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ON $(1,2)^*$-$\alpha\hat{g}$-CLOSED SETS

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Abstract. In this paper we introduce a new class of sets called $(1,2)^*$-$\alpha\hat{g}$-closed sets in bitopological spaces. We prove that this class lies between the class of $(1,2)^*$-α-closed sets and the class of $(1,2)^*$-αg-closed sets. Also, we find some basic properties of $(1,2)^*$-$\alpha\hat{g}$-closed sets. Applying these sets, we introduce a new space called $T_{(1,2)^*-\alpha\hat{g}}$-space.

1. INTRODUCTION

Levine [4], Mashhour et al. [8] and Njastad [9] have introduced the concepts of semi-open sets, preopen sets and α-open sets respectively. Levine [5] introduced generalised closed sets and studied thier properties. Bhattacharya and Lahiri [3], Arya and Nour [2], Maki et al. [6], [7] introduced semi-generalised closed sets, generalised semi-closed sets and α-generalised closed sets and generalised α-closed sets respectively. Veerakumar [11] defined $\hat{g}$-closed sets in topological spaces. Thivagar et al [10] have introduced the concepts of $(1,2)^*$-semi-open sets, $(1,2)^*$-α-open sets, $(1,2)^*$-generalised closed sets, $(1,2)^*$-semi-generalised closed sets and $(1,2)^*$-α-generalised closed sets in bitopological spaces.

In the present paper, we introduce a new class of sets called $(1,2)^*$-α-$\hat{g}$-closed sets (briefly $(1,2)^*$-$\alpha\hat{g}$-closed) sets in bitopological spaces and prove that this class lies between the class of $(1,2)^*$-α-closed sets and the class of $(1,2)^*$-αg-closed sets. Also, we find some basic properties of $(1,2)^*$-$\alpha\hat{g}$-closed sets in bitopological spaces. Applying these sets, we introduce a new space called $T_{(1,2)^*-\alpha\hat{g}}$-space.

2. PRELIMINARIES

Let A be a subset of a topological space (X,τ). Let Cl(A) and Int(A) denote the closure and the interior of A.

Definition 2.1. A subset A of a topological space (X,τ) is called

(i) semi-open, [4], if $A \subseteq \mathrm{Cl}(\mathrm{Int}(A))$
(ii) preopen, [8], if $A \subseteq \mathrm{Int}(\mathrm{Cl}(A))$
(iii) α-open, [9], if $A \subseteq \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$.

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The complement of a semi-open (resp. preopen and α-open) set is called semi-closed (resp. preclosed and α-closed). The α-closure of a subset A of X, denoted by αCl(A), is defined to be the intersection of all α-closed sets containing A.

Throughout this paper (X, τ_1, τ_2) and (Y, σ_1, σ_2) represent bitopological spaces on which no separation axioms are assumed unless otherwise mentioned.

Definition 2.2 ([10]). A subset S of a bitopological space (X, τ_1, τ_2) is said to be $\tau_{1,2}$-open if $S = A \cup B$ where $A \in \tau_1$, and $B \in \tau_2$. A subset S of X is $\tau_{1,2}$-closed if the complement of S is $\tau_{1,2}$-open.

Definition 2.3 ([10]). Let S be a subset of X. Then

(i) The $\tau_1\tau_2$-interior of S, denoted by $\tau_1\tau_2$-Int(S) is defined by $\cup\{G/G \subset S$ and G is $\tau_{1,2}$-open $\}$.

(ii) The $\tau_1\tau_2$-closure of S denoted by $\tau_1\tau_2$-Cl(S) is defined by $\cap\{F/S \subset F$ and F is $\tau_{1,2}$- closed $\}$.

Remark 2.1. (i) $\tau_1\tau_2$-Int(S) is $\tau_{1,2}$-open for each $S \subset X$ and $\tau_1\tau_2$-Cl(S) is $\tau_{1,2}$-closed for each $S \subset X$.

(ii) A set $S \subset X$ is $\tau_{1,2}$- open iff $S = \tau_1\tau_2$-Int(S) and is $\tau_{1,2}$- closed iff $S = \tau_1\tau_2$-Cl(S).

(iii) $\tau_1\tau_2$-Int(S) $=\mathrm{Int}_{\tau_1}(S) \cup \mathrm{Int}_{\tau_2}(S)$
and $\tau_1\tau_2$-Cl(S) $=\mathrm{Cl}_{\tau_1}(S) \cap \mathrm{Cl}_{\tau_2}(S)$ for any $S \subset X$

(iv) For any family $\{S_i/i \in I\}$ of subsets of X we have
  (a) $\bigcup\limits_i \tau_1\tau_2\text{-Int}(S_i) \subset \tau_1\tau_2\text{-Int}(\bigcup\limits_i S_i)$
  (b) $\bigcup\limits_i \tau_1\tau_2\text{-Cl}(S_i) \subset \tau_1\tau_2\text{ -Cl}(\bigcup\limits_i S_i)$
  (c) $\tau_1\tau_2\text{-Int}(\bigcap\limits_i S_i) \subset \bigcap\limits_i \tau_1\tau_2\text{-Int}(S_i)$
  (d) $\tau_1\tau_2\text{-Cl}(\bigcap\limits_i S_i) \subset \bigcap\limits_i \tau_1\tau_2\text{-Cl}(S_i)$

(v) $\tau_{1,2}$-open sets need not form a topology.

We recall the following definitions which are useful in the sequel.

