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AN INTEGRAL FORMULA IN QUATERNION SPACE FORM

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Abstract. In this paper, we obtained an integral formula in quaternion space form.

1. INTRODUCTION

Let $M(c)$ denote a $4n$-dimensional quaternion space form of quaternion sectional curvature c and $p(H)$ denote the $4n$-dimensional quaternionprojective space of constant quaternion sectional curvature 4. Let N be an n-dimensional Riemannian manifold isometrically immersed in $M(c)$. We call N a totally real submanifold of $M(c)$ if each tangent 2-plane of N is mapped into a totally real plane in $M(c)$. Many studies on integral formulas have been done by many mathematicians (see [4], [6], [9]). Liu [6] studied compact space-like submanifolds with constant scalar curvature and flat normal bundle. Furthermore, Zheng [9] studied spacelike hypersurfaces in the de Sitter space. Furthermore, quaternion manifolds are also studied by many mathematicians (see [2], [3], [5]). In this paper, we consider totally real submanifold with constant scalar curvature and flat normal bundle and we use the method of proof which is given in [6] and [9]. The techniques used in the paper are from [8], where similar results were obtained for totally real submanifolds in complex space forms. Thus we obtain the following theorem.

Theorem 1.1. *Let N be an n-dimensional compact Riemannian manifold isometrically immersed in $M(c)$ with constant scalar curvature. Thus, we have*

$$\int\left[|\nabla S|+\left(\frac{c}{4}(n+1)-\frac{3}{2}S\right)S-n^2|\nabla H|^2+nH\Delta H-n^2H^2c-\sum_{i}\lambda_i^{\alpha}(nH)_{ii}\right]\leq 0.$$

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2. PRELIMINARIES

We use the same notations and terminologies as in [1], [7]. Let $M(c)$ denote a $4n$-dimensional quaternion space form of quaternion sectional curvature c and let N be an n-dimensional totally real submanifold of $M(c)$. We choose a local field of orthonormal frames,

$$e_1,\dots,e_n, \qquad e_{I(1)} = Ie_1,\dots,e_{I(n)} = Ie_n, \qquad e_{J(1)} = Je_1, \lambda, e_{J(n)} = Je_n,$$

$$e_{K(1)} = Ke_1,\dots,e_{K(n)} = Ke_n,$$

is such a way that when restricted to $N, e_1,\dots,e_n$ are tangent to N. Here I, J, K are the almost Hermitian structure and satisfy

$$IJ = -JI = K, \quad JK = -KJ = I, \quad KI = -IK = J, \quad I^2 = J^2 = K^2 = -1.$$

We shall use the following convention on the range of indices:

$$A, B, \dots = 1,\dots,n, I(1),\dots,I(n), J(1),\dots,J(n), K(1),\dots,K(n),$$

$$\alpha, \beta, \dots = I(1),\dots,I(n), J(1),\dots,J(n), K(1),\dots,K(n),$$

$$i, j, \dots = 1,\dots,n, \qquad \Phi = I, J, K.$$

Let $\{w_A\}$ be the dual frame field. Then the structure equations of $M(c)$ are given by

$$dw_A = -\sum_B w_{AB} \wedge w_B, \qquad w_{AB} + w_{BA} = 0,$$

$$dw_{AB} = -\sum_C w_{AC} \wedge w_{CB} + \frac{1}{2}\sum_{CD} K_{ABCD} w_C \wedge w_D, \tag{2.1}$$

$$\begin{aligned} K_{ABCD} = \frac{c}{4}(&\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC} + I_{AC}I_{BD} - I_{AD}I_{BC} + 2I_{AB}I_{CD} \\ &+ J_{AC}J_{BD} - J_{AD}J_{BC} + 2J_{AB}J_{CD} \\ &+ K_{AC}K_{BD} - K_{AD}K_{BC} + 2K_{AB}K_{CD}). \end{aligned}$$

Restricting these forms to N, we get the following structure equations of the immersion:

$$w_\alpha = 0, \quad w_{\alpha i} = \sum_j h^\alpha_{ij} w_j, \quad h^\alpha_{ij} = h^\alpha_{ji}, \quad h^{\Phi(i)}_{jk} = h^{\Phi(j)}_{ik} = h^{\Phi(k)}_{ij},$$

$$dw_{ij} = -\sum_k w_{ik} \wedge w_{kj} + \frac{1}{2}\sum_{kl} R_{ijkl} w_k \wedge w_l,$$

$$\begin{aligned} R_{ijkl} = \frac{c}{4}(&\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} + I_{ik}I_{jl} - I_{il}I_{jk} + 2I_{ij}I_{kl} \\ &+J_{ik}J_{jl} - J_{il}J_{jk} + 2J_{ij}J_{kl} + K_{ik}K_{jl} - K_{il}K_{jk} + 2K_{ij}K_{kl}) \\ &+\sum_{\alpha}(h^{\alpha}_{ik}h^{\alpha}_{jl} - h^{\alpha}_{il}h^{\alpha}_{jk}), \end{aligned} \tag{2.2}$$

$$dw_{\alpha\beta} = -\sum_{\gamma} w_{\alpha\gamma} \wedge w_{\gamma\beta} + \frac{1}{2}\sum_{ij} R_{\alpha\beta ij} w_i \wedge w_j,$$

$$R_{\alpha\beta ij} = K_{\alpha\beta ij} + \sum_{k}(h^{\alpha}_{ik}h^{\beta}_{kj} - h^{\beta}_{ik}h^{\alpha}_{kj}). \tag{2.3}$$

