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WIENER-HOPF EQUATIONS TECHNIQUE FOR MULTIVALUED GENERAL VARIATIONAL INEQUALITIES

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ABSTRACT. In this paper, we introduce a new class of variational inequalities involving two operators. Using the projection technique, we establish the equivalence between the multivalued general variational inequalities and the fixed point problems as well as with the Wiener-Hopf equations. This equivalent formulation is used to suggest and analyze some iterative algorithms for solving the multivalued general variational inequalities. We also discuss the convergence analysis of these iterative methods. Several special cases are also discussed.

1. INTRODUCTION

Variational inequalities, which were introduced by Stampacchia [21] in early sixties, have played an important role in the development of various fields of pure and applied sciences. Variational inequalities have been generalized and extended in several directions using novel and new techniques, see [1-20] and the references therein. Noor [12] has shown that the minimum of a class of differentiable nonconvex functions on a nonconvex set can be characterized by a class of variational inequalities, which is known as general variational inequalities. Motivated and inspired by the research going on in this field, we introduced and study a new class of variational inequalities, which is called the multivalued general variational inequalities. Using the projection techniques, we establish the equivalence bewteen the multivalued general variational inequalities and the multivalued Wiener-Hopf equations. This equivalence is used to suggest and analyze a class of projection iterative methods for solving the multivalued general variational inequalities. We also study the convergence criteria of the proposed iterative methods under suitable conditions. Results obtain in this paper include the previously obtained results of Noor [12] as special cases.

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2. PRELIMINARIES

Let K be a nonempty closed and convex set in a real Hilbert space H, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ respectively. For given nonlinear operators $g \colon H \to H$ and $T \colon H \to 2^H$, consider the problem of finding $u \in H : \eta \in Tu$ such that

$$\langle \rho\eta + u - g(u), g(v) - u \rangle \geq 0, \quad \forall v \in H : g(v) \in K, \tag{2.1}$$

where $\rho > 0$ is a constant. Inequality of type (2.1) is called the multivalued general variational inequality involving two operators.

For $g \equiv I$, the identity operator, the multivalued general variational inequality (2.1) is equivalent to finding $u \in K$ such that

$$\langle \eta, v - u \rangle \geq 0, \quad \forall v \in K, \tag{2.2}$$

which is called the multivalued variational inequality.

If T is single valued then the multivalued general variational inequality (2.1) is equivalent to finding $u \in H : g(u) \in H$ such that

$$\langle \rho Tu + u - g(u), g(v) - u \rangle \geq 0, \quad \forall v \in H : g(v) \in K, \tag{2.3}$$

which is called the general variational inequality introduced and studied by Noor [12] in connection with nonconvex functions. See also Noor and Noor [15,16] for more details.

If $g \equiv I$, the identity operator, then problem (2.3) is equivalent to finding $u \in K$ such that

$$\langle Tu, v - u \rangle \geq 0, \quad \forall v \in K, \tag{2.4}$$

which is known as the classical variational inequality introduced and studied by Stampacchia [21] in 1964. For the recent trends and developments in variational inequalities, see [2,12-16] and the references therein.

We also need the following well known concepts and results.

Lemma 2.1 ([3]). *For a given $z \in H$, $u \in H$ satisfies the inequality*

$$\langle u - z, v - u \rangle \geq 0, \quad \forall v \in K,$$

if and only if

$$u = P_K z,$$

where P_K is the the projection operator of H onto the closed convex set K.

It is well known that the projection operator P_K is a nonexpansive operator.

Definition 2.1. A mapping $g \colon H \to H$ is called

(i) δ-Lipschitz, if for all $u_1, u_2 \in H$, there exists a constant $\delta > 0$, such that

$$||g(u_1) - g(u_2)|| \leq \delta ||u_1 - u_2||;$$

(ii) σ-strongly monotone, if for all $u_1, u_2 \in K$, there exists a constant $\sigma > 0$, such that

$$\langle g(u_1) - g(u_2), u_1 - u_2 \rangle \geq \sigma ||u_1 - u_2||^2.$$

Definition 2.2. A multivalued operator $T\colon H \to 2^H$ is called

(i) M-Lipschitz, if for all $u_1, u_2 \in H$, there exists a constant $\beta > 0$ such that

$$\eta \in Tu : M(Tu_1 - Tu_2) \leq \beta||u_1 - u_2||.$$

(ii) α-strongly monotone, if for all $u_1, u_2 \in K$, there exists a constant $\alpha > 0$, such that

$$\langle \eta_1 - \eta_2, u_1 - u_2 \rangle \geq \alpha||u_1 - u_2||^2, \eta_1 \in Tu_1, \eta_2 \in Tu_2$$

where $M(\cdot, \cdot)$ is the Hausdorff metric on $C(H)$.

