  $f$  is *prequasiinvex* (*prequasiincave*) etc. on  $X$  if it is prequasiinvex (*prequasiincave*) etc. at each point of  $X$ .

**Proposition 3.1.** *The domains of all types of preinvex (preincave), prepseudoinvex (prepseudoincave) and prequasiinvex (prequasiincave) functions are invex sets.*

*Proof.* From Definitions 3.1, 3.2 and 3.3, we get

$$\forall x \in X, \forall \lambda \in [0, 1] \Rightarrow u + \lambda\eta(x, u) \in X,$$

that is  $X$  is an invex set with respect the vector function  $\eta$ .  $\square$

The epigraph of the function  $f$  is defined by the set

$$E_{f,X} = \{(x, \alpha) \mid x \in X, \alpha \in \mathbb{R}, f(x) \leq \alpha\}.$$

**Theorem 3.1.** *Let  $X$  be a nonempty set in  $\mathbb{R}^n$  and  $f: X \rightarrow \mathbb{R}$  a function.  $f$  is preinvex on  $X$  if and only if its epigraph  $E_{f,X}$  is an invex set in  $\mathbb{R}^{n+1}$ .*

*Proof.* NECESSITY. Suppose that the function  $f$  is preinvex on  $X$  (Definition ( $\rho$  pI)). Let  $(x, \alpha), (u, \beta) \in E_{f,X}$ . This imply  $f(x) \leq \alpha, f(u) \leq \beta$ . Because  $f$  is preinvex on  $X$ , for all  $x, u \in X, \forall \lambda \in [0, 1], f(u + \lambda\eta(x, u)) \leq \lambda\alpha + (1 - \lambda)\beta = \beta + \lambda(\alpha - \beta)$ ; therefore  $(u + \lambda\eta(x, u), \beta + \lambda(\alpha - \beta)) \in E_{f,X}$  that is  $(u, \beta) + \lambda[\eta(x, u), \alpha - \beta] \in E_{f,X}$ . It follows that the epigraph  $E_{f,X}$  is an invex set with respect to the mapping

$$\tilde{\eta}: (X \times \mathbb{R}) \times (X \times \mathbb{R}) \rightarrow \mathbb{R}^{n+1}, \quad \tilde{\eta}((x, \alpha), (u, \beta)) = (\eta(x, u), \alpha - \beta).$$

SUFFICIENCY. If  $E_{f,X}$  is an invex set in  $\mathbb{R}^{n+1}$  with respect to  $\tilde{\eta}$ , arbitrary given, where  $X$  is invex in  $\mathbb{R}^n$  with respect to  $\eta$ , then for any  $x, u \in X$  we have  $(x, f(x)), (u, f(u)) \in E_{f,X}$  and it results

$$(u, f(u)) + \lambda\tilde{\eta}[(x, f(x)), (u, f(u))] \in E_{f,X}, \quad \forall \lambda \in [0, 1].$$

We set  $\tilde{\eta}[(x, f(x)), (u, f(u))] = [\eta(x, u), f(x) - f(u)]$  and obtain

$$\begin{aligned} (u, f(u)) + \lambda[\eta(x, u), f(x) - f(u)] &\in E_{f,X}, \\ (u + \lambda\eta(x, u), f(u) + \lambda[f(x) - f(u)]) &\in E_{f,X}. \end{aligned}$$

From these relations, we infer  $f(u + \lambda\eta(x, u)) \leq f(u) + \lambda[f(x) - f(u)]$ , that is  $f$  is preinvex on  $X$ .  $\square$

### 3.2. Operations with preinvex (preincave) functions.

**Proposition 3.2.** *If the functions  $f_1, \dots, f_m: X \rightarrow \mathbb{R}$  are  $\eta$ -preinvex ( $\eta$ -prepseudo-invex,  $\eta$ -prequasiinvex) then the function  $c_1 f_1 + \dots + c_m f_m$ ,  $c_1 > 0, \dots, c_m > 0$  is  $\eta$ -preinvex ( $\eta$ -prepseudo-invex,  $\eta$ -prequasiinvex).*

*Proof.* For an index  $i \in \{1, \dots, m\}$ ,  $x, u \in X$  and  $\lambda \in [0, 1]$  we have

$$f_i(u + \lambda\eta(x, u)) \leq \lambda f_i(x) + (1 - \lambda)f_i(u).$$

Multiplying this relation by  $c_i > 0$  and summing over  $i$  it results

$$\left( \sum_{i=1}^m c_i f_i \right) (u + \lambda\eta(x, u)) \leq \lambda \left( \sum_{i=1}^m c_i f_i \right) (x) + (1 - \lambda) \left( \sum_{i=1}^m c_i f_i \right) (u)$$

and the proposition is proved.  $\square$

**Proposition 3.3.** *If  $(f_i)_{i \in I}$  is a family of preinvex real functions on  $X$ , then the function  $\sup_{i \in I} f_i$  is a preinvex function on  $X$ .*

