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ON WEAKLY SYMMETRIC $(LCS)_n$-MANIFOLDS

ABSOS ALI SHAIKH AND TRAN QUOC BINH

Abstract. The object of the present paper is to provide the existence of weakly symmetric and weakly Ricci-symmetric $(LCS)_n$-manifolds by several non-trivial new examples and obtained various interesting results in such manifolds.

1. INTRODUCTION

In 1989, Tamássy and Binh ([10], [11]) introduced the notion of weakly symmetric and weakly Ricci-symmetric Riemannian manifolds and studied such structures on Sasakian manifolds and proved that such a structure does not always exist. Weakly symmetric and weakly Ricci-symmetric structures are also studied by Shaikh and Jana ([6], [8], [9]).

Recently the first author ([7]) introduced the notion of Lorentzian concircular structure manifolds (briefly $(LCS)_n$-manifolds) with an example, which generalizes the notion of LP-Sasakian manifolds introduced by Matsumoto ([4]). The present paper deals with a study of weakly symmetric and weakly Ricci-symmetric $(LCS)_n$-manifolds. Section 2 is concerned with weakly symmetric manifolds. Section 3 consists of some fundamental results of $(LCS)_n$-manifolds. In section 4 we investigate the nature of 1-forms of the weakly symmetric $(LCS)_n$-manifolds. Section 5 deals with weakly Ricci-symmetric $(LCS)_n$-manifolds. In the last section, the existence of weakly symmetric and weakly Ricci-symmetric $(LCS)_n$-manifolds are ensured by several new examples.

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2. WEAKLY SYMMETRIC MANIFOLDS

A non-flat Riemannian manifold (M^n, g) $(n > 2)$ is called a weakly symmetric manifold if its curvature tensor R of type (0, 4) satisfies the condition

$$\begin{aligned}(\nabla_X R)(Y,Z,U,V) = {} & \alpha(X)R(Y,Z,U,V) + \beta(Y)R(X,Z,U,V) \\ & +\gamma(Z)R(Y,X,U,V) + \delta(U)R(Y,Z,X,V) \\ & +\sigma(V)R(Y,Z,U,X),\end{aligned} \tag{2.1}$$

for all vector fields $X,\ Y,\ Z,\ U,\ V \in \chi(M)$, where $\alpha, \beta, \gamma, \delta$ and σ are 1-forms (non zero simultaneously) and ∇ is the operator of covariant differentiation with respect to the Riemannian metric g. The 1-forms are called the associated 1-forms of the manifold and an n-dimensional manifold of this kind is denoted by $(WS)_n$. Recently, De and Bandyopadhyay ([2]) proved that in a $(WS)_n$ the associated 1-forms $\beta = \gamma$ and $\delta = \sigma$. Hence (2.1) reduces to the following form:

$$\begin{aligned}(\nabla_X R)(Y,Z,U,V) = {} & \alpha(X)R(Y,Z,U,V) + \beta(Y)R(X,Z,U,V) \\ & +\beta(Z)R(Y,X,U,V) + \delta(U)R(Y,Z,X,V) \\ & +\delta(V)R(Y,Z,U,X).\end{aligned} \tag{2.2}$$

Also Shaikh and Jana [6] studied weakly symmetric manifolds with various new examples of both vanishing and non-vanishing scalar curvature. In this connection it may be mentioned that although the definition of a $(WS)_n$ is similar to that of a generalized pseudosymmetric manifold introduced by Chaki ([1]), the defining condition of a $(WS)_n$ is little weaker than that of a generalized pseudosymmetric manifold. That is, if in (2.1) the 1-form α is replaced by 2α and σ is replaced by α, then the manifold will be a generalized pseudosymmetric manifold ([1]). Again a Riemannian manifold (M^n, g) $(n > 2)$ is called a weakly Ricci-symmetric if there exist 1-forms A, B, C such that

$$(\nabla_X)S(Y,Z) = A(X)S(Y,Z) + B(Y)S(X,Z) + C(Z)S(X,Y) \tag{2.3}$$

holds for any vector fields X, Y, Z where S is the Ricci tensor of the manifold of type (0, 2).

3. $(LCS)_n$-MANIFOLDS

An n-dimensional Lorentzian manifold M is a smooth connected paracompact Hausdorff manifold with a Lorentzian metric g, that is, M admits a smooth symmetric tensor field g of type (0, 2) such that for each point $p \in M$, the tensor $g_p\colon T_pM \times T_pM \to R$ is a non-degenerate inner product of signature $(-, +, \ldots, +)$, where T_pM denotes the tangent vector space of M at p and R is the real number space. A non-zero vector $v \in T_pM$ is said to be timelike (resp., non-spacelike, null, spacelike) if it satisfies $g_p(v,v) < 0$ (resp, ≤ 0, $= 0$, > 0) ([5]). The category to which a given vector falls is called its causal character.

Let M^n be a Lorentzian manifold admitting a unit timelike concircular vector field ξ, called the characteristic vector field of the manifold. Then we have ([7])

$$g(\xi,\xi) = -1, \tag{3.1}$$

and

$$(\nabla_X \eta)(Y) = \alpha\{g(X,Y) + \eta(X)\eta(Y)\} \quad (\alpha \neq 0) \tag{3.2}$$

for all vector fields X, Y where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g , η is 1-form associated to ξ,

$$g(X,\xi) = \eta(X), \tag{3.3}$$

and α is a non-zero scalar function satisfies

$$X\alpha = \rho\eta(X), \tag{3.4}$$

ρ being a certain scalar function given by $\rho = -(\xi\alpha)$. If we put

$$\phi X = \frac{1}{\alpha}\nabla_X \xi, \tag{3.5}$$

then from (3.3) and (3.5) we have

$$\phi X = X + \eta(X)\xi, \tag{3.6}$$

from which it follows that ϕ is a symmetric (1, 1) tensor. Thus the Lorentzian manifold M^n together with the unit timelike concircular vector field ξ, its associated 1-form η and (1, 1) tensor field ϕ is said to be a Lorentzian concircular structure manifold (*briefly* $(LCS)_n$*-manifold*) ([7]). Especially, if we take $\alpha = 1$, then we can obtain the LP-Sasakian structure of Matsumoto ([4]). In a $(LCS)_n$-manifold, the following relations hold ([7]):