Definition 2.4 ([10]). A subset A of a bitopological space (X, τ_1, τ_2) is called

(i) $(1,2)^*$-semi-open if $A \subseteq \tau_1\tau_2$-Cl($\tau_1\tau_2$-Int(A))

(ii) $(1,2)^*$-preopen if $A \subseteq \tau_1\tau_2$-Int($\tau_1\tau_2$-Cl(A))

(iii) $(1,2)^*$-α-open if $A \subseteq \tau_1\tau_2$-Int($\tau_1\tau_2$ -Cl($\tau_1\tau_2$-Int(A)))

(iv) $(1,2)^*$-generalised closed (briefly $(1,2)^*$-g-closed) if $\tau_1\tau_2$-Cl(A) $\subset U$ whenever $A \subset U$ and U is $\tau_{1,2}$-open in X.

(v) The $(1,2)^*$-α-closure (resp. $(1,2)^*$- semi-closure) of a subset A of X, denoted by $(1,2)^*$-α Cl(A) (resp. $(1,2)^*$-sCl(A)) is defined to be the intersection of all $(1,2)^*$- α-closed (resp. $(1,2)^*$-semi-closed) sets containing A.

(vi) The $(1,2)^*$-α-interior of a subset A of X, denoted by $(1,2)^*$-α Int(A) is defined to be the union of all $(1,2)^*$-α-open sets contained in A.

(vii) $(1,2)^*$-semi-generalised closed (briefly $(1,2)^*$-sg-closed) if
$(1,2)^*$-$s\mathrm{Cl}(A) \subset U$ whenever $A \subset U$ and U is $(1,2)^*$-semi-open in X.

(viii) $(1,2)^*$-generalised semi-closed (briefly $(1,2)^*$-gs-closed) if
$(1,2)^*$-$s\mathrm{Cl}(A) \subset U$ whenever $A \subset U$ and U is $\tau_{1,2}$-open in X.

(ix) $(1,2)^*$-α-generalised closed (briefly $(1,2)^*$-αg-closed) if
$(1,2)^*$-$\alpha\mathrm{Cl}(A) \subset U$ whenever $A \subset U$ and U is $\tau_{1,2}$-open in X.

(x) $(1,2)^*$-generalised α-closed (briefly $(1,2)^*$-gα-closed) if
$(1,2)^*$-$\alpha\mathrm{Cl}(A) \subset U$ whenever $A \subset U$ and U is $(1,2)^*$-α-open in X.

The complement of a $(1,2)^*$-semi-open (resp. $(1,2)^*$-α-open) set is called $(1,2)^*$-semi-closed (resp. $(1,2)^*$-α-closed). The complement of a $(1,2)^*$-g-closed (resp. $(1,2)^*$-sg-closed, $(1,2)^*$-gs-closed, $(1,2)^*$-gα-closed, $(1,2)^*$-αg closed) set is called $(1,2)^*$-g-open (resp. $(1,2)^*$-sg-open, $(1,2)^*$-gs-open, $(1,2)^*$-gα-open, $(1,2)^*$-αg-open).

Remark 2.2. (i) Since arbitrary union (resp. intersection) of $(1,2)^*$-α-open (resp. $(1,2)^*$-α-closed) sets is $(1,2)^*$-α-open (resp. $(1,2)^*$-α-closed), $(1,2)^*$-α Int$A$ (resp. $(1,2)^*$-$\alpha\mathrm{Cl}(A)$) is $(1,2)^*$-α-open (resp. $(1,2)^*$-α-closed).

(ii) For a bitopological space (X, τ_1, τ_2), a subset A of X is $(1,2)^*$-α-closed (resp. $(1,2)^*$-semi-closed) if and only if $(1,2)^*$-α $\mathrm{Cl}(A) = A$ (resp. $(1,2)^*$-$s\mathrm{Cl}(A) = A$).

3. $(1,2)^*$-$\alpha\hat{g}$-CLOSED SETS

We introduce the following definitions.

Definition 3.1. A subset A of a bitopological space (X, τ_1, τ_2) is called $(1,2)^*$-$\hat{g}$-closed if $\tau_1\tau_2$-$\mathrm{Cl}(A) \subset U$ whenever $A \subset U$ and U is $(1,2)^*$-semi-open in X.

The complement of a $(1,2)^*$-$\hat{g}$-closed set is called $(1,2)^*$-$\hat{g}$-open.

Definition 3.2. A subset A of a bitopological space (X, τ_1, τ_2) is called $(1,2)^*$-α-$\hat{g}$-closed (briefly $(1,2)^*$-$\alpha\hat{g}$-closed) if $(1,2)^*$-α $\mathrm{Cl}(A) \subset U$ whenever $A \subset U$ and U is $(1,2)^*$-$\hat{g}$-open in X.

The complement of a $(1,2)^*$-$\alpha\hat{g}$-closed set is called $(1,2)^*$-$\alpha\hat{g}$-open.