3. MAIN RESULTS

In this section, we use the Wiener-Hopf equation technique to suggest and analyze an iterative method for solving the multivalued general variational inequality (2.1). We now consider the problem of solving the Wiener-Hopf equations. To be more precise, let $Q_K = I - gP_K$, where I is the identity operator and g is a nonlinear operator. For given nonlinear operators $T\colon H \to 2^H$ and $g\colon H \to H$, we consider the problem of finding $z \in H, u \in H$ such that $\eta \in Tu$ and

$$\eta + \rho^{-1} Q_k z = 0, \tag{3.1}$$

where $\rho > 0$ is a constant. Equations (3.1) are called the multivalued general Wiener-Hopf equations.

Note that if $T\colon H \to H$ is a single valued then the problem (3.1) is equivalent to finding $z \in H$ such that

$$TP_k z + \rho^{-1} TP_k z = Q_K z \tag{3.2}$$

which is known as the general Wiener-Hopf equation. We note that if $g = I$ in (3.2) then one can obtain the original Wiener-Hopf equation, which are mainly due to Shi [20]. It has been shown that the Wiener-Hopf equations have played an important and significant role in developing several numerical techniques for solving variational inequalities and related optimization problems, see [8-18].

First of all, we show that the variational inequality problem (2.1) is equivalent to the fixed point problem using Lemma 2.1, which is the main motivation of our next result.

Lemma 3.1. *The function $u \in H : \eta \in Tu$ such that $g(u) \in K$ is a solution of the multivalued general variational inequality* (2.1) *if and only if $u \in k$, $\eta \in Tu$ satisfies the relation*

$$u = P_K[g(u) - \rho\eta], \eta \in Tu,$$

where $\rho > 0$ is a constant and P_k is the projection operator.

Proof. Let $u \in H : \eta \in Tu$ be a solution of (2.1). Then

$$\langle u - (g(u) - \rho\eta), g(v) - u \rangle \geq 0, \quad \forall v \in H : g(v) \in K,$$

which is equivalent to

$$u = P_K[g(u) - \rho\eta],$$

the required result. □

Using Lemma 3.1, we have.

Lemma 3.2. *The function $u \in H : \eta \in Tu$ such that $g(u) \in K$ is a solution of the multivalued general variational inequality* (2.1) *if and only if $z \in H, u \in H : \eta \in Tu$ is a solution of the multivalued general Wiener-Hopf equation* (3.1)*, where*

$$u = P_K z \tag{3.3}$$

$$z = g(u) - \rho\eta, \tag{3.4}$$

where $\rho > 0$ is a constant and P_K is the projection operator.

Proof. Let $u \in K$ be a solution of (2.1), then from Lemma 3.1; we have

$$u = P_K[g(u) - \rho\eta].$$

Now by using the fact that $Q_K = I - gP_K$, we obtain

$$Q_K(g(u) - \rho\eta) = g(u) - \rho\eta - gP_K(g(u) - \rho\eta) = -\rho\eta$$

from which it follows that

$$\eta + \rho^{-1} Q_K z = 0,$$

where

$$z = g(u) - \rho\eta.$$

Conversely, let $z \in H, u \in H : \eta \in Tu$ such that $\eta \in Tu$ be a solution of the multivalued general Wiener-Hopf equations (3.1). Then

$$\rho\eta = -Q_K z = gP_K(z) - z.$$

From Lemma (2.1) and (3.3), for all $g(v) \in K$, we obtain

$$0 \leq \langle P_K z - z, g(v) - u\rangle = \langle \rho\eta + u - g(u), g(v) - u\rangle,$$

which implies that

$$\langle \rho Tu + u - g(u), g(v) - u\rangle \geq 0.$$

Thus $u = P_K z$ is the solution of the multivalued variational inequalities (2.1). □

Lemma 3.2 implies that the multivalued general variational inequalities (2.1) and the multivalued general Wiener-Hopf equation (3.1) are equivalent. We use this equivalent formulation to suggest a number of iterative methods for solving the multivalued general variational inequalities (2.1).