$$\text{a) } \eta(\xi) = -1, \text{ b) } \phi\xi = 0, \text{ c) } \eta(\phi X) = 0, \text{ d) } g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y), \tag{3.7}$$

$$\eta(R(X,Y)Z) = (\alpha^2 - \rho)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)], \tag{3.8}$$

$$S(X,\xi) = (n-1)(\alpha^2 - \rho)\eta(X), \tag{3.9}$$

$$R(X,Y)\xi = (\alpha^2 - \rho)[\eta(Y)X - \eta(X)Y], \tag{3.10}$$

$$(\nabla_X \phi)(Y) = \alpha\{g(X,Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X\}, \tag{3.11}$$

for any vector fields X, Y, Z where R, S denote respectively the curvature tensor and the Ricci tensor of the manifold. Throughout the paper the curvature tensor of both type (1, 3) and (0, 4) will be denoted by the same letter R.

Lemma 3.1. *In a $(LCS)_n$ -manifold, the following relation holds:*

$$X\rho = -\xi(\rho)\eta(X), \tag{3.12}$$

for any vector field X.

Proof. From (3.4), it follows that

$$\nabla(d\alpha)(X,Y)=\nabla_X(d\alpha)(Y)=X(Y\alpha)-((\nabla_XY)\alpha)$$

which implies that

$$\nabla(d\alpha)(X,Y)=(d\alpha)(Y,X). \tag{3.13}$$

Also

$$\nabla(d\alpha)(Y,X)=Y(d\alpha(X))-d\alpha(\nabla_YX)$$

which implies by virtue of (3.3) and (3.4) that

$$\nabla(d\alpha)(Y,X)=(Y\rho)\eta(X)+\rho\alpha[g(X,Y)+\eta(X)\eta(Y)].$$

This implies by virtue of (3.2) that

$$(X\rho)\eta(Y)=(Y\rho)\eta(X).$$

which yields

$$(X\rho)\eta(\xi)=\xi(\rho)\eta(X)$$

From $\eta(\xi)=-1$ it follows $X\rho=-\xi(\rho)\eta(X)$. Hence the result. □

4. WEAKLY SYMMETRIC $(LCS)_n$-MANIFOLDS

Definition 4.1. A $(LCS)_n$-manifold (M^n,g) $(n>2)$ is said to be a weakly symmetric if its curvature tensor R of type (0, 4) satisfies the condition

$$\begin{aligned}(\nabla_XR)(Y,Z,U,V)=&A^*(X)R(Y,Z,U,V)+B^*(Y)R(X,Z,U,V)\\&+B^*(Z)R(Y,X,U,V)+C^*(U)R(Y,Z,X,V)\\&+C^*(V)R(Y,Z,U,X),\end{aligned} \tag{4.1}$$

for all vector fields $X,\ Y,\ Z,\ U,\ V\in\chi(M^n)$, where $A^*,B^*$and C^* are 1-forms (non zero simultaneously) and ∇ is the operator of covariant differentiation with respect to the Lorentzian metric g.

Taking an orthonormal frame field at any point of the manifold and then contracting over Y and V in (4.1) we get

$$\begin{aligned}(\nabla_X)S(Z,U)=A^*(X)S(Z,U)+B^*(Z)S(X,U)+B^*(R(X,Z)U)\\+C^*(U)S(X,Z)+C^*(R(X,U)Z).\end{aligned} \tag{4.2}$$

Theorem 4.1. *In a weakly symmetric $(LCS)_n$-manifold the following relation holds*

$$A^*(\xi)+B^*(\xi)+C^*(\xi)=-\frac{(2\alpha\rho-\beta)}{(\alpha^2-\rho)}. \tag{4.3}$$

Proof. In a weakly Ricci-symmetric $(LCS)_n$-manifold we have the relation (4.2). Setting $U = \xi$ in (4.2) we get

$$(\nabla_X)S(Z,\xi) = (n-1)(\alpha^2-\rho)[A^*(X)\eta(Z)+B^*(Z)\eta(X)]+C^*(\xi)S(Z,X) \quad (4.4)$$
$$+B^*(R(X,Z)\xi)+C^*(R(X,\xi)Z).$$

Using (3.2) and (3.8) we get

$$(\nabla_X)S(Z,\xi) = (n-1)(\alpha^2-\rho)[A^*(X)\eta(Z)+B^*(Z)\eta(X)]+C^*(\xi)S(Z,X) \quad (4.5)$$
$$+(\alpha^2-\rho)[B^*(X)\eta(Z)-B^*(Z)\eta(X)+C^*(X)\eta(Z)$$
$$-g(X,Z)C^*(\xi)].$$

Again

$$(\nabla_X S)(Y,\xi) = \nabla_X S(Y,\xi) - S(\nabla_X Y,\xi) - S(Y,\nabla_X \xi)$$

which yields by virtue of (3.3), (3.4), (3.9) and (3.12) that

$$(\nabla_X S)(Y,\xi) = (n-1)[(2\alpha\rho-\beta)\eta(X)\eta(Y)+\alpha(\alpha^2-\rho)g(X,Y)]-\alpha S(X,Y). \quad (4.6)$$

From (4.5) and (4.6), it follows that

$$\alpha S(X,Z) = (n-1)(2\alpha\rho-\beta)\eta(X)\eta(Z)+(\alpha^2-\rho)[(n-1)\{\alpha g(X,Z)$$
$$-A^*(X)\eta(Z)-B^*(Z)\eta(X)\}-B^*(X)\eta(Z)+B^*(Z)\eta(X)$$
$$-C^*(X)\eta(Z)+C^*(\xi)g(X,Z)]-C^*(\xi)S(Z,X). \quad (4.7)$$

Setting $X = Z = \xi$ in (4.7) we obtain (4.3) (since $\alpha^2 \neq \rho$). This proves the theorem. □

Theorem 4.2. *In a weakly symmetric $(LCS)_n$-manifold $(n > 2)$ the following relation holds*

$$A^*(X)+B^*(X)+C^*(X) = \frac{(2\alpha\rho-\beta)}{(\alpha^2-\rho)}\eta(X). \quad (4.8)$$