Proposition 3.1. *Every* $\tau_{1,2}$*-open set is* $(1,2)^*$*-*$\hat{g}$*-open.*

Proof. Let A be a $\tau_{1,2}$-open set in X. Then A^c is a $\tau_{1,2}$-closed set. Therefore $\tau_1\tau_2$-$\mathrm{Cl}(A^c) = A^c$. This implies $\tau_1\tau_2$-$\mathrm{Cl}(A^c) \subset U$ whenever $A^c \subset U$ and U is a $(1,2)^*$-semi-open set in (X, τ_1, τ_2). Hence A^c is $(1,2)^*$-$\hat{g}$-closed or A is $(1,2)^*$-$\hat{g}$-open. □

Proposition 3.2. *Every* $(1,2)^*$*-*α*-closed set is* $(1,2)^*$*-*$\alpha\hat{g}$*-closed.*

Proof. Let A be a $(1,2)^*$-α-closed set and U be any $(1,2)^*$-$\hat{g}$-open set containing A. Since A is $(1,2)^*$-α-closed, $(1,2)^*$-$\alpha\mathrm{Cl}(A) = A \subset U$. Hence A is $(1,2)^*$-$\alpha\hat{g}$-closed. □

Corollary 3.1. *Every* $\tau_{1,2}$*-closed set is* $(1,2)^*$*-*$\alpha\hat{g}$*-closed.*

Proof. Every $\tau_{1,2}$-closed set is $(1,2)^*$-α-closed and hence $(1,2)^*$-$\alpha\hat{g}$-closed. □

Remark 3.1. A $(1,2)^*$-$\alpha\hat{g}$-closed set need not be $(1,2)^*$-α-closed as shown in the follwing example.

Example 3.1. Let $X = \{a, b, c\}$
$\tau_1 = \{\phi, X, \{a, b\}\}$; $\tau_2 = \{\phi, X, \{a, c\}\}$;
$\tau_{1,2}$-open sets $= \{\phi, X, \{a, b\}, \{a, c\}\}$
$\tau_{1,2}$-closed sets $= \{\phi, X, \{c\}, \{b\}\}$
{b,c} is $(1,2)^*$-$\alpha\hat{g}$-closed but not $(1,2)^*$-α-closed.

Proposition 3.3. *Every* $(1,2)^*$*-*$\alpha\hat{g}$*-closed set is* $(1,2)^*$*-*αg*-closed.*

Proof. Let A be a $(1,2)^*$-$\alpha\hat{g}$-closed set and U be any $\tau_{1,2}$-open set containing A. Since A is $(1,2)^*$-$\alpha\hat{g}$-closed and every $\tau_{1,2}$-open set is $(1,2)^*$-$\hat{g}$-open , $(1,2)^*$-$\alpha\mathrm{Cl}(A) \subset U$. Hence A is $(1,2)^*$-αg-closed. ☐

Remark 3.2. A $(1,2)^*$-αg-closed set need not be $(1,2)^*$-$\alpha\hat{g}$-closed as shown in the follwing example.

Example 3.2. Let $X = \{a, b, c\}$
$\tau_1 = \{\phi, X, \{a\}\}$; $\tau_2 = \{\phi, X, \{b, c\}\}$;
$\tau_{1,2}$-open sets $= \{\phi, X, \{a\}, \{b, c\}\}$
{a,c} is not $(1,2)^*$-$\alpha\hat{g}$-closed but $(1,2)^*$-αg-closed.

Remark 3.3. From the above results, it follows that the class of $(1,2)^*$-$\alpha\hat{g}$-closed sets lies between the class of $(1,2)^*$-α-closed sets and the class of $(1,2)^*$-αg-closed sets.

Proposition 3.4. *Every* $(1,2)^*$*-*$\alpha\hat{g}$*-closed set is* $(1,2)^*$*-gs-closed.*

Proof. The result follows since every $\tau_{1,2}$-open set is $(1,2)^*$-$\hat{g}$-open and $(1,2)^*$-$s\mathrm{Cl}(A) \subset (1,2)^*$-$\alpha$ $\mathrm{Cl}(A)$ for every subset A of X. ☐

Remark 3.4. A $(1,2)^*$-gs-closed set need not be $(1,2)^*$-$\alpha\hat{g}$-closed as shown in the follwing example.

Example 3.3. Let $X = \{a, b, c\}$
$\tau_1 = \{\phi, X, \{a\}\}$; $\tau_2 = \{\phi, X, \{b\}\}$;
$\tau_{1,2}$-open sets $= \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$
{b} is not $(1,2)^*$-$\alpha\hat{g}$-closed but $(1,2)^*$-gs-closed.

Remark 3.5. $(1,2)^*$-$\alpha\hat{g}$-closedness is independent from $(1,2)^*$-sg-closedness, $(1,2)^*$-g-closedness, $(1,2)^*$-$\hat{g}$-closedness and $(1,2)^*$-$g\alpha$-closedness as shown in the following examples.

Example 3.4. Let $X = \{a, b, c\}$
$\tau_1 = \{\phi, X, \{a\}\}$; $\tau_2 = \{\phi, X, \{a, b\}\}$;
$\tau_{1,2}$-open sets $= \{\phi, X, \{a\}, \{a, b\}\}$
{a,c} is $(1,2)^*$-$\alpha\hat{g}$-closed but neither $(1,2)^*$-$g\alpha$-closed nor $(1,2)^*$-sg-closed. Also {b} is $(1,2)^*$-$\alpha\hat{g}$-closed but neither $(1,2)^*$-g-closed nor $(1,2)^*$-$\hat{g}$-closed.

The set {a} in example 3.3 is $(1,2)^*$-sg-closed but not $(1,2)^*$-$\alpha\hat{g}$-closed. The set {b} in example 3.2 is $(1,2)^*$-g-closed, $(1,2)^*$-$\hat{g}$-closed and $(1,2)^*$-$g\alpha$-closed but not $(1,2)^*$-$\alpha\hat{g}$-closed.