Using (3.4), the Wiener-Hopf equation (3.1) can be rewritten in the form as

$$Q_K z = -\rho\eta,$$

which implies that

$$z = gP_K z - \rho\eta = g(u) - \rho\eta.$$

By an appropriate and suitable rearrangements of the terms of the general Wiener-Hopf equations (3.1), one can suggest and analyze a number of iterative methods for solving the multivalued general variational inequalities (2.1) and related optimization problems.

This fixed point formulation enables to suggest the following iterative method for solving problem (2.1).

Algorithm 3.1. For a given $z_0 \in H, u_0 \in K,\ \eta_0 \in Tu_0$ such that $g(u_0) \in K$ compute the approximate solution $z_{n+1}, u_{n+1}, \eta_{n+1}$ by the iterative schemes:

$$u_n = P_K z_n, \tag{3.5}$$

$$z_{n+1} = (1-a_n)z_n + a_n P_K[g(u_n) - \rho\eta_n], n = 0,1,2\dots \tag{3.6}$$

$$\eta_n \in Tu_n : ||\eta_n - \eta_{n-1}|| \leq M(Tu_n - Tu_{n-1}), \tag{3.7}$$

where $\rho > 0$ is a constant and $a_n \in [0,1]$ for all $n \geq 0$. Algorithm 3.1 is known as Mann iteration for solving the Wiener-Hopf equations.

Note that if $g = I$, the identity operator, then Algorithm 3.1 reduces to the following iterative method for solving the multivalued variational inequalities (2.2).

Algorithm 3.2. For a given $z_0 \in H, u_0 \in K,\ \eta_0 \in Tu_0$, compute the approximate solution $z_{n+1}, u_{n+1}, \eta_{n+1}$ by the iterative schemes:

$$u_n = P_K z_n$$

$$z_{n+1} = (1-a_n)z_n + a_n P_K[u_n - \rho\eta_n], n = 0,1,2.$$

$$\eta_n \in Tu_n : ||\eta_n - \eta_{n-1}|| \leq M(Tu_n - Tu_{n-1})$$

where $\rho > 0$ is a constant and $a_n \in [0,1]$ for all $n \geq 0$.

If T is single valued operator then Algorithm 3.1 reduces to the following Algorithm 3.3 for solving problem (2.3).

Algorithm 3.3. For a given $z_0 \in H, u_0 \in K$ such that $g(u_0) \in K$, compute the approximate solution z_{n+1}, u_{n+1} by the iterative schemes:

$$u_n = P_K z_n$$

$$z_{n+1} = (1-a_n)z_n + a_n P_K[g(u_n) - \rho Tu_n], n = 0,1,2.$$

where $\rho > 0$ is a constant and $a_n \in [0,1]$ for all $n \geq 0$.

If $g = I$ and T is single valued, then Algorithm 3.1 reduces to the following Algorithm 3.4 for solving the problem (2.4).

Algorithm 3.4. For a given $z_0 \in H, u_0 \in K$, compute the approximate solution z_{n+1}, u_{n+1} by the iterative schemes

$$u_n = P_K z_n$$

$$z_{n+1} = (1-a_n)z_n + a_n P_K[u_n - \rho Tu_n], n = 0,1,2.$$

where $\rho > 0$ is a constant and $a_n \in [0,1]$ for all $n \geq 0$.

We now consider the convergence analysis of Algorithm 3.1 and this is the main motivation of our next result.