**Theorem 3.2.** *If the functions  $f$  and  $g$  are  $\eta$ -preinvex and positive on the set  $X$ , then the function  $fg$  is  $(-1)$ -preinvex on  $X$  with respect to  $\eta, \theta: X \times X \rightarrow \mathbb{R}^n$ , where  $\|\theta(x, u)\|^2 = f(x)g(u) + f(u)g(x)$ .*

*Proof.* For  $x, u \in X$  and  $\lambda \in [0, 1]$ , we have

$$\begin{aligned} f(u + \lambda\eta(x, u)) &\leq \lambda f(x) + (1 - \lambda)f(u), \\ g(u + \lambda\eta(x, u)) &\leq \lambda g(x) + (1 - \lambda)g(u). \end{aligned}$$

Multiplying side by side these two relations, it results

$$\begin{aligned} (fg)(u + \lambda\eta(x, u)) &\leq \lambda^2 f(x)g(x) + (1 - \lambda)^2 f(u)g(u) + \lambda(1 - \lambda)[f(x)g(u) + f(u)g(x)] \\ &\leq \lambda(fg)(x) + (1 - \lambda)(fg)(u) + [f(x)g(u) + f(u)g(x)]. \end{aligned}$$

Therefore we have

$$(fg)(u + \lambda\eta(x, u)) \leq \lambda(fg)(x) + (1 - \lambda)(fg)(u) - (-1)\|\theta(x, u)\|^2,$$

where  $\theta(x, u)$  satisfies the condition  $\|\theta(x, u)\|^2 = f(x)g(u) + f(u)g(x)$ .  $\square$

**Theorem 3.3.** *Let the invex set  $X \subseteq \mathbb{R}^n$ , the convex set  $C \subseteq \mathbb{R}^m$ , the vector function  $u = (u_1, \dots, u_m): X \rightarrow C$ , the scalar function  $f: C \rightarrow \mathbb{R}$  be and the compound function  $F = f \circ u: X \rightarrow \mathbb{R}$ .*

*If the function  $f(u_1, \dots, u_m)$  is convex (strictly convex) and  $f$  is increasing (strictly increasing) with respect to each preinvex (strictly preinvex) component  $u_i$ , or  $f$  is decreasing (strictly decreasing) with respect to each preincave (strictly preincave) component  $u_j$ , then the function  $F$  is preinvex (strictly preinvex) on  $X$ .*

*Conversely, if the function  $F$  is preinvex (strictly preinvex) and  $u$  is an affine vector function, then the function  $f$  is preinvex (strictly preinvex) on  $C$ .*

*Proof.* Similar to those of Bereanu's theorem, [3], replacing  $x - u$  by  $\eta(x, u)$ .  $\square$

**Theorem 3.4.** *Let the functions  $f$  and  $g$  be  $\rho$ -preinvex on the set  $X$  with respect to the mappings  $\eta$ ,  $\theta_1$ , respectively  $\eta$ ,  $\theta_2$ . If  $f$  and  $g$  are positive on  $X$  and  $\rho > 0$ , then the function  $fg$  is  $(-1)$ -preinvex on  $X$  with respect to  $\eta$  and  $\theta$ , where*

$$\|\theta(x, u)\|^2 = f(x)g(u) + f(u)g(x) + \rho^2\|\theta_1(x, u)\|^2\|\theta_2(x, u)\|.$$

*Proof.* For  $x, u \in X$  and  $\lambda \in [0, 1]$ , we have

$$\begin{aligned} f(u + \lambda\eta(x, u)) &\leq \lambda f(x) + (1 - \lambda)f(u) - \lambda\rho\|\theta_1(x, u)\|^2, \\ g(u + \lambda\eta(x, u)) &\leq \lambda g(x) + (1 - \lambda)g(u) - \lambda\rho\|\theta_2(x, u)\|^2. \end{aligned}$$

Multiplying side by side, we deduce

$$\begin{aligned} (fg)(u + \lambda\eta(x, u)) &\leq [\lambda f(x) + (1 - \lambda)f(u)] \cdot [\lambda g(x) + (1 - \lambda)g(u)] \\ &\quad - \lambda\rho\|\theta_2\|^2[\lambda f(x) + (1 - \lambda)f(u)] \\ &\quad - \lambda\rho\|\theta_1\|^2[\lambda g(x) + (1 - \lambda)g(u)] + \lambda^2\rho^2\|\theta_1\|^2\|\theta_2\|^2 \\ &\leq \lambda(fg)(x) + (1 - \lambda)(fg)(u) + f(x)g(u) + f(u)g(x) + \rho^2\|\theta_1\|^2\|\theta_2\|^2, \end{aligned}$$

that is the conclusion.  $\square$

**Theorem 3.5.** *Let the functions  $f$  and  $g$  be defined and positive on the set  $X$ . We suppose that  $f$  is  $\rho$ -preinvex with respect to  $\eta$  and  $\theta_1$ , and  $g$  is  $\rho$ -preincave with respect to  $\eta$  and  $\theta_2$ . Then the function  $f/g$  is  $\rho$ -preinvex with respect to  $\eta$  and a mapping  $\theta$  which can be found.*