Remark 3.6. From the above discussions and known results we have the following implications. $A \to B$ (resp. $A \nleftrightarrow B$) represents A implies B but not conversely (resp. A and B are independent of each other) and $(1) = \tau_{1,2}$-closed set, $(2) = (1,2)^*$-α-closed set, $(3) = (1,2)^*$-$\hat{g}$-closed set, $(4) = (1,2)^*$-$\alpha\hat{g}$-closed set, $(5) = (1,2)^*$-g-closed set ,$(6) = (1,2)^* - \alpha g$-closed set , $(7) = (1,2)^*$-$g\alpha$-closed set, $(8) = (1,2)^*$-sg-closed set, $(9) = (1,2)^*$-gs-closed set.

$$\begin{tikzcd}
1 \arrow[r] \arrow[d] \arrow[dr] & 3 \arrow[r] \arrow[d, leftrightarrow, "-" marking] & 5 \arrow[d] \arrow[dl, leftrightarrow, "\times" marking] \\
2 \arrow[r] \arrow[d] & 4 \arrow[r] \arrow[d, leftrightarrow, "-" marking] \arrow[dl, leftrightarrow, "\times" marking] \arrow[dr] & 6 \arrow[d] \\
7 & 8 \arrow[r] & 9
\end{tikzcd}$$

Remark 3.7. The union of two $(1,2)^*$-$\alpha\hat{g}$-closed sets need not be $(1,2)^*$-$\alpha\hat{g}$-closed as shown in the following example.

Example 3.5. Let $X = \{a,b,c\}$
$\tau_1 = \{\phi, X, \{a,b\}\}$; $\tau_2 = \{\phi, X, \{b,c\}, \{a,c\}, \{c\}\}$;
$\tau_{1,2}$-open sets $= \{\phi, X, \{a,b\}, \{b,c\}, \{a,c\}, \{c\}\}$
{b} and {c} are $(1,2)^*$-$\alpha\hat{g}$-closed but {b,c} is not $(1,2)^*$-$\alpha\hat{g}$-closed.

Remark 3.8. The intersection of two $(1,2)^*$-$\alpha\hat{g}$-closed sets need not be $(1,2)^*$-$\alpha\hat{g}$-closed as shown in the following example.

Example 3.6. Let $X = \{a,b,c\}$
$\tau_1 = \{\phi, X, \{a\}\}$; $\tau_2 = \{\phi, X\}$;
$\tau_{1,2}$-open sets $= \{\phi, X, \{a\}\}$
{a,b} and {a,c} are $(1,2)^*$-$\alpha\hat{g}$-closed, but {a} is not $(1,2)^*$-$\alpha\hat{g}$-closed.

Proposition 3.5. *If a set A is* $(1,2)^*$-$\alpha\hat{g}$-*closed, then* $(1,2)^*$-$\alpha Cl(A) - A$ *contains no nonempty* $\tau_{1,2}$-*closed set.*

Proof. Let A be $(1,2)^*$-$\alpha\hat{g}$-closed and F a $\tau_{1,2}$-closed subset of $(1,2)^*$-$\alpha Cl(A) - A$. Then $A \subset F^c$; F^c is $\tau_{1,2}$-open and hence $(1,2)^*$-$\hat{g}$-open. Since A is $(1,2)^*$-$\alpha\hat{g}$-closed, $(1,2)^*$-α $\mathrm{Cl}(A) \subset F^c$. Consequently $F \subset (1,2)^*$-α $\mathrm{Cl}(A) \cap ((1,2)^*$-$\alpha$ $\mathrm{Cl}(A))^c$ and this implies F is ϕ. □

Remark 3.9. The converse of Proposition 3.5 need not be true.

Example 3.7. Let $X = \{a,b,c\}$
$\tau_1 = \{\phi, X, \{a\}\}$; $\tau_2 = \{\phi, X\{b,c\}\}$;
Let $A = \{b\}$. $(1,2)^*$-α $\mathrm{Cl}(A) - A$ contains no nonempty $\tau_{1,2}$-closed set. However A is not $(1,2)^*$-$\alpha\hat{g}$-closed.

Proposition 3.6. *If a set A is $(1,2)^*$-$\alpha\hat{g}$-closed, then $(1,2)^*$-$\alpha\mathrm{Cl}(A)$-A contains no nonempty $(1,2)^*$-$\hat{g}$-closed set.*

Proof. Let A be $(1,2)^*$-$\alpha\hat{g}$-closed and F a $(1,2)^*$-$\hat{g}$-closed subset of $(1,2)^*$-$\alpha\mathrm{Cl}(A)$-A. Then $A \subset F^c$ and F^c is $(1,2)^*$-$\hat{g}$-open. So $(1,2)^*$-$\alpha\mathrm{Cl}(A) \subset F^c$. Consequently $F \subset (1,2)^*$-$\alpha\mathrm{Cl}(A) \cap ((1,2)^*$-$\alpha\mathrm{Cl}(A))^c$ and this implies F is ϕ. ☐

Corollary 3.2. *If a set A is $(1,2)^*$-$\alpha\hat{g}$-closed, then $(1,2)^*$-$\alpha\mathrm{Cl}(A)$-A contains no nonempty $\tau_{1,2}$-closed set.*

Proof. It follows since every $\tau_{1,2}$-closed set is $(1,2)^*$-$\hat{g}$-closed. ☐

Proposition 3.7. *If A is a $(1,2)^*$-$\hat{g}$-open and $(1,2)^*$-$\alpha\hat{g}$-closed subset of (X,τ_1,τ_2) then A is a $(1,2)^*$-α-closed subset of (X,τ_1,τ_2).*

Proof. Since A is $(1,2)^*$-$\hat{g}$-open and $(1,2)^*$-$\alpha\hat{g}$-closed, $(1,2)^*$-$\alpha\mathrm{Cl}(A) \subset A$. Hence A is $(1,2)^*$-α-closed. ☐

Definition 3.3. A bitopological space (X,τ_1,τ_2) is said to be a $(1,2)^*$-T_1-space if for every $x, y \in X$, $x \neq y$, there exists a $\tau_{1,2}$-open set U containing x but not y.