Theorem 3.1. *Let the multivalued operator* $T\colon H \to 2^H$ *be* α*-strongly monotone and M-Lipschitz with constant* $\beta > 0$*, and* $g\colon H \to H$ *be* σ*-strongly monotone and* δ*-Lipschitz. If*

$$\left|\rho - \frac{\alpha}{\beta^2}\right| < \frac{\sqrt{\alpha^2 - \mu^2(2k-k^2)}}{\beta^2}, \tag{3.8}$$

$$\alpha > \beta\sqrt{k(2-k)}, \quad k < 1,$$

where

$$k = \sqrt{1 - 2\sigma + \delta^2}, \tag{3.9}$$

and $a_n \in [0,1]$, $\sum_{n=0}^{\infty} a_n = \infty$*, then the approximate solutions* $\{z_n\}, \{u_n\}$ *and* $\{\eta_n\}$ *obtained from Algorithm* 3.1 *converges strongly to a solution* $z \in H, u \in K$ $, \eta \in Tu$ *satisfying the multivalued general Wiener-Hopf equations* (3.1)*.*

Proof. Let $u \in K, \eta \in Tu$ such that $g(u) \in K$ be a solution of (2.1). Then, using Lemma 3.2, we have

$$z = (1 - a_n)z + a_n P_K\{g(u) - \rho\eta\}, \tag{3.10}$$

where $a_n \in [0,1]$ and $\sum_{n=0}^{\infty} a_n = \infty$. To prove the result, we need first to evaluate $||z_{n+1} - z||$ for all $n \geq 0$. From (3.4) and (3.6), and the nonexpansivity of P_K, we have

$$\begin{aligned} ||z_{n+1} - z|| &= ||(1-a_n)z_n + a_n P_K\{g(u_n) - \rho\eta_n\} - (1-a_n)z - a_n P_K\{g(u) - \rho\eta\}|| \\ &\leq (1-a_n)||z_n - z|| + a_n||g(u_n) - g(u) - \rho(\eta_n - \eta)|| \\ &\leq (1-a_n)||z_n - z|| + a_n||u_n - u - (g(u_n) - g(u))|| \\ &\quad + a_n||u_n - u - \rho(\eta_n - \eta)||. \end{aligned} \tag{3.11}$$

From the α-strongly monotonicity and M-Lipschitz continuity of the operator T, we have

$$\begin{aligned} &||u_n - u - \rho(\eta_n - \eta)||^2 \\ &\quad = ||u_n - u||^2 - 2\rho\langle \eta_n - \eta, u_n - u\rangle + \rho^2||\eta_n - \eta||^2 \\ &\quad \leq ||u_n - u||^2 - 2\rho\alpha||u_n - u\|^2 + \rho^2(M(Tu_1 - Tu_2))^2 \\ &\quad \leq ||u_n - u||^2 - 2\rho\alpha||u_n - u||^2 + \rho^2\beta^2||u_n - u||^2] \\ &\quad = (1 - 2\rho\alpha + \rho^2\beta^2)||u_n - u||^2. \end{aligned} \tag{3.12}$$

In a similar way, we have

$$\begin{aligned} \|u_n - u - (g(u_n) - g(u))\| &\leq [1 - 2\sigma + \delta^2]||u_n - u||^2 \\ &= k^2||u_n - u||^2, \end{aligned} \tag{3.13}$$

where k is defined by (3.9).

Combining (3.11), (3.12) and (3.13), we have

$$||z_{n+1} - z|| \leq (1 - a_n)||z_n - z|| + a_n\theta||u_n - u||, \tag{3.14}$$

where $\theta = \sqrt{1 - 2\rho\alpha + \rho^2\beta^2} + k$.

Also from (3.3) and (3.5) and the nonexpansivity of the projection operator P_K, we have

$$||u_n - u|| \leq ||P_K z_n - P_K z|| \leq ||z_n - z||. \tag{3.15}$$

Combining (3.14), and (3.15), we have

$$\begin{aligned} ||z_{n+1} - z|| &\leq (1 - a_n)||z_n - z|| + a_n\theta||z_n - z|| \\ &\leq [1 - (1 - \theta)a_n]||z_n - z||, \\ &\leq \prod_{i=0}^{n}[1 - (1 - \theta)a_i]||z_0 - z||. \end{aligned}$$

Since $\sum_{n=0}^{\infty} a_n$ diverges and $1 - \theta > 0$, we have $\lim_{n\to\infty} \prod_{i=0}^{n}[1 - (1 - \theta)]a_i = 0$ and then $\lim_{n\to\infty} ||z_{n+1} - z|| = 0$. Consequently the sequences $\{z_n\}$ and $\{u_n\}$ converges strongly to z in H and u in K.

