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THE GEODESICS OF A PSEUDO-RIEMANNIAN MANIFOLD

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Abstract. Given an arbitrary pseudo-Riemannian manifold (M, g), consider a real function f, defined on M, whose Hessian with respect to the initial pseudo-Riemannian metric g is non-degenerate. Then we obtain a new pseudo-Riemannian manifold (M, h), where $h = \nabla_g^2 f$. The aim of this work is twofold. First we provide the general form of geodesic equations of a given pseudo-Riemannian manifold. Then we suggest a possible numerical study. We indicate an implicit parametric solution and use a numerical method to produce the direction geodesic field plots as well as a phase portrait of solution curves.

1. INTRODUCTION AND PRELIMINARIES

According to [12], [13], a pseudo-Riemannian metric of signature (p, q) on a smooth manifold M of dimension $n = p + q$ is a smooth symmetric differentiable 2-form g on M such that, at each point x of M, g_x is non-degenerate on T_xM with the signature (p, q). We call (M, g) *a pseudo-Riemannian manifold.*

Let be given a pseudo-Riemannian manifold (M, g). The fundamental theorem of pseudo-Riemannian geometry states that there exists an unique linear connection ∇_g on M, called the Levi-Civita connection (of g), such that the following two assertions hold good:

a) ∇_g is metric (i.e. $\nabla_g g = 0$); b) ∇_g is torsion-free (i.e. $T = 0$).

If $(U, x^1, \dots, x^n)$ is a coordinate chart on M, then the Christoffel symbols Γ^k_{ij} of the Levi-Civita connection are related to the functions g_{ij} by the formulas

$$\Gamma^k_{ij} = \frac{1}{2} g^{k\ell} \left(\frac{\partial g_{\ell i}}{\partial x^j} + \frac{\partial g_{j\ell}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^\ell} \right).$$

Also, the curvature R has the components R^ℓ_{ijk} given by

$$R^\ell_{ijk} = \frac{\partial \Gamma^\ell_{ki}}{\partial x^j} - \frac{\partial \Gamma^\ell_{ji}}{\partial x^k} + \Gamma^r_{ki}\Gamma^\ell_{jr} - \Gamma^r_{ji}\Gamma^\ell_{kr}.$$

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Note that in local coordinates a geodesic $\gamma(t) = (x^i(t))_{i=1,\dots,n}$ satisfies a system of n second order differential equations

$$\ddot{x}^i + \Gamma^i_{jk}\dot{x}^j\dot{x}^k = 0, \qquad i = 1,\dots,n.$$

If $f\colon M \to \mathbb{R}$ is a smooth function, then the second covariant derivative

$$\nabla^2_g f = \left(\frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma^k_{ij}\frac{\partial f}{\partial x^k}\right) dx^i \otimes dx^j$$

is called the Hessian of f, [14], [17].

Let us suppose that the Hessian $h = \nabla^2_g f$ is non-degenerate. Then h is a pseudo-Riemannian metric which produces the Levi-Civita connection ∇_h and the Christoffel symbols $\bar{\Gamma}^k_{ij}$.

Throughout this paper, we shall use the following notations:

$$f_{,i} = \frac{\partial f}{\partial x^i}; \qquad f_{,ij} = \frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma^m_{ij} f_{,m}; \qquad f_{,ijk} = \frac{\partial f_{,ij}}{\partial x^k} - \Gamma^\ell_{ki} f_{,\ell j} - \Gamma^\ell_{kj} f_{,\ell i}.$$

We have, [3]

Theorem 1.1. *Let $f,^{pk}$ be the contravariant components of the pseudo-Riemannian metric $h_{pk} = f_{,pk}$ and R^m_{ijk} be the components of the curvature tensor field produced by the pseudo-Riemannian metric g_{ij}. Then the components of Levi-Civita connection ∇_h are given by the following formula*

$$\bar{\Gamma}^p_{ij} = \Gamma^p_{ij} + \frac{1}{2} f,^{kp} \left[f_{,ijk} + (R^m_{ikj} + R^m_{jki}) f_{,m}\right].$$

Corollary 1.1. *The differential system of geodesics is*

$$\ddot{x}^p + \left[\Gamma^p_{ij} + f,^{pk}\left(\frac{1}{2} f_{,ijk} + R^\ell_{ikj} f_{,\ell}\right)\right]\dot{x}^i\dot{x}^j = 0, \quad p = 1,\dots,n.$$

We remark that Corollary 1.1 is the generalization of Theorem 2.1 from [9] in the pseudo-Riemannian case.