Theorem 3.1. *In a $(1,2)^*$-T_1-space (X,τ_1,τ_2) every $(1,2)^*$-$\alpha\hat{g}$-closed set is $(1,2)^*$-α-closed.*

Proof. Let (X,τ_1,τ_2) be a $(1,2)^*$-T_1-space and A be $(1,2)^*$-$\alpha\hat{g}$-closed subset of X. Let $y \notin A$. Then for every $x \in A$, there exists a $\tau_{1,2}$-open set U_x such that $x \in U_x$ and $y \notin U_x$. Then $\cup_{x\in A} U_x = U$ is $\tau_{1,2}$-open and $(1,2)^*$-$\hat{g}$-open. Also $A \subseteq U$ and $y \notin U$. Since A is $(1,2)^*$-$\alpha\hat{g}$-closed, $(1,2)^*$-$\alpha\mathrm{Cl}(A) \subset U$ and therefore $y \notin (1,2)^*$-$\alpha\mathrm{Cl}(A)$. Then $(1,2)^*$-$\alpha\mathrm{Cl}(A) \subset A$ and A is $(1,2)^*$-α-closed. ☐

Proposition 3.8. *Let A be a $(1,2)^*$-$\alpha\hat{g}$-closed subset of (X,τ_1,τ_2).
If $A \subset B \subset (1,2)^*$-$\alpha\mathrm{Cl}(A)$ then B is also a $(1,2)^*$-$\alpha\hat{g}$-closed subset of (X,τ_1,τ_2).*

Proof. Let U be a $(1,2)^*$-$\hat{g}$-open set of (X,τ_1,τ_2) such that $B \subset U$. Then $A \subset U$. Since A is a $(1,2)^*$-$\alpha\hat{g}$-closed set, $(1,2)^*$-$\alpha\mathrm{Cl}(A) \subset U$. Also since $B \subset (1,2)^*$-$\alpha\mathrm{Cl}(A)$, $(1,2)^*$-$\alpha\mathrm{Cl}(B) \subset (1,2)^*$-$\alpha\mathrm{Cl}((1,2)^*$-$\alpha\mathrm{Cl}(A) = (1,2)^*$-$\alpha\mathrm{Cl}(A)$. Thus $(1,2)^*$-$\alpha\mathrm{Cl}(B) \subset U$. Hence B is also a $(1,2)^*$-$\alpha\hat{g}$-closed subset of (X,τ_1,τ_2). ☐

Proposition 3.9. *Let A be a $(1,2)^*$-$\alpha\hat{g}$-closed subset of (X,τ_1,τ_2).
If $A \subset Y \subset X$ and Y is $\tau_{1,2}$-open then A is $(1,2)^*$-αg-closed relative to Y.*

Proof. Let $A \subset U_1$, a $\tau_{1,2}$-open set in Y. Then $U_1 = U \cap Y$, where U is a $\tau_{1,2}$-open set in X. Any $\tau_{1,2}$-open set is $(1,2)^*$-$\hat{g}$-open and A is $(1,2)^*$-$\alpha\hat{g}$-closed, $(1,2)^*$-$\alpha\mathrm{Cl}(A) \subset U$. Hence $(1,2)^*$-$\alpha\mathrm{Cl}(A)$ (relative to Y) $\subset U \cap Y = U_1$ and A is $(1,2)^*$-αg-closed relative to Y. ☐

Proposition 3.10. *For each $a \in X$ either $\{a\}$ is $(1,2)^*$-$\hat{g}$-closed or $\{a\}^c$ is $(1,2)^*$-$\alpha\hat{g}$-closed in X.*

Proof. Suppose that {a} is not $(1,2)^*$-$\hat{g}$-closed in X. Then $\{a\}^c$ is not $(1,2)^*$-$\hat{g}$-open. Therefore the only $(1,2)^*$-$\hat{g}$-open set containing $\{a\}^c$ is X and $(1,2)^*$-$\alpha\mathrm{Cl}(\{a\}^c) \subset X$. Hence $\{a\}^c$ is $(1,2)^*$-$\alpha\hat{g}$-closed. ☐

Theorem 3.2. *Let A be $(1,2)^*$-$\alpha\hat{g}$-closed in X. Then A is $(1,2)^*$-α-closed if and only if $(1,2)^*$-$\alpha\mathrm{Cl}(A) - A$ is $\tau_{1,2}$-closed.*