Now we prove that $\eta_n \to \eta \in Tu$. From (3.7), we have

$$||\eta_n - \eta_{n-1}|| \leq M(Tu_n - Tu_{n-1}) \leq \beta||u_n - u_{n-1}||,$$

which implies that $\{\eta_n\}$ is a Cauchy sequence in H, so there exists $\eta \in H$ such that $\eta_n \to \eta$. Further,

$$\begin{aligned} d(\eta, Tu) = Inf\{||\eta - t|| : t \in Tu\} &\leq ||\eta - \eta_n|| + d(\eta_n, Tu) \\ &\leq ||\eta - \eta_n|| + M(Tu_n - Tu_{n-1}) \\ &\leq ||\eta - \eta_n|| + \beta||u_n - u|| \to 0. \end{aligned}$$

Hence, since Tu is closed, we have $\eta \in Tu$, which completes the proof. □

REFERENCES

[1] E. Al-Shemas and S. Billups: *An iterative method for generalized set-valued nonlinear mixed quasi-variational inequalities*, J. Com. Appl. Math., **170**(2004), 423-432.

[2] E. Al-Shemas: *Projection iterative methods for multivalued general variational inequalities*, Appl. Math. Inf. Sci., **3**(2009), No. 2, 177-184.

[3] C. Baiocchi, A. Capelo: *Variational and Quasi-Variational Inequalities*, J. Wiley and Sons, New York, London, 1984.

[4] F. Giannessi, A. Maugeri: *Variational Inequalities and Network Equilibrium Problems,* Plenum Press, New York, 1995.

[5] R. Glowinski, J. L. Lions, R. Trémolières: *Numerical Analysis of Variational Inequalities*, North-Holland, Amsterdam, 1981.

[6] M. Aslam Noor: *Some algorithms for general monotone mixed variational inequalities*, Math. Computer Modelling, **29**(1999), 1-9.

[7] M. Aslam Noor: *Some recent advances in variational inequalities, Part I - Basic concepts*, New Zealand J. Math., **26**(1997), 53-80.

[8] M. Aslam Noor: *Some recent advances in variational inequalities, Part II - Other concepts*, New Zealand J. Math., **26**(1997), 229-255.

[9] M. Aslam Noor: *New approximation schemes for general variational inequalities*, J. Math. Anal. Appl., **251**(2000), 217-229.

[10] M. Aslam Noor: *A Wiener-Hopf dynamical system for variational inequalities*, New Zealand J. Math., **31**(2002), 173-182.

[11] M. Aslam Noor: *Some developments in general variational inequalities*, Appl. Math. Comput., **152**(2004), 199-277.

[12] M. Aslam Noor: *Differentiable nonconvex functions and general variational inequalities*, Appl. Math. Comput., **199**(2008), 623-630.

[13] M. Aslam Noor: *On a class of general variational inequalities*, J. Adv. Math. Studies, **1**(2008), No. 1-2, 75-86.

[14] M. Aslam Noor: *On Hermite-Hadamard integral inequalities for product of two nonconvex functions*, J. Adv. Math. Studies, **2**(2009), No. 1, 53-62.

[15] M. Aslam Noor and K. Inayat Noor: *On sensitivity analysis of general variational inequalities*, Math. Comm., **13**(2008), 75-83.

[16] M. Aslam Noor and K. Inayat Noor: *Projection algorithms for solving a system of general variational inequalities*, Nonl. Anal., **70**(2009), 2700-2706.

[17] M. Aslam Noor, Y. J. Wang and N. Xiu: *Some new projection methods for variational inequalities*, Appl. Math. Computation, **137**(2003), 423-435.

[18] M. Patriksson: *Nonlinear Programming and Variational Inequalities: A Unified Approach*, Kluwer Academic Publishers, Dordrecht, 1998.

[19] A. Pitonyok, P. Shi and M. Shillor: *On an iterative method for variational inequalities*, Numer. Math., **58**(1990), 231-242.

[20] P. Shi: *Equivalence of Wiener-Hopf equations with variational inequalities*, Proc. Amer. Math. Soc., **111**(1991), 339- 346.

[21] G. Stampacchia: *Formes bilineaires coercivities sur les ensembles convexes*, C. R. Acad. Sci. Paris, **258**(1964), 4413-4416.

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