J. Adv. Math. Studies
Vol. **2**(2009), No. 2, 17-24
`http://journal.fairpartners.ro`

ON THE 2-DIMENSIONAL INTERPOLATION

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Abstract. Generally speaking, the interpolation means a way to get global approximative information from a reduced number of data. It is well known the Lagrange polynomial interpolation, extended to splines, in order to avoid the Runge phenomenon. In the multidimensional case, the solution either does not exist or is not unique. In this paper, one study the set of Lagrange type (characterized in Proposition 2.1) and perturbations of the interpolation formula (Propositions 3.1 and 3.2). In §4 we introduce a new kind of spline, suggested by the theory of deformable plates, permitting a new approach of the interpolation.

1. PRELIMINARIES

Consider integers $n \geq 1$, $r \geq 1$ and a field K of null characteristics (for instance, $K = \mathbb{R}$ or $\mathbb{C}$).

Fix a finite subset $A \subset K^n$, $A = \{a_1, a_2, \ldots, a_N\}$, with N elements, called *ad-hoc nodes*. The direct n-dimensional extension of the classical Lagrange interpolation problem requires to determine a polynomial $P \in K_r[X_1, \ldots, X_n]$ in n undeterminates of degree at most r, having the values $P(a_i), 1 \leq i \leq N$, prescribed (taken from K). The solution cannot exist and cannot be unique, a special part being incumbent upon the "geometry" of the set A. In the differential calculus of functions of several variables, a main idea was to restrict the functions to convenient right-lines and to consider the partial derivatives. Such an idea does not hold for the interpolation.

2. THE POLYNOMIAL INTERPOLATION

A subset $A \subset K^n$, $A = \{a_1, a_2, \ldots, a_N\}$ is said Lagrange-type set if the K-linear map

$$f\colon K_r[X_1, \ldots, X_n] \to K^N, \quad f(P) = (P(a_1), \ldots, P(a_N))$$

is an isomorphism. In this case, for any $b_1, \ldots, b_N \in K$ there is and is unique a polynomial P of degree $\leq r$, with coefficients in K, such that

$$P(a_i) = b_i, \text{ for any } 1 \leq i \leq N. \tag{2.1}$$

Received: May 21, 2009.

2000 *Mathematics Subject Classification*: 65D05; 41A05.

Key words and phrases: Polynomial interpolation, Lagrange-type set, multidimensional interpolation, perturbation estimate, interpolation by thin-plate splines.

If $P(X_1,\dots,X_n)=\sum\limits_{|p|\le r} c_p X_1^{p_1}\cdots X_n^{p_n}$, with $c_p\in K$, $p=(p_1,\dots,p_n)$ and $|p|=p_1+\dots+p_n$, the total number of the coefficients c_p is C_{n+r}^n (as many as the monomials of degree $\le r$ in n undeterminates). By putting $a_i=(a_i^1,\dots,a_i^n)$, the relations (2.1) become

$$\sum_{|p|\le r} c_p\left(a_i^1\right)^{p_1}\cdots(a_i^n)^{p_n}=b_i,\quad 1\le i\le N. \tag{2.2}$$

Necessarily, $N=C_{n+r}^n$. The set of the indices $p=(p_1,\dots,p_n)$ with $|p|\le r$ will be lexicographically ordered.

Proposition 2.1. *Fix n, r as above and a set $A=\{a_1,a_2,\dots,a_N\}\subset K^n$, with $N=C_{n+r}^n$. Consider the matrix $M=(a_{ip})\in M_N(K)$, where $a_{ip}=\left(a_i^1\right)^{p_1}\dots(a_i^n)^{p_n}$, for $1\le i\le N$ and $p=(p_1,\dots,p_n)$ with $|p|\le r$.*

a) *The set A is a Lagrange-type set if and only if M is invertible.*

b) *If $K=\mathbb{R}$ or $\mathbb{C}$, the property of a set $A\subset K^n$ with $N=C_{n+r}^n$ elements to be Lagrange is generic (≡a.e.).*

Proof. a) By definition, A is Lagrange-type set if and only if the system (2.2) has a unique solution for any $b_i\in K$. This is a linear system by the form $M\cdot C=B$, where C and B are the N-dimensional column vectors having the components (c_p) and (b_i). This is true if and only if M is invertible.

b) M invertible means $\det M\neq 0$. But the probability to have $\det M\neq 0$ is equal to 1 (relatively to the Lebesgue measure), hence the fact that a subset A with $N=C_{n+r}^n$ elements is Lagrange-type is almost everywhere. □

Consider now some special cases.