Proof. Necessity: Let A be a $(1,2)^*$-α-closed subset of X. Then $(1,2)^*$-$\alpha\mathrm{Cl}(A) = A$ and $(1,2)^*$-$\alpha\mathrm{Cl}(A) - A = \phi$ which is $\tau_{1,2}$-closed.
Sufficiency: Since A is $(1,2)^*$-$\alpha\hat{g}$-closed, by Proposition 3.5 $(1,2)^*$-$\alpha\mathrm{Cl}(A) - A$ contains no nonempty $\tau_{1,2}$-closed set. But $(1,2)^*$-$\alpha\mathrm{Cl}(A) - A$ is $\tau_{1,2}$-closed. This implies $(1,2)^*$-$\alpha\mathrm{Cl}(A) - A = \phi$, which means $(1,2)^*$-$\alpha\mathrm{Cl}(A) = A$ and hence A is $(1,2)^*$-α-closed. □

4. APPLICATIONS

Definition 4.1. A subset A of (X, τ_1, τ_2) is called $(1,2)^*$-$\alpha\hat{g}$-open if and only if A^c is $(1,2)^*$-$\alpha\hat{g}$-closed in (X, τ_1, τ_2).

Remark 4.1. For a subset A of (X, τ_1, τ_2), $(1,2)^*$-$\alpha\mathrm{Cl}(A^c) = [(1,2)^*\text{-}\alpha\mathrm{Int}(A)]^c$

Theorem 4.1. *A subset A of (X, τ_1, τ_2) is $(1,2)^*$-$\alpha\hat{g}$-open if and only if $F \subset (1,2)^*$-$\alpha\mathrm{Int}(A)$ whenever F is $(1,2)^*$-$\hat{g}$-closed and $F \subset A$.*

Proof. Necessity: Let A be a $(1,2)^*$-$\alpha\hat{g}$-open set in (X, τ_1, τ_2). Let F be $(1,2)^*$-$\hat{g}$-closed and $F \subset A$. Then $F^c \supseteq A^c$ and F^c is $(1,2)^*$-$\hat{g}$-open. Since A^c is $(1,2)^*$-$\alpha\hat{g}$-closed, $(1,2)^*$-$\alpha\mathrm{Cl}(A^c) \subseteq F^c$. By remark 4.1 $[(1,2)^*\text{-}\alpha\mathrm{Int}(A)]^c \subseteq F^c$. That is $F \subset (1,2)^*$-$\alpha\mathrm{Int}(A)$.
Sufficiency: Let $A^c \subseteq U$ where U is $(1,2)^*$-$\hat{g}$-open. Then $U^c \subset A$ where U^c is $(1,2)^*$-$\hat{g}$-closed. By the hypothesis $U^c \subseteq (1,2)^*$-$\alpha\mathrm{Int}(A)$. That is $[1,2)^*\text{-}\alpha\mathrm{Int}(A)]^c \subseteq U$. By remark 4.1 $(1,2)^*$-$\alpha\mathrm{Cl}(A^c) \subseteq U$. This implies A^c is $(1,2)^*$-$\alpha\hat{g}$-closed. Hence A is $(1,2)^*$-$\alpha\hat{g}$-open. □

Proposition 4.1. *If $(1,2)^*$-$\alpha\mathrm{Int}(A) \subseteq B \subseteq A$ and A is $(1,2)^*$-$\alpha\hat{g}$-open then B is $(1,2)^*$-$\alpha\hat{g}$-open.*

Proof. $(1,2)^*$-$\alpha\mathrm{Int}(A) \subseteq B \subseteq A$ implies $A^c \subseteq B^c \subseteq [(1,2)^*\text{-}\alpha\mathrm{Int}(A)]^c$. By remark 4.1 $A^c \subseteq B^c \subseteq [(1,2)^*\text{-}\alpha\mathrm{Cl}(A^c)$. Also A^c is $(1,2)^*$-$\alpha\hat{g}$-closed. By Proposition 3.8 B^c is $(1,2)^*$-$\alpha\hat{g}$-closed. Hence B is $(1,2)^*$-$\alpha\hat{g}$-open. □

Remark 4.2. Every $\tau_{1,2}$-open set is $(1,2)^*$-$\alpha\hat{g}$-open. But the converse may not be true as shown in the following example.

Example 4.1. Let $X = \{a, b, c\}$
$\tau_1 = \{\phi, X, \{a, b\}\}$; $\tau_2 = \{\phi, X\{a, c\}\}$;
$\tau_{1,2}$-open sets $= \{\phi, X, \{a, b\}, \{a, c\}\}$; {a} is $(1,2)^*$-$\alpha\hat{g}$-open but not $\tau_{1,2}$-open.

As an application of $(1,2)^*$-$\alpha\hat{g}$-closed sets we introduce the following definition.

Definition 4.2. A space (X, τ_1, τ_2) is called a $T_{(1,2)^*-\alpha\hat{g}}$-space if every $(1,2)^*$-$\alpha\hat{g}$-closed set in it is $(1,2)^*$-α-closed.

Theorem 4.2. *For a space (X, τ_1, τ_2) the following conditions are equivalent.*

(i) (X, τ_1, τ_2) is a $T_{(1,2)^*-\alpha\hat{g}}$-space.
(ii) Every singleton set of (X, τ_1, τ_2) is either $(1,2)^*$-$\hat{g}$-closed or $(1,2)^*$-α-open.