2. MAIN RESULT

Let us take $M = \mathbb{R}^2_+$ be the positive quadrant endowed with the metric $g = (g_{ij})$, where

$$g_{ij}(x, y) = \operatorname{diag}\left(\frac{1}{x^2}, \frac{1}{y^2}\right).$$

It is known that this metric has the Christoffel coefficients

$$\Gamma^1_{11} = -\frac{1}{x}, \quad \Gamma^2_{22} = -\frac{1}{y}, \quad \Gamma^2_{11} = \Gamma^1_{21} = \Gamma^1_{12} = \Gamma^2_{12} = \Gamma^2_{21} = \Gamma^1_{22} = 0.$$

We choose the function

$$f\colon \mathbb{R}^2_+ \to \mathbb{R}, \quad f(x, y) = \frac{1}{x} + \sqrt{x} + \frac{1}{y} + \sqrt{y},$$

and we prove that $\nabla^2 f = (f_{,ij})$ is positive definite on (M, g).

Indeed, after straightforward calculation, we get

$$\nabla^2 f = \operatorname{diag}\left(\frac{1}{x^3} + \frac{1}{4x\sqrt{x}}, \frac{1}{y^3} + \frac{1}{4y\sqrt{y}}\right)$$

which is positive definite on $\mathbb{R}^2_+$, therefore $\left(\mathbb{R}^2_+, \nabla^2 f\right)$ is a Riemannian manifold.

According to Theorem 1.1, we have

$$\bar{\Gamma}^1_{11} = -\frac{3}{x}\left(\frac{1}{4} + \frac{1}{4 + x\sqrt{x}}\right), \quad \bar{\Gamma}^2_{22} = -\frac{3}{y}\left(\frac{1}{4} + \frac{1}{4 + y\sqrt{y}}\right),$$
$$\bar{\Gamma}^2_{11} = 0, \quad \bar{\Gamma}^1_{12} = \bar{\Gamma}^1_{21} = 0, \quad \bar{\Gamma}^2_{12} = \bar{\Gamma}^2_{21} = 0, \quad \bar{\Gamma}^1_{22} = 0,$$

and with $p = 1$ in Corollary 1.1, we find $x''(t) = -\bar{\Gamma}^1_{11}(x'(t))^2$. We put $x = x(t)$ and state

Theorem 2.1. *The component x of geodesic curves of the Riemannian manifold $\left(\mathbb{R}^2_+, \nabla^2 f\right)$ is solution of the* 2nd *order* ODE

$$x'' = \frac{3}{x}\left(\frac{1}{4} + \frac{1}{4 + x\sqrt{x}}\right)\left(x'\right)^2, \tag{2.1}$$

with arbitrary initial values.

Remark 2.1. With $p = 2$ in Corollary 1.1, we find $y''(t) = -\bar{\Gamma}^2_{22}(y'(t))^2$, therefore y satisfies the ODE in Theorem 2.1 too.

To find a more convenient form of equation (2.1), we arrange it in the form

$$\frac{x''}{x'} = \frac{3}{4}\frac{x'}{x} + \frac{3x'}{x\left(4 + x\sqrt{x}\right)},$$

and, after an integration, in the form

$$\ln x'(t) = \frac{3}{4}\ln x(t) + 3\int \frac{x'(t)}{x(t)\left(4 + x(t)\sqrt{x(t)}\right)}\, dt. \tag{2.2}$$

We denote by I the integral in the right side of (2.2). Changing the variable $\sqrt{x} = u$ reduces this integral to the form

$$I = 2\int \frac{1}{u\left(4 + u^3\right)}\, du.$$

Using the general procedure for the integration of rational functions, after straightforward calculation, we obtain a primitive

$$I = \ln\frac{\sqrt[4]{x}}{\sqrt[6]{x\sqrt{x} + 4}} + \ln k_1,$$

where k_1 is any positive real constant. Using this primitive, we can write (2.2) as

$$\ln x'(t) = \frac{3}{4}\ln x(t) + 3\ln\frac{\sqrt[4]{x(t)}}{\sqrt[6]{x(t)\sqrt{x(t)}+4}} + \ln k_1. \tag{2.3}$$

After an arrangement in (2.3) and using for y the same form, we can state our main result. This is given in

Theorem 2.2. *The geodesic curves of the Riemannian manifold* $\left(\mathbb{R}^2_+, \nabla^2 f\right)$ *are solutions of the first order* ODE *system*

$$x'(t) = \frac{k_1 x(t)^{\frac{3}{2}}}{\sqrt{x(t)^{\frac{3}{2}}+4}}, \quad y'(t) = \frac{k_2 y(t)^{\frac{3}{2}}}{\sqrt{y(t)^{\frac{3}{2}}+4}}, \tag{2.4}$$

where k_1 and k_2 are any positive real constants, and the initial values are arbitrary.

