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NEW TYPE DUALITIES IN PDI AND PDE CONSTRAINED OPTIMIZATION PROBLEMS

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ABSTRACT. In a previous paper [8], Pitea, Udrişte and Mititelu considered the problem (MFP) of minimizing a vector of quotients of functionals of curvilinear integrals subject to PDE and/or PDI constraints and studied efficiency necessary conditions for the problem (MFP). The goal of the present work is to associate to this problem a dual, (MFD), and to establish some connections between these two programs. We state a weak duality theorem proving that the minimum value of the objective function of the primal is not less than the maximum value of the objective function of the dual. Then we give direct and converse duality theorems which prove that the values of primal and dual programs are equal in certain conditions. While Section 1 is introductory [7], Section 2 is new as a whole, containing our results.

1. INTRODUCTION AND PRELIMINARIES

Let (T, h) and (M, g) be Riemannian manifolds of dimensions p and n, respectively. The local coordinates on T and M will be written $t = (t^\alpha)$ and $x = (x^i)$, respectively. Let $J^1(T, M)$ be the first order jet bundle associated to T and M.

Using the product order relation on $\mathbb{R}^p$, the hyperparallelepiped Ω_{t_0,t_1}, in $\mathbb{R}^p$, with the diagonal opposite points $t_0 = (t_0^1, \dots, t_0^p)$ and $t_1 = (t_1^1, \dots, t_1^p)$, can be written as being the interval $[t_0, t_1]$. Suppose γ_{t_0,t_1} is a piecewise C^1-class curve joining the points t_0 and t_1.

The closed Lagrange 1-forms densities of C^∞-class

$$f_\alpha = (f_\alpha^\ell)\colon J^1(T, M) \to \mathbb{R}^r, \quad \ell = \overline{1, r}, \quad \alpha = \overline{1, p},$$

and

$$k_\alpha = (k_\alpha^\ell)\colon J^1(T, M) \to \mathbb{R}^r, \quad \ell = \overline{1, r}, \quad \alpha = \overline{1, p}$$

determine the following path independent functionals

$$F^\ell(x(\cdot)) = \int_{\gamma_{t_0,t_1}} f_\alpha^\ell(t, x(t), x_\gamma(t))\, dt^\alpha, \quad K^\ell(x(\cdot)) = \int_{\gamma_{t_0,t_1}} k_\alpha^\ell(t, x(t), x_\gamma(t))\, dt^\alpha,$$

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where $x_\gamma(t) = \dfrac{\partial x}{\partial t^\gamma}(t)$, $\gamma = \overline{1,p}$ are partial velocities.

The closeness conditions (complete integrability conditions) are $D_\beta f^\ell_\alpha = D_\alpha f^\ell_\beta$, and $D_\beta k^\ell_\alpha = D_\alpha k^\ell_\beta$, $\alpha, \beta = \overline{1,p}$, $\alpha \neq \beta$, $\ell = \overline{1,r}$, where D_β is the total derivative. Suppose

$$K^\ell(x(\cdot)) = \int_{\gamma_{t_0,t_1}} k^\ell_\alpha(t, x(t), x_\gamma(t))\, dt^\alpha > 0.$$

Also we accept that the Lagrange matrix densities

$$g = (g^b_a)\colon J^1(T, M) \to \mathbb{R}^{ms}, \quad a = \overline{1,s}, \quad b = \overline{1,m}, \quad m < n,$$

of C^∞-class defines the partial differential inequations (PDI) (of evolution)

$$g(t, x(t), x_\gamma(t)) \leqq 0, \quad t \in \Omega_{t_0,t_1}, \tag{1.1}$$

and the Lagrange matrix densities

$$h = (h^b_a)\colon J^1(T, M) \to \mathbb{R}^{qs}, \quad a = \overline{1,s}, \quad b = \overline{1,q},\ q < n,$$

defines the partial differential equations (PDE) (of evolution)

$$h(t, x(t), x_\gamma(t)) = 0, \quad t \in \Omega_{t_0,t_1}. \tag{1.2}$$

The aim of this work is to introduce and study two dual programs:

(1) the multi-time multi-objective fractional variational problem (MFP) of minimizing a vector of quotients of path independent curvilinear functionals

$$\text{(MFP)} \quad \min_{x(\cdot)} \left(\frac{F^1(x(\cdot))}{K^1(x(\cdot))}, \dots, \frac{F^r(x(\cdot))}{K^r(x(\cdot))} \right),$$

(2) the multi-objective variational dual problem

$$\text{(MFD)} \quad \begin{cases} \max\limits_{y(\cdot)} \left(\dfrac{F^1(y(\cdot))}{K^1(y(\cdot))}, \dots, \dfrac{F^r(y(\cdot))}{K^r(y(\cdot))} \right) \\ \quad \text{subject to} \\ \quad \Lambda^{10}_\ell \dfrac{\partial f^\ell_\alpha}{\partial y}(t, y(t), y_\gamma(t)) - \Lambda^{20}_\ell \dfrac{\partial k^\ell_\alpha}{\partial y}(t, y(t), y_\gamma(t)) \\ + < \mu_\alpha(t), \dfrac{\partial g}{\partial y}(t, y(t), y_\gamma(t)) > + < \nu_\alpha(t), \dfrac{\partial h}{\partial y}(t, y(t), y_\gamma(t)) > \\ -D_\gamma \Big(\Lambda^{10}_\ell \dfrac{\partial f^\ell_\alpha}{\partial y_\gamma}(t, y(t), y_\gamma(t)) - \Lambda^{20}_\ell \dfrac{\partial k^\ell_\alpha}{\partial y_\gamma}(t, y(t), y_\gamma(t)) \\ + < \mu_\alpha(t), \dfrac{\partial g}{\partial y_\gamma}(t, y(t), y_\gamma(t)) > + < \nu_\alpha(t), \dfrac{\partial h}{\partial y_\gamma}(t, y(t), y_\gamma(t)) > \Big) = 0, \\ \qquad\qquad t \in \Omega_{t_0,t_1},\ \alpha = \overline{1,p} \\ \quad < \mu_\alpha(t), g(t, y(t), y_\gamma(t)) > + < \nu_\alpha(t), h(t, y(t), y_\gamma(t)) > \ \geqq 0, \\ \qquad\qquad \alpha = \overline{1,p},\ t \in \Omega_{t_0,t_1} \\ \quad \Lambda^{10} \geq 0, \end{cases}$$

taking into account that the functions $x(t), y(t)$ have to satisfy the boundary conditions $x(t_0) = x_0$, $x(t_1) = x_1$, or $x(t)|_{\partial\Omega_{t_0,t_1}}$ =given, respectively, $y(t_0) = y_0$, $y(t_1) = y_1$, or $y(t)|_{\partial\Omega_{t_0,t_1}}$ = given, the partial differential inequations of evolution (1.1), and the partial differential equations of evolution (1.2).

Let $C^\infty(\Omega_{t_0,t_1}, M)$ be the space of all functions $x\colon \Omega_{t_0,t_1} \to M$ of C^∞-class, with the norm

$$\|x\| = \|x\|_\infty + \sum_{\alpha=1}^{p} \|x_\alpha\|_\infty.$$

If

$$\mathcal{F}(\Omega_{t_0,t_1}) = \{x \in C^\infty(\Omega_{t_0,t_1}, M) \,|\, x(t_0) = x_0,\ x(t_1) = x_1,\ g(t, x(t), x_\gamma(t)) \leqq 0, \\ h(t, x(t), x_\gamma(t)) = 0,\ t \in \Omega_{t_0,t_1}\}$$

is the *set of all feasible solutions* of the previous problems, we can write

$$x(\cdot) \in \mathcal{F}(\Omega_{t_0,t_1}), \qquad y(\cdot) \in \mathcal{F}(\Omega_{t_0,t_1}).$$

The paper [8] introduced and studied the problem (MFP), given necessary conditions for the efficiency of a feasible solution.

To develop the theory in our main section, we need the notion of efficient solution.