1) The one-dimensional case $n=1$, $r\ge 1$. In this case, $N=r+1$ and any set $A=\{a_1,a_2,\dots,a_N\}$ of distinct elements from K is Lagrange type, since the matrix M is a Vandermonde matrix, hence nonsingular.

2) The 2-dimensional case $n=2$, $r\ge 1$. In this case, $N=C_{r+2}^2$ and necessarily any plane subset of Lagrange type has $\dfrac{(r+1)(r+2)}{2}$ elements. For instance, if $K=\mathbb{R}$ and $A=\{a_1,a_2,a_3\}$, then $N=3$ and $r=1$. If $a_1=(x_1,y_1)$, $a_2=(x_2,y_2)$, $a_3=(x_3,y_3)$, the corresponding matrix is

$$M=\begin{pmatrix}1 & x_1 & y_1\\ 1 & x_2 & y_2\\ 1 & x_3 & y_3\end{pmatrix}.$$

By Proposition 2.1, the set A is Lagrange-type if and only if the nodes a_1, a_2, a_3 are not collinear. If $A=\{a_1,a_2,a_3,a_4,a_5,a_6\}$, $a_p=(x_p,y_p)$, $1\le p\le 6$, the matrix

M becomes

$$M = \begin{pmatrix} 1 & x_1 & y_1 & x_1^2 & x_1y_1 & y_1^2 \\ 1 & x_2 & y_2 & x_2^2 & x_2y_2 & y_2^2 \\ \vdots & & \vdots & & \vdots & \\ 1 & x_6 & y_6 & x_6^2 & x_6y_6 & y_6^2 \end{pmatrix}$$

and A is Lagrange type if and only if the nodes a_i are not situated on the same plane conic.

Let us return to the general case. In this respect, fix $r \geq 1$, and consider $A = \{a_1, a_2, \ldots, a_N\} \subset K^n$ a Lagrange type set with $N = C^n_{n+r}$ elements. Then there are unique polynomials $L_i \in K_r[X_1, \ldots, X_N]$, $1 \leq i \leq N$ which satisfy $L_i(a_j) = \delta_{ij}$, for $1 \leq i, j \leq N$. These polynomials correspond, by the above linear isomorphism f, to the vectors of the canonical base of K^N. Then the unique solution of the system (2.1) is just $P = \sum\limits_{i=1}^{N} b_i L_i$. Indeed,

$$P(a_j) = \left(\sum_{i=1}^{N} b_i L_i\right)(a_j) = \sum_{i=1}^{N} b_i L_i(a_j) = \sum_{i=1}^{N} b_i \delta_{ij} = b_j,$$

for $1 \leq j \leq N$. The polynomials $L_1, \ldots, L_N$ can be called the basic polynomials associated to the Lagrange type set A.

If $K = \mathbb{R}$ and $f(x_1, \ldots, x_n)$, $f\colon \mathbb{R}^n \to \mathbb{R}$ is a function such that the values $f(a_i)$, $1 \leq i \leq N$ in the points of a Lagrange type set A are known, then the classical Lagrange interpolation suggests the approximative formula

$$f \simeq \sum_{i=1}^{N} f(a_i) \cdot L_i. \tag{2.3}$$

3. PERTURBATIONS OF THE INTERPOLATIONS FORMULAS

There are some difficulties to apply the Lagrange interpolation. One is that the formula (2.3) cannot be derived (in the sense that one gets a bed approximative formula). Another one is the following: whenever $B \subset K^n$ is a finite set such that $A \cup B$ is Lagrange type, it is not easy to establish a connection between the basic polynomials associated to $A \cup B$ and to A. At last, recall the Runge phenomenon which proves the dangers of polynomial interpolation for even the least oscillatory data.

Fix a Lagrange type set $A \subset \mathbb{R}^n$, $A = \{a_1, a_2, \ldots, a_N\}$, with $N = C^n_{n+r}$. For any $\varepsilon > 0$, denote by $L_\varepsilon(A)$ the family of all Lagrange-type sets $C = \{c_1, c_2, \ldots, c_N\} \subset \mathbb{R}^n$ such that $||a_p - c_p|| < \varepsilon$, for $1 \leq p \leq N$. By the Proposition 2.1 b), this family is rich. Denote by $C^2(\mathbb{R}^n)$ the set of all functions $f\colon \mathbb{R}^n \to \mathbb{R}$ of class C^2 and consider the $\mathbb{R}$-linear map $\varphi_A\colon C^2(\mathbb{R}^n) \to \mathbb{R}_r[X_1, \ldots, X_n]$, $\varphi_A(f) = \sum\limits_{i=1}^{N} f(a_i) L_i$, where $L_1, \ldots, L_N$ are the basic polynomials associated to A.