The equations (2.4) cannot be integrated using analytical methods because we meet a Chebyshev integral of this form $\int u^{-2}\sqrt{u^3+4}\,du$. However, a numerical study of this system could be useful.

3. A SOLUTION STUDY

In this section, we find an implicit parametric solution of (2.4), and use a numerical method to produce the direction geodesic field plots as well as a phase portrait of solution curves.

For our first purpose, it is enough to make the study of the first equation in (2.4). We consider both x and k_1 be positive, and we write the first equation in (2.4) as

$$k_1 dt = \frac{\sqrt{x\sqrt{x}+4}}{x\sqrt{x}}\,dx. \tag{3.1}$$

If we integrate (3.1) side by side, we obtain

$$k_1 t + c_1 = \int \frac{\sqrt{x\sqrt{x}+4}}{x\sqrt{x}}\,dx.$$

Changing the variable $\sqrt{x} = v$ reduces this equality to the form

$$k_1 t + c_1 = 2\int \frac{\sqrt{v^3+4}}{v^2}\,dv.$$

or, more convenient, to the form

$$k_1 t + c_1 = \int \left(v + \frac{4}{v^2}\right)\cdot\frac{1}{\sqrt{1+\dfrac{v^3}{4}}}\,dv. \tag{3.2}$$

If $v \in \left(0, \sqrt[3]{4}\right)$, then we can make use of Taylor expansion and write (3.2) as

$$k_1 t + c_1 = \int \left(\left(v + \frac{4}{v^2} \right) \sum_{n \geq 0} (-1)^n \frac{(2n)!}{2^{2n} (n!)^2} \left(\frac{v^3}{4} \right)^n \right) dv. \tag{3.3}$$

If we commute the sum and the integral in (3.3), after integration we obtain

$$k_1 t + c_1 = \sum_{n \geq 0} (-1)^n \frac{(2n)!}{2^{4n} (n!)^2} \left(\frac{v^{3n+2}}{3n+2} + \frac{4 v^{3n-1}}{3n-1} \right).$$

Finally, we return to x as variable. So, if $x \in \left(0, \sqrt[3]{16}\right)$, we have

$$k_1 t + c_1 = \sum_{n \geq 0} (-1)^n \frac{(2n)!}{2^{4n} (n!)^2} \left(\frac{x^{\frac{1}{2}(3n+2)}}{3n+2} + \frac{4 x^{\frac{1}{2}(3n-1)}}{3n-1} \right). \tag{3.4}$$

We underly that equation (3.4) gives an implicit parametric form of the solution $x = x\,(t, k_1, c_1)$ of the ODEs in Theorem 2.2. A similar form can be written for $y = y\,(t, k_2, c_2)$, where k_2 and c_2 are real constants, $k_2 > 0$. Therefore, we have obtained this result.

Theorem 3.1. *Let* $0 < x, y < \sqrt[3]{16}$. *The implicit parametric solution of* ODEs *in Theorem* 2.2 *is given by*

$$\begin{aligned} k_1 t + c_1 &= \sum_{n \geq 0} (-1)^n \frac{(2n)!}{2^{4n} (n!)^2} \left(\frac{x^{\frac{1}{2}(3n+2)}}{3n+2} + \frac{4 x^{\frac{1}{2}(3n-1)}}{3n-1} \right); \\ k_2 t + c_2 &= \sum_{n \geq 0} (-1)^n \frac{(2n)!}{2^{4n} (n!)^2} \left(\frac{y^{\frac{1}{2}(3n+2)}}{3n+2} + \frac{4 y^{\frac{1}{2}(3n-1)}}{3n-1} \right), \end{aligned}$$

k_1, c_1, k_2 *and* c_2 *being real constants,* $k_1 > 0$, $k_2 > 0$.

We could imagine a computer aided study of ODEs in (2.4). This may be performed by asymptotic methods or by means of numerical procedures. For our purposes, we choose the numerical way with MAPLE (in [4], such kind of study is made using MAPLE for PDEs).

Since the system in (2.4) is determined to be autonomous, we can produce direction geodesic field plots (grid of arrows tangential to solution curves) as indicated in Figures 1, 2 and 3.