We call $h = \sum_{ij\alpha} h^{\alpha}_{ij} w_i w_j e_{\alpha}$ the second fundamental form of the immersed manifold N. We denote by $S = \sum_{ij\alpha}(h^{\alpha}_{ij})^2$ the square of the length of h. If we use the equation (2.1) and (2.2), we have

$$\tau = \frac{c}{4}n(n+1) + n^2H^2 - S, \tag{2.4}$$

where τ is the scalar curvature of N and H is the length of meancurvature vector of N and H is given by:

$$H = \frac{1}{n}trh_{\alpha} = \frac{1}{n}\sum_{i} h^{\alpha}_{ii} \geq 0.$$

Let h^{α}_{ijk} and h^{α}_{ijkl} denote the covariant derivative and the second covariant derivative of h^{α}_{ij}. We define h^{α}_{ijk} and h^{α}_{ijkl} by

$$\sum h^{\alpha}_{ijk} w_k = dh^{\alpha}_{ij} - \sum_{l} h^{\alpha}_{il} w_{lj} - \sum_{l} h^{\alpha}_{lj} w_{li} + \sum_{\beta} h^{\beta}_{ij} w_{\alpha\beta}$$

and

$$\sum h^{\alpha}_{ijkl} w_l \;\; = \;\; dh^{\alpha}_{ijk} - \sum_{l} h^{\alpha}_{ljk} w_{li} - \sum_{l} h^{\alpha}_{ilk} w_{lj} - \sum_{l} h^{\alpha}_{ijl} w_{lk} + \sum_{\beta} h^{\beta}_{ijk} w_{\alpha\beta},$$

respectively, where

$$h^{\alpha}_{ijk} = h^{\alpha}_{ikj} \tag{2.5}$$

and

$$h^{\alpha}_{ijkl} - h^{\alpha}_{ijlk} = \sum_{m} h^{\alpha}_{im} R_{mjkl} + \sum_{m} h^{\alpha}_{mj} R_{mikl} - \sum_{\beta} h^{\beta}_{ij} R_{\alpha\beta kl}, \tag{2.6}$$

where $R_{\alpha\beta kl}$ are the components of the normal curvature tensor of N. If $R_{\alpha\beta kl} = 0$ at point x of N, we say that the normal connection of N is flat at x. Let H_{α} and

$\triangle$ denote the $n \times n$ matrix (h_{ij}^{α}) and the Laplacian on N, respectively. By a simple calculation, we have

$$\begin{aligned} \frac{1}{2} \triangle S &= \sum_{ijk\alpha} (h_{ijk}^{\alpha})^2 + \frac{c}{4}(n+1)S + \sum_{\alpha,\beta} tr(H_{\alpha}H_{\beta} - H_{\beta}H_{\alpha})^2 \\ &\quad - \sum_{\alpha,\beta} (trH_{\alpha}H_{\beta})^2 + nH \triangle H - n^2H^2c. \end{aligned} \tag{2.7}$$

3. PROOF OF THEOREM

We know from (2.4) that

$$n^2H^2 - S = \tau - \frac{c}{4}n(n+1) \tag{3.1}$$

where τ is the scalar curvature of N. Taking the covariant derivative of (3.1) and using the fact that $\tau =$ const., we obtain

$$n^2HH_k = \sum_{i,j,\alpha} h_{ij}^{\alpha}.h_{ijk}^{\alpha},$$

and hence, by Cauchy-Schwarz inequality, we have

$$\sum_k n^4H^2(H_k)^2 = \sum_k \left(\sum_{i,j,\alpha} h_{ij}^{\alpha} \cdot h_{ijk}^{\alpha} \right)^2 \leq \sum_{i,j,\alpha} (h_{ij}^{\alpha})^2 \sum_{i,j,k,\alpha} (h_{ijk}^{\alpha})^2,$$

that is

$$n^4H^2 \left\| \bigtriangledown H \right\|^2 \leq S \cdot \left\| \bigtriangledown S \right\|. \tag{3.2}$$