Proposition 3.1. *If $C \in L_\varepsilon(A)$ and $f \in C^2(\mathbb{R}^n)$, then there is $M > 0$ depending on ε, such that*

$$||\varphi_A(f) - \varphi_C(f)|| \leq M \cdot ||f||, \tag{3.1}$$

for the euclidian norms.

Proof. For any $d = (d_1, \ldots, d_n) \in \mathbb{R}^n$, $f(x) = f(d) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(d) \cdot (x_i - d_i) + R_1(x)$ and apply φ_A:

$$\varphi_A f(x) = f(d) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(d) \cdot (x_i - d_i) + \varphi_A R_1(x).$$

Therefore $f(x) - \varphi_A f(x) = R_1(x) - \varphi_A R_1(x)$, for any $x \in \mathbb{R}^n$.

Similarly, $f(x) - \varphi_C f(x) = R_1(x) - \varphi_C R_1(x)$, whence $\varphi_A f(x) - \varphi_C f(x) = \varphi_A R_1(x) - \varphi_C R_1(x)$. By making use of the integral Taylor formula, with the Lagrange rest, one gets (3.1). □

The Proposition 3.1 can be used to prove some convergence results.

Suppose now a bounded open subset $D \subset \mathbb{R}^n$ and a function $f\colon D \to \mathbb{R}$ such that are known the values $f(a_i)$, $1 \leq i \leq N$. For any $\varepsilon > 0$, denote by F_ε the set of all functions $\psi\colon D \to \mathbb{R}$ of class C^2 such that $|f(a_i) - \psi(a_i)| < \varepsilon$ for any $1 \leq i \leq N$.

Here is a method to get such functions ψ: one chooses a covering of D by n-dimensional cubes K_i of centers a_i, $1 \leq i \leq N$ and a suitable partition of the unity $\psi_i \in C^2(D)$, null outside K_i, $0 \leq \psi_i \leq 1$, $\sum_i \psi_i = 1$. For any i, one also chooses $f_i \in C^2(D)$ such that $|f(a_i) - f_i(a_i)| < \varepsilon$; taking $\psi = \sum f_i \psi_i$ and keeping into account that $f = \sum_i f \psi_i$, it will follow that

$$\begin{aligned} |f(a_i) - \varphi(a_i)| &= \left| \sum_i (f(a_i) - f_i(a_i)) \cdot \psi_i(a_i) \right| \leq \sum_i |f(a_i) - f_i(a_i)| \cdot \psi_i(a_i) \\ &\leq \varepsilon \sum_i \psi_i(a_i) = \varepsilon. \end{aligned}$$

If the matrix $(\psi_j(a_i))$, $1 \leq i, j \leq N$ is invertible, one can directly take the real numbers $q_1, \ldots, q_N$, such that $\sum_{j=1}^{N} \psi_j(a_i) \cdot q_j = f(a_i)$ and take $\psi = \sum_{i=1}^{N} q_i \psi_i$.

By replacing the basic polynomials by the functions associated to a partition of the unity as above, one can consider the operator

$$\psi_A\colon C^2(D) \to C^2(D), \quad \psi_A(f) = \sum_{i=1}^{N} f(a_i) \cdot \psi_i.$$

Similarly to Proposition 3.1, we have:

Proposition 3.2. *If A and C are two Lagrange-type sets in D with the same number of elements, then for any $f \in C^2(D)$,*

$$||\psi_A(f) - \psi_C(f)|| \le 4M \cdot ||f||, \tag{3.2}$$

where $||f|| = \sup\limits_{x \in D} |f(x)|$ *and* $M = \max\limits_{1 \le i \le N} ||\psi_i||$.

Proof. It is sufficient to remark that for any $x \in D$,

$$|f(x) - \psi_A f(x)| = \left| f(x) - \sum_{i=1}^{N} (f(x) - f(a_i)) \cdot \psi_i(x) \right| \le 2M \cdot ||f||.$$

$\square$

4. INTERPOLATION BY THIN-PLATE SPLINES

In data fitting, computer graphics or medical imaging applications, the 1-dimensional setting cannot be always used. For instance, in image restoration problem, one must construct an invertible smooth transformation $T \colon \mathbb{R}^n \to \mathbb{R}^n$ (for $n = 2$ or 3), which aligns two images in the best possible way. The polynomial interpolants may not have this property, even the data points used form a Lagrange-type set. Instead, the spline interpolation can be extended to higher dimensions, by simply multiplying a basic function in one coordinate with a basic function in another (this working well if the points lie on same kind of rectangular grid).