Definition 1.1. A feasible solution $x^\circ(\cdot) \in \mathcal{F}(\Omega_{t_0,t_1})$ is called *efficient solution* for the program (MFP) if and only if for any feasible solution $x(\cdot) \in \mathcal{F}(\Omega_{t_0,t_1})$, we have the implication

$$\frac{F(x(\cdot))}{K(x(\cdot))} \geq \frac{F(x^\circ(\cdot))}{K(x^\circ(\cdot))} \Longrightarrow \frac{F(x(\cdot))}{K(x(\cdot))} = \frac{F(x^\circ(\cdot))}{K(x^\circ(\cdot))},$$

where

$$\frac{F(x(\cdot))}{K(x(\cdot))} = \left(\frac{F^1(x(\cdot))}{K^1(x(\cdot))}, \ldots, \frac{F^r(x(\cdot))}{K^r(x(\cdot))}\right).$$

Remark 1.1. To study significant basic ideas of optimization problems of path independent curvilinear integrals with PDE constraints or with isoperimetric constraints as multiple integrals or path independent curvilinear integrals, the reader is encouraged to see [3], [4], [5], [7], [8], [12], [13], [14] and [15].

It is well known that such kind of optimization problems arise in wide areas of research in pure and applied sciences and in the new technology as well. First of all, we have in mind the material sciences where many times optimal estimation of material parameters is required, either non-destructive determination of faults is needed. Next, chemistry which provides a huge class of constrained optimization problems such as the determination of contamination sources given the flow model and the variance of the source. Last, but not least, games theory where the main study is finding optimal wining strategies. All of these considerations motivate the theoretical research given in this paper.

In the problems of our study the objective function is of curvilinear integral type and could mould the entropy when we have in mind physics or minimizing the cost or maximizing the profit when we discuss economical problems.

2. MAIN RESULTS

Let ρ be a real number and $b\colon C^\infty(\Omega_{t_0,t_1}, M) \times C^\infty(\Omega_{t_0,t_1}, M) \to [0,\infty)$ a functional. To any closed 1-form $a = (a_\alpha)$ we associate the path independent curvilinear functional

$$A(x(\cdot)) = \int_{\gamma_{t_0,t_1}} a_\alpha(t, x(t), x_\gamma(t))\, dt^\alpha.$$

The following definition of the quasiinvexity [2], [6], [9], [10], [16] [17], helps us to state the results included in this section.

Definition 2.1. The functional A is called [strictly] (ρ, b)-quasiinvex at the point $x^\circ(\cdot)$ if there is a vector function

$$\eta\colon J^1(\Omega_{t_0,t_1}, M) \times J^1(\Omega_{t_0,t_1}, M) \to \mathbb{R}^n,$$

such that

$$\eta(t, x^\circ(t), x^\circ_\gamma(t), x^\circ(t), x^\circ_\gamma(t)) = 0,$$

and the functional $\theta\colon C^\infty(\Omega_{t_0,t_1}, M) \times C^\infty(\Omega_{t_0,t_1}, M) \to \mathbb{R}^n$, such that for any $x(\cdot)$ $[x(\cdot) \neq x^\circ(\cdot)]$, the following implication holds

$$(A(x(\cdot)) \leqq A(x^\circ(\cdot))) \Rightarrow \Bigg(b(x((\cdot), x^\circ(\cdot)) \int_{\gamma_{t_0,t_1}} \Bigg[< \eta(t, x(t), x_\gamma(t), x^\circ(t), x^\circ_\gamma(t)),$$
$$\frac{\partial a_\alpha}{\partial x}(t, x^\circ(t), x^\circ_\gamma(t)) > + < D_\gamma \eta(t, x(t), x_\gamma(t), x^\circ(t), x^\circ_\gamma(t)),$$
$$\frac{\partial a_\alpha}{\partial x_\gamma}(t, x^\circ(t), x^\circ_\gamma(t)) > \Bigg] dt^\alpha [<] \leqq -\rho b(x(\cdot), x^\circ(\cdot)) \|\theta(x(\cdot), x^\circ(\cdot))\|^2 \Bigg).$$

Let us denote by $\pi(x^\circ(\cdot))$ the minimizing functional vector of the problem (MFP) at the point $x^\circ(\cdot) \in \mathcal{F}(\Omega_{t_0,t_1})$ and by $\delta(y(\cdot), y_\gamma(\cdot), \Lambda^{10}, \Lambda^{20}, \mu(\cdot), \nu(\cdot))$ the maximizing functional vector of the dual problem (MFD) at the point $(y(\cdot), y_\gamma(\cdot), \Lambda^{10}, \Lambda^{20}, \mu(\cdot), \nu(\cdot)) \in \Delta$, where Δ is the domain of the problem (MFD).