Proof. $(i) \Rightarrow (ii)$ Let $x \in X$. Suppose {x} is not a $(1,2)^*$-$\hat{g}$-closed set of (X, τ_1, τ_2). Then $X - \{x\}$ is not a $(1,2)^*$-$\hat{g}$-open set. So X is the only $(1,2)^*$-$\hat{g}$-open set containing $X - \{x\}$. So $X - \{x\}$ is a $(1,2)^*$-$\alpha\hat{g}$-closed set of (X, τ_1, τ_2). Since (X, τ_1, τ_2) is a $T_{(1,2)^*-\alpha\hat{g}}$-space, $X - \{x\}$ is a $(1,2)^*$-α-closed set of (X, τ_1, τ_2) or equivalently {x} is a $(1,2)^*$-α-open set of (X, τ_1, τ_2).
$(ii) \Rightarrow (i)$ Let A be a $(1,2)^*$-$\alpha\hat{g}$-closed set of X. Trivially $A \subset (1,2)^*$-αCl(A). Let $x \in (1,2)^*$-αCl(A).By (ii) {x} is either $(1,2)^*$-$\hat{g}$-closed or $(1,2)^*$-α-open.

(a) Suppose that {x} is $(1,2)^*$-$\hat{g}$-closed. If $x \notin A$, $(1,2)^*$-αCl$(A) - A$ contains a nonempty $(1,2)^*$-$\hat{g}$-closed set {x}. By Proposition 3.6 we arrive at a contradiction. Thus $x \in A$.
(b) Suppose that {x} is $(1,2)^*$-α-open. Since $x \in (1,2)^*$-αCl(A), $\{x\} \cap A \neq \phi$. This implies that $x \in A$.
Thus in any case $x \in A$. So $(1,2)^*$-αCl$(A) \subset A$. Therefore $(1,2)^*$-αCl$(A) = A$ or equivalently A is $(1,2)^*$-α-closed. Hence (X, τ_1, τ_2) is a $T_{(1,2)^*-\alpha\hat{g}}$-space. □

Definition 4.3. A bitopological space (X, τ_1, τ_2) is called

(i) $T_{(1,2)^*b}$- space if every $(1,2)^*$-gs-closed set in X is $\tau_{1,2}$-closed.
(ii) $\alpha T_{(1,2)^*b}$- space if every $(1,2)^*$-αg-closed set in X is $\tau_{1,2}$-closed.

Theorem 4.3. (i) *Every* $T_{(1,2)^*b}$- *space is a* $T_{(1,2)^*-\alpha\hat{g}}$-*space.*
(ii) *Every* $\alpha T_{(1,2)^*b}$- *space is a* $T_{(1,2)^*-\alpha\hat{g}}$-*space.*

Proof. (i) Let A be a $(1,2)^*$-$\alpha\hat{g}$-closed set. Then A is $(1,2)^*$-gs-closed. Since (X, τ_1, τ_2) is $T_{(1,2)^*b}$- space, A is $\tau_{1,2}$-closed. It is true that every $\tau_{1,2}$-closed is $(1,2)^*$-α-closed. Therefore X is $T_{(1,2)^*-\alpha\hat{g}}$-space.
(ii) Let A be a $(1,2)^*$-$\alpha\hat{g}$-closed set. Then A is $(1,2)^*$-αg-closed. Since X is a $\alpha T_{(1,2)^*b}$- space, A is $\tau_{1,2}$-closed. Therefore A is $(1,2)^*$-α-closed which shows X is $T_{(1,2)^*-\alpha\hat{g}}$- space. □

Remark 4.3. The reverse implications in the above theorem are not true in general as shown in the following examples.

Example 4.2. Let $X = \{a, b, c\}$
$\tau_1 = \{\phi, X, \{a\}\}$; $\tau_2 = \{\phi, X\{b\}\}$;
$\tau_{1,2}$-open sets $= \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$; (X, τ_1, τ_2) is a $T_{(1,2)^*-\alpha\hat{g}}$-space but not a $T_{(1,2)^*b}$- space since {b} is $(1,2)^*$-gs-closed but not $\tau_{1,2}$-closed.

Example 4.3. Let $X = \{a, b, c\}$
$\tau_1 = \{\phi, X, \{a\}\}$; $\tau_2 = \{\phi, X\{b, c\}\}$;
$\tau_{1,2}$-open sets $= \{\phi, X, \{a\}, \{b, c\}\}$; (X, τ_1, τ_2) is a $T_{(1,2)^*-\alpha\hat{g}}$-space but not a $\alpha T_{(1,2)^*b}$-space since {a,c} is $(1,2)^*$-αg-closed but not $\tau_{1,2}$-closed.

Definition 4.4. A bitopological space (X,τ_1,τ_2) is called $T_{(1,2)^*\hat{g}}$- space if every $(1,2)^*$-$\hat{g}$-closed set in X is $\tau_{1,2}$-closed.

Proposition 4.2. *Let (X,τ_1,τ_2) be a bitopological space. If a set A is $(1,2)^*$-$\hat{g}$-closed then $\tau_1\tau_2$-Cl$(A)-A$ contains no nonempty $(1,2)^*$-semi-closed set.*

Proof. Suppose $\tau_1\tau_2$-Cl$(A)-A$ contains the $(1,2)^*$-semi-closed set F. Then $A\subseteq F^c$. F^c is $(1,2)^*$-semi-open and A is $(1,2)^*$-$\hat{g}$-closed. Therefore $\tau_1\tau_2$-Cl$(A)\subseteq F^c$. Then $F\subseteq(\tau_1\tau_2\text{-Cl}(A))^c$.
Hence $F\subset\tau_1\tau_2$-Cl$(A)\cap(\tau_1\tau_2\text{-Cl}(A))^c=\phi$ which implies $F=\phi$. □