In what follows, we propose a new kind of transform, inspired from Mechanics of deformable plates. This can be visualized on the transformation providing the optimal solution for the problem of deforming a flat piece of sheet metal at a finite number of locations. Thus, for a given set of N points $a_i = (x_i, y_i)$, $1 \le i \le N$ and a given set of real numbers $b_1, \dots, b_N$, one proposes a tps-transform (thin-plate spline) $T \colon \mathbb{R}^2 \to \mathbb{R}$ such that $T(a_i) = b_i$, $1 \le i \le N$, similarly with (2.1), but by making use of the concept of bending energy.

If $F \colon \mathbb{R}^2 \to \mathbb{R}$ is a function of class $C_c^2(\mathbb{R}^n)$ with compact support, one defines the *bending energy* associated to F, as being the integral

$$E(F) = \iint_{\mathbb{R}^2} \left(F''^2_{xx} + 2F''^2_{xy} + F''^2_{yy} \right) dx\, dy. \tag{4.1}$$

Obviously, $E(F) \ge 0$ and if F is a polynomial of degree 1 in x and y, then $E(F) = 0$. If $F(x, y) = f(x)$, $x \in [a, b]$, then

$$E(F) = \int_a^b f''^2(x) dx \quad \text{(up a multiplicative constant).} \tag{4.2}$$

Such an integral reminds us the cubic-spline functions.

Recall that if $I = [a, b]$ and $\Delta \colon a = x_0 < x_1 < \cdots < x_n < x_{n+1} = b$ is a partition with n interior nodes, a function $s \colon I \to \mathbb{R}$ is called a spline of degree k $(k \ge 1)$ relatively to Δ if $s \in C^{k-1}(I)$, restriction of which to any subinterval

$[x_i, x_{i+1}]$, $0 \leq i \leq n$ is equal to a polynomial of degree at most k. Putting for any $\alpha \in \mathbb{R}$,

$$(x-\alpha)_+^k = \begin{cases} (x-\alpha)^k, & x > \alpha \\ 0, & x \leq \alpha, \end{cases}$$

one knows that the linear space $S_{n,k}(\Delta)$ of all above spline functions has as basis the set $\mathcal{B} = \{1, x, \ldots, x^k, (x-x_1)_+^k, \ldots, (x-x_n)_+^k\}$. Thus, in order to determine a spline $s \in S_{n,k}(\Delta)$, $n+k+1$ conditions are necessary and sufficient. A special part have the cubic splines ($k=3$). If $f \in C^2_{[a,b]}$, then there is a unique spline f_s of degree 3 verifying the following $n+4$ conditions: $f_s(x_i) = f(x_i)$, $1 \leq i \leq n$, $f_s(a) = f(a)$, $f_s'(a) = f'(a)$, $f_s(b) = f(b)$, $f_s'(b) = f'(b)$. This $f_s \in S_{n,3}(\Delta)$ is called the cubic spline of interpolation associated to f. Of course, one can propose the approximative formula: $f(x) \simeq f_s(x)$ for any $x \in [a,b]$. It holds the following

Proposition 4.1. *The minimum of the integral* (4.2) *for* $f \in C^2_{[a,b]}$ *holds for* f_s.

Proof. We have to prove that $\int_a^b f_s''(x)^2 dx \leq \int_a^b f''(x)^2 dx = E(F)$. Indeed, if we integrate by parts and keep into account that the third derivative is constant on pieces, then $\int_a^b f_s''(x) \cdot \left(f''(x) - f_s''(x)\right) dx = -\int_a^b f'''(x) \cdot \left(f'(x) - f_s'(x)\right) dx = 0$, hence $\int_a^b f_s''(x) \cdot f''(x) dx = \int_a^b f_s''(x)^2 dx$, therefore

$$0 \leq \int_a^b \left(f''(x) - f_s''(x)\right)^2 dx = \int_a^b f''(x)^2 dx - \int_a^b f_s''(x)^2 dx,$$

whence the result. ☐

One naturally put the problem to minimize the energy functional $E\colon C^2_{[a,b]} \to \mathbb{R}$ given by (4.1). Any function φ of class C^2 (with compact support) which minimizes E must satisfy the Euler-Lagrange-Gauss equation and applying this, one deduces that $\frac{\partial^4 \varphi}{\partial x^4} + 2\frac{\partial^4 \varphi}{\partial x^2 \partial y^2} + \frac{\partial^4 \varphi}{\partial y^4} = 0$, well-known in Elasticity Theory. The first term is also denoted by $\Delta^2 \varphi$ and called bilaplacian of φ. Thus, we have to solve the biharmonic equation $\Delta^2 \varphi = 0$. We do not check for all the solutions, but only a special ones.