*Proof.* For  $x, u \in X$  and  $\lambda \in [0, 1]$  successively we have

$$\begin{aligned} f(u + \lambda\eta(x, u)) &\leq \lambda f(x) + (1 - \lambda)f(u) - \lambda\rho\|\theta_1(x, u)\|^2, \\ g(u + \lambda\eta(x, u)) &\geq \lambda g(x) + (1 - \lambda)g(u) + \lambda\rho\|\theta_2(x, u)\|^2, \\ \frac{f}{g}(u + \lambda\eta(x, u)) &\leq \frac{\lambda f(x) + (1 - \lambda)f(u) - \lambda\rho\|\theta_1(x, u)\|^2}{\lambda g(x) + (1 - \lambda)g(u) + \lambda\rho\|\theta_2(x, u)\|^2}. \end{aligned}$$

Let  $E(x, u, \rho, \lambda)$  which satisfies the equality

$$\frac{\lambda f(x) + (1 - \lambda)f(u) - \lambda\rho\|\theta_1\|^2}{\lambda g(x) + (1 - \lambda)g(u) + \lambda\rho\|\theta_2\|^2} = \lambda \frac{f(x)}{g(x)} + (1 - \lambda) \frac{f(u)}{g(u)} - \lambda\rho E(x, u, \rho, \lambda).$$

CASE  $\rho > 0$ . If  $E(x, u, \rho, \lambda) > 0$  for  $\lambda \in \Lambda_1 \subseteq [0, 1]$ , then we can choose  $\theta(x, u) = \theta(x, u, \rho)$  such that  $\|\theta(x, u)\|^2 = \inf_{\lambda \in \Lambda_1} E(x, u, \rho, \lambda)$ .

If  $E(x, u, \rho, \lambda) < 0$  for  $\lambda \in \Lambda_2 \subseteq [0, 1]$ , then we choose  $\theta(x, u)$  such that  $\|\theta(x, u)\|^2 = - \inf_{\lambda \in \Lambda_2} E(x, u, \rho, \lambda)$ .

CASE  $\rho < 0$ . Determine  $\theta(x, u) = \theta(x, u, \rho)$  in a similar way to  $\rho > 0$ .  $\square$

**Corollary 3.1** ([1]). *If the function  $f$  is  $\eta$ -preincave on the set  $X$  such that  $f(x) > 0$  for any  $x \in X$ , then the function  $\frac{1}{f}$  is  $\eta$ -preinvex on  $X$ .*

## 4. NONSMOOTH INVEX FUNCTIONS ON OPEN SETS

Let be given  $A \subseteq \mathbb{R}^n$  a nonempty open set and  $f: A \rightarrow \mathbb{R}$  a function. Suppose that the upper Dini directional derivative  $f'_+$  is finite. In the following, we consider vector functions of the form  $\eta: A \times A \rightarrow \mathbb{R}^n$ , where  $\eta \neq 0$ .

**Definition 4.1** ([7]). 1) The function  $f$  is (*strictly*) *invex* at  $u \in A$  if there exists a vector function  $\eta$  such that

$$\forall x \in A [x \neq u]: f(x) - f(u) [ > ] \geq f'_+(u; \eta(x, u)).$$

2) The function  $f$  is *invex* on  $A$  (*invex*) if it is invex at each point of  $A$ .

**Definition 4.2** ([7]). 1) The function  $f$  is (*strictly*) *pseudoinvex* at  $u \in A$  if there exists a vector function  $\eta$  such that

$$\forall x \in A [x \neq u]: f'_+(u; \eta(x, u)) \geq 0 \Rightarrow f(x) \geq [ > ] f(u).$$

2) The function  $f$  is *pseudoinvex* on  $A$  if it is invex at each of  $A$ .

**Definition 4.3** ([7]). 1) The function  $f$  is *quasiinvex* at  $u \in A$  if there exists a vector function  $\eta$  such that

$$\forall x \in A: f(x) \leq f(u) \Rightarrow f'_+(u; \eta(x, u)) \leq 0.$$

2) The function  $f$  is *quasiinvex* on  $A$  if it is quasiinvex at each point of  $A$ .

3) The function  $f$  is (*strictly*) *semistrictly quasiinvex* at  $u \in A$  if there exists a vector function  $\eta$  such that

$$\forall x \in A, x \neq u: f(x) [ \leq ] < f(u) \Rightarrow f'_+(u; \eta(x, u)) < 0.$$

Concerning the relations between the invex, pseudoinvex and quasiinvex functions and those preinvex, prepseudoinvex and prequasiinvex functions respectively, we recall the following results:

**Theorem 4.1** (Mititelu, [10, 12]). *Let  $A \subset \mathbb{R}^n$  be an open set and  $f: A \rightarrow \mathbb{R}$  a function. Then for  $f$  the following relations are true:  $\text{invex} \Leftrightarrow \text{preinvex}$ ,  $\text{pseudoinvex} \Leftrightarrow \text{prepseudoinvex}$ ,  $\text{quasiinvex} \Leftrightarrow \text{prequasiinvex}$ .*

According to Theorem 4.1, the properties of the (generalized) preinvex functions defined on open sets transfer to the (generalized) invex functions, respectively.

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