Theorem 4.4. *For a bitopological space (X,τ_1,τ_2) the following are equivalent.*

(i) *(X,τ_1,τ_2) is a $T_{(1,2)^*\hat{g}}$ space.*
(ii) *Every singleton set $\{x\}$ is either $(1,2)^*$-semi-closed or $\tau_{1,2}$-open.*

Proof. (i)⇒(ii): Let $x\in X$. If {x} is not $(1,2)^*$-semi-closed, then $X-\{x\}$ is not $(1,2)^*$-semi-open. So X is the only $(1,2)^*$-semi-open set containing $X-\{x\}$. Therefore $X-\{x\}$ is $(1,2)^*$-$\hat{g}$-closed. Since X is $T_{(1,2)^*-\hat{g}}$- space, $X-\{x\}$ is $\tau_{1,2}$-closed or {x} is $\tau_{1,2}$-open.
(ii)⇒(i): Let A be a $(1,2)^*$-$\hat{g}$-closed set of (X,τ_1,τ_2). Let $x\in\tau_1\tau_2$-Cl(A). By (ii) {x} is either $(1,2)^*$-semi-closed or $\tau_{1,2}$-open.
Case(i): {x} is $(1,2)^*$-semi-closed. If $x\notin A$, $\tau_1\tau_2$-Cl$(A)-A$ contains a nonempty $(1,2)^*$-semi-closed set {x}. By Proposition 4.2 we arrive at a contradiction. Thus $x\in A$.
Case(ii): Suppose that {x} is $\tau_{1,2}$-open. Since $x\in\tau_1\tau_2$-Cl(A), $\{x\}\cap A\neq\phi$. This implies that $x\in A$.
Thus in any case $x\in A$. So $\tau_1\tau_2$-Cl$(A)\subset A$. Therefore $\tau_1\tau_2$-Cl$(A)=A$ or equivalently A is $\tau_{1,2}$-closed. Hence (X,τ_1,τ_2) is a $T_{(1,2)^*-\hat{g}}$- space. □

Remark 4.4. $T_{(1,2)^*-\alpha\hat{g}}$-spaces and $T_{(1,2)^*-\hat{g}}$-spaces are independent of one another as the following examples show.

Example 4.4. Let $X=\{a,b,c\}$
$\tau_1=\{\phi,X,\{a\}\}$; $\tau_2=\{\phi,X\}$;
$\tau_{1,2}$-open sets $=\{\phi,X,\{a\}\}$; (X,τ_1,τ_2) is a $T_{(1,2)^*-\hat{g}}$-space but not a $T_{(1,2)^*-\alpha\hat{g}}$-space because {a,b} is $(1,2)^*$-$\alpha\hat{g}$-closed but not $(1,2)^*$-α-closed.

Example 4.5. Let $X=\{a,b,c\}$
$\tau_1=\{\phi,X,\{a\}\}$; $\tau_2=\{\phi,X\{b,c\}\}$;
$\tau_{1,2}$-open sets $=\{\phi,X,\{a\},\{b,c\}\}$; (X,τ_1,τ_2) is a $T_{(1,2)^*-\alpha\hat{g}}$-space but not a $T_{(1,2)^*-\hat{g}}$-space since {a,c} is $(1,2)^*$-$\hat{g}$-closed but not $\tau_{1,2}$-closed.

REFERENCES

[1] M. E. Abd El-Monsef, S. Rose Mary and M. Lellis Thivagar: *$\alpha\hat{g}$-closed sets in topological spaces*, Assiut Univ. J. of Mathematics and Computer Science, **36**(2007), No. 1, 43-51.

[2] S. P. Arya and T. Nour: *Characterizations of S-normal spaces*, Indian J. Pure. Appl. Math., **21**(1990), No. 8, 717-719.

[3] P. Bhattacharya and B.K. Lahiri: *Semi-generalised closed sets in topology*, Indian J. Math., **29**(1987), 375-382 .

[4] N. Levine: *Semi-open sets and semi-continuity in topological spaces*, Amer. Math. Monthly, **70**(1963), 36-41.

[5] N. Levine: *Generalised closed sets in topology*, Rend. circ. Mat. Palermo, **19**(1970), 89-96.

[6] H. Maki, R. Devi and K. Balachandran: *Semi-generalised closed maps and generalised semi-closed maps*, Mem. Fac. Sci. Kochi Univ. Ser. A Math., **14**(1993), 41-54.

[7] H. Maki, R. Devi and K. Balachandran: *Generalised α-closed maps and α-generalised closed maps*, Indian J. Pure. Appl. Math., **29**(1998), No. 1, 37-49.

[8] A. S. Mashhour, M. E. Abd El-Monsef and S. N. El Deeb: *On pre continuous and weak pre continuous mappings*, Proc. Math. Phys. Soc. Egypt., **53**(1982), 47-53.

[9] O. Njastad: *On some Classes of nearly open sets*, Pacific J. Math., **15**(1965), 961-970.

[10] O. Ravi and M. Lellis Thivagar, *On Stronger forms of* $(1,2)^*$*-quotient mappings in bitopological spaces*, Internat. J. Math. Game Theory and Algebra, **14**(2004), No. 6, 481-492.

[11] M. K. R. S. Veerakumar: *$\hat{g}$-closed sets in topological spaces*, Bull. Allah. Math. Soc., **18**(2003), 99-112.

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