$k_1 = 10$ and $k_2 = 1$; the component x is dominant and the direction field is almost horizontal, see Figure 1.

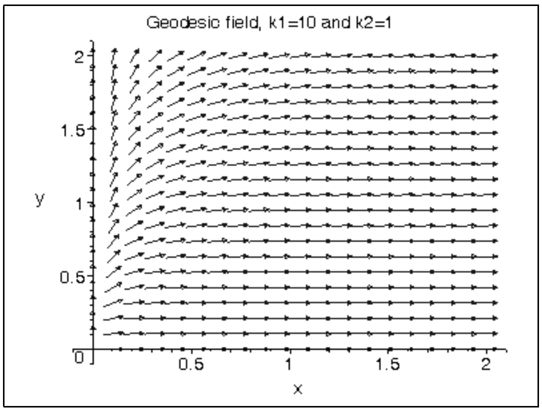

Figure 1

$k_1 = 2$ and $k_2 = 1$; no one component is dominant; the direction field has the shape plotted in Figure 2.

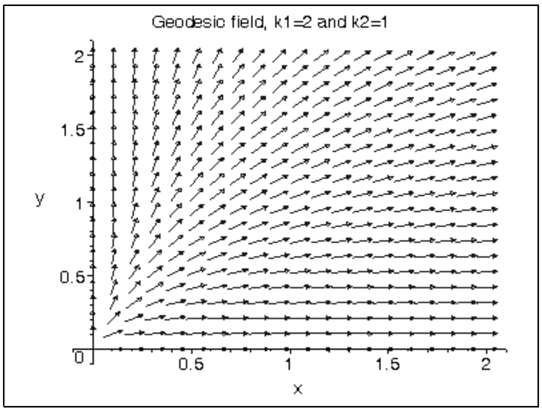

Figure 2

$k_1 = 1$ and $k_2 = 10$; the component y is dominant and the direction field is almost vertical, see Figure 3.

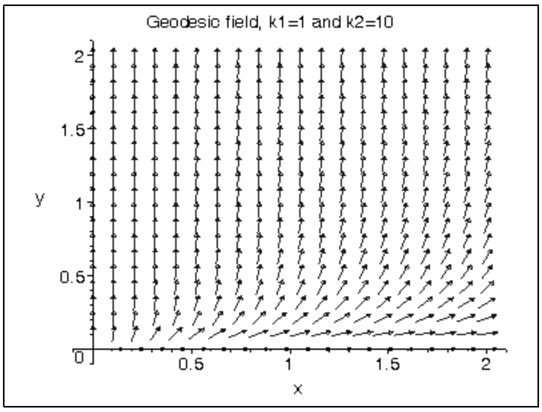

Figure 3

For the system of first order differential equations (2.4), in Figure 4 is plotted a phase portrait of solution curves, $x(t)$, by a numerical method. We used $k_1 = 1$ and the list of initial conditions $x(1) \in \{0.33; 0.66; 1.0\}$. Remark that the solution curve $y(t)$ has the same shape.

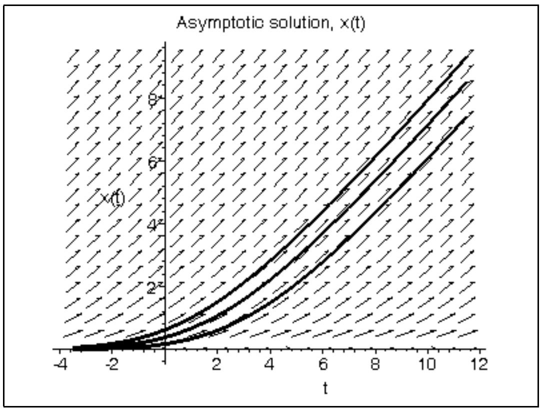

Figure 4

4. CONCLUSION

In this paper we solved the problem of finding the geodesic equations [3], [18] of a class of pseudo-Riemannian manifolds, but the problem is open for other classes of manifolds. For this case study, we indicate an implicit parametric solution and use a numerical method to produce the direction geodesic field plots as well as a phase portrait of solution curves. The results our work give a relevant link between differential geometry and applied (experimental) sciences, see [1] for geometrical methods in Statistics, [2] for mathematical modeling in Ecology, [15] for optimization methods on manifolds. Regarding different but related viewpoints, the authors address the reader to these treatises and to the research works [5]÷[8], [10], [11], [16] and [19] as well.

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