Proof. In a weakly symmetric $(LCS)_n$-manifold we have the relation (4.7). Setting $X = \xi$ in (4.7) we get

$$(n-2)B^*(Z) = -(n-1)(2\alpha\rho-\beta)\eta(Z)-(\alpha^2-\rho)(n-1)[A^*(\xi)+C^*(\xi)]\eta(Z)$$
$$-B^*(\xi)\eta(Z). \quad (4.9)$$

In view of (4.3), the relation (4.9) reduces to

$$(n-2)(\alpha^2-\rho)B^*(Z) = -(n-2)(\alpha^2-\rho)B^*(\xi)\eta(Z) \quad (4.10)$$

which yields (since $\alpha^2 \neq \rho$)

$$B^*(Z) = -B^*(\xi)\eta(Z). \quad (4.11)$$

Again, taking an orthonormal frame field at any point of the manifold and then contracting over Z and U in (4.1) we get

$$(\nabla_X S)(Y,V) = A^*(X)S(Y,V) + B^*(Y)S(X,V) + B^*(R(X,Y)V) \qquad (4.12)$$
$$+C^*(V)S(X,Y) + C^*(R(X,V)Y).$$

Setting $Y = \xi$ in (4.12) and using (3.2), (3.3), (3.4), (3.8), (3.9), (3.12) and (4.6) we get

$$\begin{aligned}\alpha S(X,V) = (n-1)(2\alpha\rho - \beta)\eta(X)\eta(V) + (\alpha^2 - \rho)[(n-1)\{\alpha g(X,V)\\ -A^*(X)\eta(V) - C^*(V)\eta(X)\} - C^*(X)\eta(V) + C^*(V)\eta(X)\\ -B^*(X)\eta(V) + B^*(\xi)g(X,V)] - B^*(\xi)S(V,X). \qquad (4.13)\end{aligned}$$

Replacing X by ξ in (4.13), it can be easily seen that

$$C^*(V) = -C^*(\xi)\eta(V). \qquad (4.14)$$

Also (4.13) yields by setting $V = \xi$ that

$$A^*(X) = -A^*(\xi)\eta(X). \qquad (4.15)$$

Adding (4.11), (4.14) and (4.15) and then using (4.3), we obtain (4.8). Hence the theorem. □

Theorem 4.3. *A weakly symmetric $(LCS)_n$-manifold is an η-Einstein manifold provided that $\alpha + B^*(\xi) \neq 0$.*

Proof. In a weakly Ricci-symmetric $(LCS)_n$-manifold we have the relation (4.13). In view of (4.11), (4.14) and (4.15 the relation (4.13) yields

$$\begin{aligned}(\alpha + B^*(\xi))S(Y,V) = (\alpha^2 - \rho)[\alpha(n-1) + B^*(\xi)]g(X,V) \qquad (4.16)\\ +[(n-1)\{(2\alpha\rho - \beta) + (\alpha^2 - \rho)(A^*(\xi) + C^*(\xi))\}\\ +(\alpha^2 - \rho)B^*(\xi)]\eta(Y)\eta(V).\end{aligned}$$

Using (4.3) in (4.16) we get

$$S(Y,V) = \lambda g(X,V) + \mu\eta(Y)\cdot\eta(V) \qquad (4.17)$$

where $\lambda = \dfrac{(\alpha^2 - \rho)[\alpha(n-1) + B^*(\xi)]}{(\alpha + B^*(\xi))}$ and $\mu = -\dfrac{(n-1)(\alpha^2 - \rho)B^*(\xi)}{(\alpha + B^*(\xi))}$. This proves the theorem. □

5. WEAKLY RICCI-SYMMETRIC $(LCS)_n$-MANIFOLDS

Definition 5.1. A $(LCS)_n$-manifold (M^n, g) $(n > 2)$ is called a weakly Ricci-symmetric if there exist 1-forms A, B, C such that

$$(\nabla_X S)(Y,Z) = A(X)S(Y,Z) + B(Y)S(X,Z) + C(Z)S(X,Y) \qquad (5.1)$$

holds for any vector fields X, Y, Z where A, B and C are 1-forms (non zero simultaneously) and ∇ is the operator of covariant differentiation with respect to the Lorentzian metric g..

Theorem 5.1. *In a weakly Ricci-symmetric $(LCS)_n$-manifold $(n > 2)$ the following relation holds*

$$A(\xi) + B(\xi) + C(\xi) = -\frac{(2\alpha\rho - \beta)}{(\alpha^2 - \rho)}. \tag{5.2}$$

Proof. In a weakly Ricci-symmetric $(LCS)_n$-manifold we have the relation (5.1). Setting $Z = \xi$ in (5.1) we get

$$(\nabla_X S)(Y, \xi) = (n-1)(\alpha^2 - \rho)[A(X)\eta(Y) + B(Y)\eta(X)] + C(\xi)S(X, Y). \tag{5.3}$$

Again

$$(\nabla_X S)(Y, \xi) = \nabla_X S(Y, \xi) - S(\nabla_X Y, \xi) - S(Y, \nabla_X \xi)$$

which yields by virtue of (3.3), (3.4), (3.9) and (3.12) that

$$(\nabla_X S)(Y, \xi) = (n-1)[(2\alpha\rho - \beta)\eta(X)\eta(Y) + \alpha(\alpha^2 - \rho)g(X, Y)] - \alpha S(X, Y). \tag{5.4}$$

From (5.3) and (5.4), it follows that

$$\begin{aligned}\alpha S(X, Y) = (n-1)[(2\alpha\rho - \beta)\eta(X)\eta(Y) + \alpha(\alpha^2 - \rho)g(X, Y)\\ -(\alpha^2 - \rho)\{A(X)\eta(Y) + B(Y)\eta(X)\}] - C(\xi)S(X, Y).\end{aligned} \tag{5.5}$$

Setting $X = Y = \xi$ in (5.5) we obtain (5.2) (since $\alpha^2 \neq \rho$). This proves the theorem. □

Theorem 5.2. *In a weakly Ricci-symmetric $(LCS)_n$-manifold $(n > 2)$ the following relation holds*

$$A(X) + B(X) + C(X) = \frac{(2\alpha\rho - \beta)}{(\alpha^2 - \rho)}\eta(X). \tag{5.6}$$