On the other hand, the Laplacian $\triangle h_{ij}^{\alpha}$ of the fundamental form h_{ij}^{α} is defined to be $\sum_k h_{ijkk}^{\alpha}$, and hence, using (2.5), (2.6) and the assumption that N has flat normal bundle, we have

$$\begin{aligned} \triangle h_{ij}^{\alpha} &= \sum_k (h_{ijkk}^{\alpha} - h_{ikjk}^{\alpha}) + \sum_k (h_{ikjk}^{\alpha} - h_{ikkj}^{\alpha}) + \sum_k (h_{ikkj}^{\alpha} - h_{kkij}^{\alpha}) + (trh_{\alpha})_{ij} \\ &= \sum_{m,k} h_{im}^{\alpha} R_{mkjk} + \sum_{m,k} h_{jm}^{\alpha} R_{mijk} + (trh_{\alpha})_{ij}, \end{aligned}$$

where $(trh_{\alpha})_{ij}$ denotes the second covariant derivative of (trh_{α}). Since the normal bundle of N is flat, we choose $e_1, \ldots, e_n$ such that

$$h_{ij}^{\alpha} = \lambda_i^{\alpha} \delta_{ij}.$$

We define an operator $\Box$ acting on f by:

$$\Box f = \sum_{i,j} (nH\delta_{ij} - h_{ij}^{\alpha}) f_{ij}. \tag{3.3}$$

Since $(nH\delta_{ij} - h_{ij}^{\alpha})$ is trace-free it follows from [4] that the operator $\Box$ is self-adjoint to the L^2-inner product of N, i.e.,

$$\int_N f \cdot \Box g = \int_N g \cdot \Box f.$$

Thus we have the following computation by use of (3.3) and (2.7)

$$\begin{aligned}\Box(nH) &= nH \triangle (nH) - \sum_i \lambda_i^{\alpha}(nH)_{ii} \\ &= \frac{1}{2} \triangle (nH)^2 - \sum_i (nH)_i^2 - \sum_i \lambda_i^{\alpha}(nH)_{ii} \\ &= \frac{1}{2} \triangle S + \frac{1}{2} n(n+1) \triangle \tau - n^2 \left|\bigtriangledown H\right|^2 - \sum_i \lambda_i^{\alpha}(nH)_{ii}. \qquad (3.4)\end{aligned}$$

Putting (2.7) in (3.4), we have

$$\begin{aligned}\Box(nH) &= \left|\bigtriangledown S\right| + \frac{c}{4}(n+1)S + \sum_{\alpha,\beta} tr(H_\alpha H_\beta - H_\beta H_\alpha)^2 \\ &\quad - \sum_{\alpha,\beta}(trH_\alpha H_\beta)^2 + \frac{1}{2}n(n+1) \triangle \tau - n^2 \left|\bigtriangledown H\right|^2 - \sum_i \lambda_i^{\alpha}(nH)_{ii} \\ &\quad + nH \triangle H - n^2H^2c. \qquad (3.5)\end{aligned}$$

Now we assume that N is compact and we obtain the following key formula by integrating (3.5) and by noting $\int_N \triangle \tau = 0$ and $\int_N \Box(nH)dV = 0$ and by using the Lemma in [7],

$$\int [\left|\nabla S\right| + \left(\frac{c}{4}(n+1) - \frac{3}{2}S\right) S - n^2 \left|\nabla H\right|^2 + nH\Delta H - n^2H^2c - \sum_{\dot{I}} \lambda_i^{\alpha} (nH)_{ii}] \le 0.$$

REFERENCES

[1] M. Bektaş: *Totally real submanifolds in aquaternion space form*, Czech. Math. J., **54(129)**(2004), 341-346.

[2] B. Y. Chen and C. S. Houh: *Totally real submanifolds of a quaternion projective space*, Ann. Mat. Pura Appl., IV. Ser. 120 (1979), 185-199.

[3] S. Funabashi: *Totally real submanifolds of a quaternionic Kaehlerian manifold*, Kodai Math. Semin. Rep., **29**(1978), 261-270.

[4] L. Haizhong: *Rigidity theorems of hypersurfaces in a sphere*, Publications De L'institut Mathematique, Nouvelle serie, t. **67(81)**(2000), 112-120.

[5] S. Ishihara: *Quaternion Kählerian manifolds*, J. Diff. Geom., **9**(1974), 483-500.

[6] X. Liu: *Space-like submanifolds with constant scalarcurvature*, C. R. Acad. Sci. Paris, t. **332**, Serie 1(2001), 729-734.

[7] H. Sun: *Totally real pseudo-umbilical submanifolds of a quaternion space form*, Glasgow Math. J., **40**(1998), 109-115.

[8] K. Yano and M. Kon: *Totally real submanifolds of complex space forms*, I. II. Tohoku Math. J., II. Ser. 28 (1976), 215-225, Kodai Math. Sem. Reports, **27**(1976), 385-399.

[9] Y. Zheng: *On space-like hypersurfaces in the de sitterspace*, Annals of Global Analysis and Geometry, **13**(1995), 317-321.

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