In polar coordinates ρ, θ, one knows that the laplacian is

$$\Delta = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \theta^2}.$$

The solutions of the above biharmonic equation which do not depends on θ (also called radial) satisfy the equation $\left(\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho}\right)^2 \varphi = 0$. In order, to solve this, put $\psi = \varphi'' + \frac{1}{\rho}\varphi'$. Then $\psi'' + \frac{1}{\rho}\psi' = 0$, $\rho\psi'' + \psi' = 0$, that is $\frac{\partial}{\partial \rho}\left(\rho\psi'(\rho)\right) = 0$

and $\psi(\rho) = A\ln\rho + B$, with A and B real constants. Therefore, $\varphi''(\rho) + \frac{1}{\rho}\varphi'(\rho) = A\ln\rho + B$. Put $\nu(\rho) = \varphi'(\rho)$; it follows that $\nu'(\rho) + \frac{1}{\rho}\nu(\rho) = A\ln\rho + B$, a first order linear differential equation. One gets $\varphi'(\rho) = \frac{A}{2}\rho\ln\rho + \left(\frac{B}{2} - \frac{A}{4}\right)\rho + \frac{C}{\rho}$, with C a new real constant. Finally, $\varphi(x,y) = A\rho^2 \cdot \ln(\rho^2) + B\ln\rho + C\rho^2$ with $\rho = \left(x^2 + y^2\right)^{\frac{1}{2}}$ and A, B, C real constants. If $\rho = 0$, take $\varphi(0,0) = 0$.

These radial biharmonic functions suggest the construction of the following thin-plate splines.

Consider a finite set A of N nodes $a_i = (x_i, y_i)$, $1 \leq i \leq N$ and fix any N real numbers $b_1, \ldots, b_N$. Instead to find polynomials P such that (2.1) hold, for $n = 2$ one checks functions which are linear combinations of the following N basic functions: $\mathcal{B} = \{\varphi(x - x_1, y - y_1), \ldots, \varphi(x - x_N, y - y_N)\}$, where φ is any fixed radial function as above. So, if $f(x,y) = \sum_{j=1}^{N} c_j \varphi(x - x_j, y - y_j)$ and denote $\varphi_{ij} = \varphi(x_i - x_j, y_i - y_j)$, $1 \leq i, j \leq N$ (hence $\varphi_{ii} = 0$), the relations $f(a_i) = b_i$, $1 \leq i \leq n$ become $\sum_{j=1}^{N} \varphi_{ij} c_j = b_i$, with unknowns $c_1, \ldots, c_N$. Considering the matrices

$$\Phi = (\varphi_{ij}), \quad X = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_N \end{pmatrix}, \quad B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_N \end{pmatrix},$$

one gets the linear system $\Phi \cdot X = B$, with N equations and N unknowns, similar to that from Proposition 2.1.

In this case, there is no restriction of type $N = C_{n+r}^{n}$ as in the case of Lagrange-type sets of nodes. For any fixed set of nodes $\{a_1, \ldots, a_N\}$as above and any data $b_1, \ldots, b_N$, one can consider the corresponding basis $\mathcal{B}$ and one gets the local function f. Such functions can be concatenated to describe a plate $D \subset \mathbb{R}^2$, with some regions of much interest; for instance, in the case of the 2-dimensional medical imaging, one can apply such interpolation by special splines for regions modeling the heart or the tubes of the lungs.

5. CONCLUSIONS

The interpolation means to obtain a global information from a restricted finite number of numerical data, taken in some discrete nodes. In the 1-dimensional case, a classical method is the Lagrange interpolation by polynomials, extended to splines; but in the multidimensional case, there are many difficulties which impose rigid restrictions on the set of chosen nodes. In the first part of the paper, we have studied the Lagrange-type interpolation by polynomials in the multidimensional case

(Proposition 2.1), also giving some results on the perturbation of the set of nodes (Propositions 3.1 and 3.2); by making use of the energetical functional taken from the study of deformable plates, we propose another class of interpolation functions, different from polynomials, which allow to enlarge the class of splines. These can be used in the description and processing different 2D configurations.

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