Proof. In a weakly Ricci-symmetric $(LCS)_n$-manifold we have the relation (5.5). Setting $Y = \xi$ in (5.5) we get

$$(\alpha^2 - \rho)A(X) = [(\alpha^2 - \rho)\{B(\xi) + C(\xi)\} + (2\alpha\rho - \beta)]\eta(X). \tag{5.7}$$

In view of (5.2), the relation (5.7) reduces to

$$(\alpha^2 - \rho)A(X) = -(\alpha^2 - \rho)A(\xi)\eta(X) \tag{5.8}$$

which yields (since $\alpha^2 \neq \rho$)

$$A(X) = -A(\xi)\eta(X). \tag{5.9}$$

In a similar manner we can obtain

$$B(X) = -B(\xi)\eta(X) \tag{5.10}$$

and

$$C(X) = -C(\xi)\eta(X). \tag{5.11}$$

Adding (5.9), (5.10), (5.11) and then using (5.2) we obtain (5.6). This proves the theorem. □

Theorem 5.3. *A weakly Ricci-symmetric* $(LCS)_n$*-manifold* $(n > 2)$ *is an* η*-Einstein manifold provided that* $\alpha + C(\xi) \neq 0$.

Proof. In a weakly Ricci-symmetric $(LCS)_n$-manifold we have the relation (5.5). In view of (5.9), (5.10) and (5.11, the relation (5.5) yields

$$\begin{aligned}[\alpha + C(\xi)]S(X,Y) = (n-1)\alpha(\alpha^2 - \rho)g(X,Y) + (n-1)[(2\alpha\rho - \beta) \\ +(\alpha^2 - \rho)\{A(\xi) + B(\xi)\}]\eta(X)\eta(Y).\end{aligned} \tag{5.12}$$

Using (5.2) in (5.12) we obtain

$$S(X,Y) = \lambda g(X,Y) + \mu\eta(X)\eta(Y) \tag{5.13}$$

where $\lambda = \dfrac{(n-1)\alpha(\alpha^2 - \rho)}{\alpha + C(\xi)}$ and $\mu = -\dfrac{(n-1)(\alpha^2 - \rho)C(\xi)}{\alpha + C(\xi)}$ are smooth functions such that $\alpha + C(\xi) \neq 0$. This proves the theorem. □

6. EXAMPLES OF $(LCS)_n$-MANIFOLDS

Example 6.1. We consider the 3-dimensional manifold

$$M = \{(x,y,z) \in R^3 : z \neq 0\},$$

where (x,y,z) are the standard coordinates in R^3. Let $\{E_1, E_2, E_3\}$ be linearly independent global frame on M given by

$$E_1 = e^z\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right), \qquad E_2 = e^z\frac{\partial}{\partial y}, \qquad E_3 = e^{2z}\frac{\partial}{\partial z}.$$

Let g be the Lorentzian metric defined by $g(E_1,E_3) = g(E_2,E_3) = g(E_1,E_2) = 0$, $g(E_1,E_1) = g(E_2,E_2) = 1, g(E_3,E_3) = -1$.

Let η be the 1-form defined by $\eta(U) = g(U,E_3)$ for any $U \in \chi(M)$. Let ϕ be the (1, 1) tensor field defined by $\phi E_1 = E_1$, $\phi E_2 = E_2$, $\phi E_3 = 0$. Then using the linearity of ϕ and g we have $\eta(E_3) = -1$, $\phi^2 U = U + \eta(U)E_3$ and

$$g(\phi U, \phi W) = g(U,W) + \eta(U)\eta(W)$$

for any $U, W \in \chi(M)$. Thus for $E_3 = \xi$, (ϕ, ξ, η, g) defines a Lorentzian paracontact structure on M.

Let ∇ be the Levi-Civita connection with respect to the Lorentzian metric g and R be the curvature tensor of g. Then we have

$$[E_1,E_2] = -e^z E_2, \qquad [E_1,E_3] = -e^{2z}E_1, \qquad [E_2,E_3] = -e^{2z}E_2.$$

Taking $E_3 = \xi$ and using Koszul formula for the Lorentzian metric g, we can easily calculate

$$\nabla_{E_1}E_3 = -e^{2z}E_1, \qquad \nabla_{E_3}E_3 = 0, \qquad \nabla_{E_2}E_3 = -e^{2z}E_2,$$

$$\nabla_{E_1}E_1 = -e^{2z}E_3, \qquad \nabla_{E_1}E_2 = 0, \qquad \nabla_{E_2}E_1 = -e^{2z}E_2,$$
$$\nabla_{E_2}E_2 = -e^{2z}E_3 - e^{z}E_1, \qquad \nabla_{E_3}E_2 = 0, \qquad \nabla_{E_3}E_1 = 0.$$

From the above it can be easily seen that (ϕ,ξ,η,g) is an $(LCS)_3$ structure on M. Consequently $M^3(\phi,\xi,\eta,g)$ is an $(LCS)_3$-manifold with $\alpha = -e^{2z} \neq 0$ and $\rho = 2e^{4z}$.

Using the above relations, we can easily calculate the non-vanishing components of the curvature tensor as follows:

$$R(E_2,E_3)E_3 = e^{4z}E_2, \qquad R(E_1,E_3)E_3 = e^{4z}E_1,$$
$$R(E_1,E_3)E_1 = e^{4z}E_3, \qquad R(E_2,E_3)E_2 = e^{4z}E_3,$$
$$R(E_1,E_2)E_2 = e^{4z}E_1 - e^{2z}E_1, \qquad R(E_1,E_2)E_1 = -e^{4z}E_2 + e^{2z}E_2$$

and the components which can be obtained from these by the symmetric properties. Using the above relation we can easily calculate the following:

$$R(E_1,E_3,E_1,E_3) = -e^{4z}, \qquad R(E_2,E_3,E_2,E_3) = -e^{4z},$$
$$R(E_1,E_2,E_1,E_2) = e^{2z} - e^{4z},$$

and their covariant derivatives are given by

$$(\nabla_{E_3}R)(E_1,E_3,E_1,E_3) = -4e^{6z}, \qquad (\nabla_{E_3}R)(E_2,E_3,E_2,E_3) = -4e^{6z},$$
$$(\nabla_{E_3}R)(E_1,E_2,E_1,E_2) = 2e^{4z} - 4e^{6z}.$$

We shall show that this $(LCS)_3$-manifold is weakly symmetric, i.e., it satisfies the relation (4.1). Let us now consider the non-vanishing 1-forms

$$\begin{aligned} A^*(E_i) &= 0 \quad \text{for} \quad i = 1,2,\\ &= \frac{2e^{2z}(2e^{2z}-1)}{e^{2z}-1} \quad \text{for} \quad i = 3, \end{aligned}$$

$$\begin{aligned} B^*(E_i) &= 0 \quad \text{for} \quad i = 1,2,\\ &= \frac{2e^{2z}}{e^{2z}-1} \quad \text{for} \quad i = 3, \end{aligned}$$

$$\begin{aligned} C^*(E_i) &= 0, \quad \text{for} \quad i = 1,2,\\ &= -\frac{4e^{2z}}{e^{2z}-1} \quad \text{for} \quad i = 3 \end{aligned}$$

at any point $x \in M$. In our M^3, (4.1) reduces with these 1-forms to the following equations:

$$\begin{aligned}(\nabla_{E_3}R)(E_1,E_2,E_1,E_2) = {} & A^*(E_3)R(E_1,E_2,E_1,E_2) + B^*(E_1)R(E_3,E_2,E_1,E_2)\\ & +B^*(E_2)R(E_1,E_3,E_1,E_2) + C^*(E_1)R(E_1,E_2,E_3,E_2)\\ & +C^*(E_2)R(E_1,E_2,E_1,E_3) \end{aligned} \tag{6.1}$$

$$\begin{aligned}(\nabla_{E_3}R)(E_i,E_3,E_i,E_3) = A^*(E_3)R(E_i,E_3,E_i,E_3) + B^*(E_i)R(E_3,E_3,E_i,E_3)\\ +B^*(E_3)R(E_i,E_3,E_i,E_3) + C^*(E_i)R(E_i,E_3,E_3,E_3)\\ +C^*(E_3)R(E_i,E_3,E_i,E_3), \quad i = 1, 2. \qquad (6.2)\end{aligned}$$

This implies that the manifold under consideration is a weakly symmetric $(LCS)_3$-manifold, which is neither recurrent nor locally symmetric. This leads to the following:

Theorem 6.1. *There exists weakly symmetric $(LCS)_3$-manifold which is neither recurrent nor locally symmetric.*

Example 6.2. We consider the 4-dimensional manifold

$$M = \{(x, y, z, u) \in R^4 : u \neq 0, 1, -1\},$$

where (x, y, z, u) are the standard coordinates in R^4. Let $\{E_1, E_2, E_3, E_4\}$ be linearly independent global frame on M given by

$$E_1 = u\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right), \quad E_2 = u\frac{\partial}{\partial y}, \quad E_3 = u\frac{\partial}{\partial z}, \quad E_4 = u^3\frac{\partial}{\partial u}.$$

Let g be the Lorentzian metric defined by $g(E_1, E_1) = g(E_2, E_2) = g(E_3, E_3) = 1$, $g(E_4, E_4) = -1$ and $g(E_i, E_j) = 0$ for $i \neq j = 1, 2, 3, 4$. Let η be the 1-form defined by $\eta(U) = g(U, E_4)$ for any $U \in \chi(M)$. Let ϕ be the (1, 1) tensor field defined by $\phi E_1 = E_1$, $\phi E_2 = E_2$, $\phi E_3 = E_3$ $\phi E_4 = 0$. Then using the linearity of ϕ and g we have $\eta(E_4) = -1$, $\phi^2 U = U + \eta(U)E_4$ and $g(\phi U, \phi W) = g(U, W) + \eta(U)\eta(W)$ for any $U, W \in \chi(M)$. Thus for $E_4 = \xi$, (ϕ, ξ, η, g) defines a Lorentzian paracontact structure on M.

Let ∇ be the Levi-Civita connection with respect to the Lorentzian metric g and R be the curvature tensor of g. Then we have

$$[E_1, E_2] = -uE_2, \quad [E_1, E_4] = -u^2E_1, \quad [E_2, E_4] = -u^2E_2, \quad [E_3, E_4] = -u^2E_3.$$

Taking $E_4 = \xi$ and using Koszul formula for the Lorentzian metric g, we can easily calculate

$$\nabla_{E_1}E_4 = -u^2E_1, \qquad \nabla_{E_2}E_4 = -u^2E_2, \qquad \nabla_{E_3}E_4 = -u^2E_3,$$

$$\nabla_{E_1}E_1 = -u^2E_4, \qquad \nabla_{E_2}E_1 = uE_2, \qquad \nabla_{E_3}E_3 = -u^2E_4,$$

$$\nabla_{E_2}E_2 = -u^2E_4 - uE_1, \qquad \nabla_{E_4}E_1 = 0, \qquad \nabla_{E_3}E_2 = 0,$$

$$\nabla_{E_1}E_3 = 0, \qquad \nabla_{E_1}E_2 = 0, \qquad \nabla_{E_3}E_1 = 0,$$

$$\nabla_{E_4}E_2 = 0, \qquad \nabla_{E_4}E_3 = 0, \qquad \nabla_{E_4}E_4 = 0.$$

From the above it can be easily seen that (ϕ, ξ, η, g) is an $(LCS)_3$ structure on M. Consequently $M^4(\phi, \xi, \eta, g)$ is an $(LCS)_4$-manifold with $\alpha = -u^2 \neq 0$ and $\rho = 2u^4$.

Using the above relations, we can easily calculate the non-vanishing components of the curvature tensor as follows:

$$R(E_2,E_3)E_2 = -u^4E_3, \qquad R(E_2,E_3)E_3 = u^4E_2, \qquad R(E_1,E_4)E_1 = u^4E_4,$$

$$R(E_1,E_3)E_1 = -u^4E_3, \qquad R(E_1,E_4)E_4 = u^4E_1, \qquad R(E_2,E_4)E_2 = u^4E_4,$$

$$R(E_1,E_3)E_3 = u^4E_1, \qquad R(E_3,E_4)E_4 = u^4E_3, \qquad R(E_3,E_4)E_3 = u^4E_4,$$

$$R(E_1,E_2)E_2 = u^4E_1-u^2E_1, \quad R(E_2,E_4)E_4 = u^4E_4, \quad R(E_1,E_2)E_1 = -u^4E_2+u^2E_2$$

and the components which can be obtained from these by the symmetric properties. Using the above relation we can easily calculate the following:

$$R(E_1,E_3,E_1,E_3) = -u^4, \qquad R(E_1,E_4,E_1,E_4) = -u^4,$$

$$R(E_2,E_3,E_2,E_3) = -u^4, \qquad R(E_2,E_4,E_2,E_4) = -u^4,$$

$$R(E_3,E_4,E_3,E_4) = -u^4, \qquad R(E_1,E_2,E_1,E_2) = -u^4+u^2$$

and their covariant derivatives are given by

$$(\nabla_{E_4}R)(E_2,E_3,E_2,E_3) = -4u^6, \qquad (\nabla_{E_4}R)(E_2,E_4,E_2,E_4) = -4u^6,$$

$$(\nabla_{E_4}R)(E_1,E_3,E_1,E_3) = -4u^6, \qquad (\nabla_{E_4}R)(E_1,E_4,E_1,E_4) = -4u^6,$$

$$(\nabla_{E_4}R)(E_3,E_4,E_3,E_4) = -4u^6, \qquad (\nabla_{E_4}R)(E_1,E_2,E_1,E_2) = -4u^6+2u^4$$

We shall now show that this $(LCS)_4$-manifold is weakly symmetric, i.e., it satisfies the relation (4.1). Let us now consider the non-vanishing 1-forms

$$\begin{aligned} A^*(E_i) &= 0 \quad \text{for} \quad i=1,2,3,\\ &= \frac{2u^2(1-2u^2)}{1-u^2} \quad \text{for} \quad i=4, \end{aligned}$$

$$\begin{aligned} B^*(E_i) &= 0 \quad \text{for} \quad i=1,2,3,\\ &= \frac{4u^2}{1-u^2} \quad \text{for} \quad i=4, \end{aligned}$$

$$\begin{aligned} C^*(E_i) &= 0 \quad \text{for} \quad i=1,2,3,\\ &= -\frac{2u^2}{1-u^2} \quad \text{for} \quad i=4 \end{aligned}$$

at any point $x \in M^4$. In our M^3,(4.1) reduces with these 1-forms to the following equations:

$$\begin{aligned} (\nabla_{E_4}R)(E_1,E_2,E_1,E_2) = A^*(E_4)R(E_1,E_2,E_1,E_2) + B^*(E_1)R(E_4,E_2,E_1,E_2)\\ +B^*(E_2)R(E_1,E_4,E_1,E_2) + C^*(E_1)R(E_1,E_2,E_4,E_2)\\ +C^*(E_2)R(E_1,E_2,E_1,E_4), \qquad (6.3) \end{aligned}$$

$$\begin{aligned}(\nabla_{E_4}R)(E_i,E_3,E_i,E_3) = A^*(E_4)R(E_i,E_3,E_i,E_3) &+ B^*(E_i)R(E_4,E_3,E_i,E_3)\\ &+B^*(E_3)R(E_i,E_4,E_i,E_3)+C^*(E_i)R(E_i,E_3,E_4,E_3)\\ &+C^*(E_3)R(E_i,E_3,E_i,E_4), \quad i=1,2, \qquad (6.4)\end{aligned}$$

$$\begin{aligned}(\nabla_{E_4}R)(E_i,E_4,E_i,E_4) = A^*(E_4)R(E_i,E_4,E_i,E_4) &+ B^*(E_i)R(E_4,E_4,E_i,E_4)\\ &+B^*(E_4)R(E_i,E_4,E_i,E_4)+C^*(E_i)R(E_i,E_4,E_4,E_4)\\ &+C^*(E_4)R(E_i,E_4,E_i,E_4), \quad i=1,2,3. \qquad (6.5)\end{aligned}$$

This implies that the manifold under consideration is a weakly symmetric $(LCS)_4$-manifold which is neither recurrent nor locally symmetric. This leads to the following:

Theorem 6.2. *There exists weakly symmetric $(LCS)_4$-manifold which is neither recurrent nor locally symmetric.*

Example 6.3. We consider the 3-dimensional manifold

$$M = \{(x,y,z) \in R^3 : z \neq 0\},$$

where (x,y,z) are the standard coordinates in R^3. Let $\{E_1,E_2,E_3\}$ be linearly independent global frame on M given by

$$E_1 = z\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right), \qquad E_2 = z\frac{\partial}{\partial y}, \qquad E_3 = \frac{\partial}{\partial z}.$$

Let g be the Lorentzian metric defined by $g(E_1,E_3) = g(E_2,E_3) = g(E_1,E_2) = 0$, $g(E_1,E_1) = g(E_2,E_2) = 1, g(E_3,E_3) = -1$. Let η be the 1-form defined by $\eta(U) = g(U,E_3)$ for any $U \in \chi(M)$. Let ϕ be the (1, 1) tensor field defined by $\phi E_1 = E_1$, $\phi E_2 = E_2$, $\phi E_3 = 0$. Then using the linearity of ϕ and g we have $\eta(E_3) = -1$, $\phi^2 U = U + \eta(U)E_3$ and $g(\phi U, \phi W) = g(U,W) + \eta(U)\eta(W)$ for any $U, W \in \chi(M)$. Thus for $E_3 = \xi$, (ϕ, ξ, η, g) defines a Lorentzian paracontact structure on M.

Let ∇ be the Levi-Civita connection with respect to the Lorentzian metric g and R be the curvature tensor of g. Then we have

$$[E_1,E_2] = -zE_2, \qquad [E_1,E_3] = -\frac{1}{z}E_1, \qquad [E_2,E_3] = -\frac{1}{z}E_2.$$

Taking $E_3 = \xi$ and using Koszul formula for the Lorentzian metric g, we can easily calculate

$$\nabla_{E_1}E_3 = -\frac{1}{z}E_1, \qquad \nabla_{E_3}E_3 = 0, \qquad \nabla_{E_2}E_3 = -\frac{1}{z}E_2,$$

$$\nabla_{E_1}E_1 = -\frac{1}{z}E_3, \qquad \nabla_{E_1}E_2 = 0, \qquad \nabla_{E_2}E_1 = zE_2,$$

$$\nabla_{E_2}E_2 = -\frac{1}{z}E_3 - zE_1, \qquad \nabla_{E_3}E_2 = 0, \qquad \nabla_{E_3}E_1 = 0.$$

From the above it can be easily seen that (ϕ, ξ, η, g) is an $(LCS)_3$ structure on M. Consequently $M^3(\phi, \xi, \eta, g)$ is an $(LCS)_3$-manifold with $\alpha = -\frac{1}{z} \neq 0$ and $\rho = -\frac{1}{z^2}$.

Using the above relations, we can easily calculate the non-vanishing components of the curvature tensor as follows:

$$R(E_2, E_3)E_3 = -\frac{2}{z^2}E_2, \qquad R(E_1, E_3)E_1 = -\frac{2}{z^2}E_3, \qquad R(E_1, E_3)E_3 = -\frac{2}{z^2}E_1,$$

$$R(E_2, E_3)E_2 = -\frac{2}{z^2}E_3, \quad R(E_1, E_2)E_2 = \frac{1}{z^2}E_1 - z^2E_1, \quad R(E_1, E_2)E_1 = z^2E_2 - \frac{1}{z^2}E_2$$

and the components which can be obtained from these by the symmetric properties. Using the above relation we can easily calculate the following:

$$S(E_1, E_1) = -\left(z^2 + \frac{1}{z^2}\right), \qquad S(E_2, E_2) = -\left(z^2 + \frac{1}{z^2}\right), \qquad S(E_3, E_3) = -\frac{4}{z^2}$$

and their covariant derivatives are given by

$$(\nabla_{E_3}S)(E_1, E_1) = -2z + \frac{2}{z^3}, \qquad (\nabla_{E_3}S)(E_2, E_2) = -2z + \frac{2}{z^3},$$

$$(\nabla_{E_3}S)(E_3, E_3) = \frac{8}{z^3}.$$

We now verify that this $(LCS)_3$-manifold is weakly Ricci-symmetric, i.e., it satisfies the relation (5.1). Let us now consider the non-vanishing 1-forms

$$\begin{aligned} A^*(E_i) &= 0 \quad \text{for} \quad i = 1, 2, \\ &= \frac{2(z^4 - 1)}{z(z^4 + 1)} \quad \text{for} \quad i = 3, \\ B^*(E_i) &= 0 \quad \text{for} \quad i = 1, 2, \\ &= -\frac{5z^3}{z^4 + 1} \quad \text{for} \quad i = 3, \\ C^*(E_i) &= 0 \quad \text{for} \quad i = 1, 2, \\ &= \frac{z^3}{(z^4 + 1)} \quad \text{for} \quad i = 3 \end{aligned}$$

at any point $x \in M$. In our M^3, (5.1) reduces with these 1-forms to the following equations:

$$(\nabla_{E_3}S)(E_1, E_1) = A(E_3)S(E_1, E_1) + B(E_1)S(E_3, E_1) + C(E_1)S(E_1, E_3), \quad (6.6)$$

$$(\nabla_{E_3}S)(E_2, E_2) = A(E_3)S(E_2, E_2) + B(E_2)S(E_3, E_2) + C(E_2)S(E_2, E_3), \quad (6.7)$$

$$(\nabla_{E_3}S)(E_3, E_3) = A(E_3)S(E_3, E_3) + B(E_3)S(E_3, E_3) + C(E_3)S(E_3, E_3). \quad (6.8)$$

This implies that with respect to the 1-forms under consideration the manifold is a weakly Ricci-symmetric $(LCS)_3$-manifold which is neither Ricci-recurrent nor Ricci-symmetric. Hence we can state the following:

Theorem 6.3. *There exists weakly Ricci-symmetric $(LCS)_3$-manifold which is neither Ricci-recurrent nor Ricci-symmetric.*

Example 6.4. We consider the 4-dimensional manifold

$$M = \{(x, y, z, u) \in R^4 : u \neq 0\},$$

where (x, y, z, u) are the standard coordinates in R^4. Let $\{E_1, E_2, E_3, E_4\}$ be linearly independent global frame on M given by

$$E_1 = \cosh u\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right), \quad E_2 = \cosh u\frac{\partial}{\partial y}, \quad E_3 = \cosh u\frac{\partial}{\partial z}, \quad E_4 = \frac{\partial}{\partial u}.$$

Let g be the Lorentzian metric defined by $g(E_1, E_1) = g(E_2, E_2) = 1$, $g(E_3, E_3) = 1$, $g(E_4, E_4) = -1$ and $g(E_i, E_j) = 0$ for $i \neq j = 1, 2, 3, 4$. Let η be the 1-form defined by $\eta(U) = g(U, E_4)$ for any $U \in \chi(M)$. Let ϕ be the (1, 1) tensor field defined by $\phi E_1 = E_1$, $\phi E_2 = E_2$, $\phi E_3 = E_3$ $\phi E_4 = 0$. Then using the linearity of ϕ and g we have $\eta(E_4) = -1$, $\phi^2 U = U + \eta(U)E_4$ and $g(\phi U, \phi W) = g(U, W) + \eta(U)\eta(W)$ for any $U, W \in \chi(M)$. Thus for $E_4 = \xi$, (ϕ, ξ, η, g) defines a Lorentzian paracontact structure on M.

Let ∇ be the Levi-Civita connection with respect to the Lorentzian metric g and R be the curvature tensor of g. Then we have

$$[E_1, E_2] = -\cosh uE_2, \quad [E_1, E_3] = -\cosh uE_3, \quad [E_1, E_4] = -\tanh uE_1,$$
$$[E_2, E_4] = -\tanh uE_2, \quad [E_3, E_4] = -\tanh uE_3.$$

Taking $E_4 = \xi$ and using Koszul formula for the Lorentzian metric g, we can easily calculate

$$\nabla_{E_1}E_4 = -\tanh uE_1, \quad \nabla_{E_2}E_4 = -\tanh uE_2, \quad \nabla_{E_3}E_4 = -\tanh uE_3,$$
$$\nabla_{E_1}E_1 = -\tanh uE_4, \quad \nabla_{E_2}E_1 = \cosh uE_2, \quad \nabla_{E_3}E_1 = \cosh uE_3,$$
$$\nabla_{E_2}E_2 = -\tanh uE_4 - \cosh uE_1, \quad \nabla_{E_3}E_3 = -\tanh uE_4 - \cosh uE_1,$$
$$\nabla_{E_1}E_3 = 0, \quad \nabla_{E_4}E_1 = 0, \quad \nabla_{E_1}E_2 = 0, \quad \nabla_{E_3}E_2 = 0,$$
$$\nabla_{E_4}E_2 = 0, \quad \nabla_{E_4}E_3 = 0, \quad \nabla_{E_4}E_4 = 0, \quad \nabla_{E_2}E_3 = 0.$$

From the above it can be easily seen that (ϕ, ξ, η, g) is an $(LCS)_4$ structure on M. Consequently $M^4(\phi, \xi, \eta, g)$ is an $(LCS)_4$-manifold with $\alpha = -\tanh u \neq 0$ and $\rho = \mathrm{sech}^2 u$. Using the above relations, we can easily calculate the non-vanishing components of the curvature tensor as follows:

$$R(E_2, E_3)E_2 = (\cosh^2 u - \tanh^2 u)E_3, \qquad R(E_2, E_3)E_3 = (\tanh^2 u - \cosh^2)uE_2,$$
$$R(E_1, E_3)E_1 = (\cosh^2 u - \tanh^2 u)E_1, \qquad R(E_1, E_3)E_3 = (\tanh^2 u - \cosh^2 u)E_1,$$
$$R(E_3, E_4)E_4 = (\mathrm{sech}^2 u - \tanh^2 u)E_3, \qquad R(E_3, E_4)E_3 = (\mathrm{sech}^2 u - \tanh^2 u)E_4,$$
$$R(E_1, E_2)E_2 = (\tanh^2 u - \cosh^2 u)E_1, \qquad R(E_1, E_2)E_1 = (\cosh^2 u - \tanh^2 u)E_2$$
$$R(E_1, E_4)E_1 = (\mathrm{sech}^2 u - \tanh^2 u)E_4, \qquad R(E_1, E_4)E_4 = (\mathrm{sech}^2 u - \tanh^2 u)E_1,$$

$$R(E_2,E_4)E_2=(\mathrm{sech}^2u-\tanh^2 u)E_4, \qquad R(E_2,E_4)E_4=(\mathrm{sech}^2u-\tanh^2 u)E_2,$$

and the components which can be obtained from these by the symmetric properties. Using the above relation we can easily calculate the following:

$$S(E_1,E_1)=1-2\cosh^2 u, \qquad S(E_2,E_2)=1-2\cosh^2 u,$$

$$S(E_3,E_3)=1-2\cosh^2 u, \qquad S(E_4,E_4)=3(\mathrm{sech}^2u-\tanh^2 u)$$

and their covariant derivatives are given by

$$(\nabla_{E_4}S)(E_1,E_1)=-4\cosh u\sinh u, \qquad (\nabla_{E_4}S)(E_2,E_2)=-4\cosh u\sinh u,$$

$$(\nabla_{E_4}S)(E_3,E_3)=-4\cosh u\sinh u, \qquad (\nabla_{E_4}S)(E_4,E_4)=-12\sec h^2u\tanh u.$$

We shall show that this $(LCS)_4$-manifold is weakly Ricci-symmetric, i.e., it satisfies the relation (5.1). Let us now consider the non-vanishing 1-forms

$$\begin{aligned}A^*(E_i)&=0 \quad \text{for} \quad i=1,2,3,\\ &= -\frac{4\cosh u\sinh u}{1-2\cosh^2 u} \quad \text{for} \quad i=4,\end{aligned}$$

$$\begin{aligned}B^*(E_i)&=0 \quad \text{for} \quad i=1,2,3,\\ &= \frac{4\cosh u\sinh u}{1-2\cosh^2 u} \quad \text{for} \quad i=4,\end{aligned}$$

$$\begin{aligned}C^*(E_i)&=0 \quad \text{for} \quad i=1,2,3,\\ &= -\frac{4\sec h^2u\tanh u}{(\sec h^2u-\tanh^2 u)} \quad \text{for} \quad i=4\end{aligned}$$

at any point $x\in M$. In our M^4, (5.1) reduces with these 1-forms to the following equations:

$$(\nabla_{E_4}S)(E_1,E_1)=A(E_4)S(E_1,E_1)+B(E_1)S(E_4,E_1)+C(E_1)S(E_1,E_4), \tag{6.9}$$

$$(\nabla_{E_4}S)(E_2,E_2)=A(E_4)S(E_2,E_2)+B(E_2)S(E_4,E_2)+C(E_2)S(E_2,E_4), \tag{6.10}$$

$$(\nabla_{E_4}S)(E_3,E_3)=A(E_4)S(E_3,E_3)+B(E_3)S(E_4,E_3)+C(E_3)S(E_3,E_4), \tag{6.11}$$

$$(\nabla_{E_4}S)(E_4,E_4)=A(E_4)S(E_4,E_4)+B(E_4)S(E_4,E_4)+C(E_4)S(E_4,E_4). \tag{6.12}$$

This implies that the manifold under consideration is a weakly Ricci-symmetric $(LCS)_4$-manifold which is neither Ricci-recurrent nor Ricci symmetric. Thus we can state the following:

Theorem 6.4. *There exists weakly Ricci-symmetric $(LCS)_3$-manifold which is neither Ricci-recurrent nor Ricci symmetric.*

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University of Burdwan, Golapbag
Department of Mathematics
Burdwan 713104*, West Bengal, India*
E-mail address: `aask2003@yahoo.co.in` *or* `aask@epatra.com`

University of Debrecen
Institute of Mathematics
*H-*4010*, Debrecen, P. O. Box* 12*, Hungary*
E-mail address: `binh@math.klte.hu`