Theorem 2.1 (Weak duality)**.** *Let $x^\circ(\cdot)$ be a feasible solution of the problem* (MFP) *and $y(\cdot)$ be an efficient solution of the problem* (MFD)*. Assume that the following conditions are fulfilled:*

a) $\Lambda^{10}_\ell > 0,\ \Lambda^{20}_\ell > 0,\ \ell = \overline{1,r},\ \Lambda^{10}_\ell F^\ell(y(\cdot)) - \Lambda^{20}_\ell K^\ell(y(\cdot)) = 0$;

b) *for any $\ell = \overline{1,r}$, the functional $F^\ell(x(\cdot))$ is (ρ'^ℓ, b)-quasiinvex at the point $y(\cdot)$ and the functional $-K^\ell(x(\cdot))$ is (ρ''^ℓ, b)-quasiinvex at the point $y(\cdot)$ with respect to η and θ;*

c) $\displaystyle\int_{\gamma_{t_0,t_1}} [< \mu_\alpha(t), g(t, x(t), x_\gamma(t)) > + < \nu_\alpha(t), h(t, x(t), x_\gamma(t)) >] dt^\alpha$ *is (ρ''', b) - quasiinvex at $y(\cdot)$ with respect to η and θ;*

d) *one of the functionals of* b), c) *is strictly quasiinvex;*

e) $\rho'^{\ell}\Lambda_{\ell}^{10}+\rho''^{\ell}\Lambda_{\ell}^{20}+\rho''' \geq 0.$

Then, the inequality $\pi(x^{\circ}(\cdot)) \leq \delta(y(\cdot), y_{\gamma}(\cdot), \Lambda^{10}, \Lambda^{20}, \mu(\cdot), \nu(\cdot))$ *is false.*

Proof. From b) it follows

$$(F^{\ell}(x^{\circ}(\cdot)) \leqq F^{\ell}(y(\cdot)))\Rightarrow\Big(b(x^{\circ}(\cdot),y(\cdot))\int_{\gamma_{t_0,t_1}}\Big[<\eta(t,x^{\circ}(t),x^{\circ}_{\gamma}(t),y(t),y_{\gamma}(t)), \frac{\partial f^{\ell}_{\alpha}}{\partial y}(t,y(t),y_{\gamma}(t))>+< D_{\gamma}\eta(t,x^{\circ}(t),x^{\circ}_{\gamma}(t),y(t),y_{\gamma}(t)), \frac{\partial f^{\ell}_{\alpha}}{\partial y_{\gamma}}(t,y(t),y_{\gamma}(t))>\Big]dt^{\alpha} \leqq -\rho'^{\ell}b(x^{\circ}(\cdot),y(\cdot))\|\theta(x^{\circ}(\cdot),y(\cdot))\|^{2}\Big), \quad \ell=\overline{1,r}, \tag{2.1}$$

and

$$(-K^{\ell}(x^{\circ}(\cdot)) \leqq - K^{\ell}(y(\cdot)))\Rightarrow\Big(b(x^{\circ}(\cdot),y(\cdot))\int_{\gamma_{t_0,t_1}}\Big[<-\eta(t,x^{\circ}(t),x^{\circ}_{\gamma}(t),y(t),y_{\gamma}(t)), \frac{\partial k^{\ell}_{\alpha}}{\partial y}(t,y(t),y_{\gamma}(t))>+<-D_{\gamma}\eta(t,x^{\circ}(t),x^{\circ}_{\gamma}(t),y(t),y_{\gamma}(t)), \frac{\partial k^{\ell}_{\alpha}}{\partial y_{\gamma}}(t,y(t),y_{\gamma}(t))>\Big]dt^{\alpha} \leqq \rho''^{\ell}b(x^{\circ}(\cdot),y(\cdot))\|\theta(x^{\circ}(\cdot),y(\cdot))\|^{2}\Big), \quad \ell=\overline{1,r}. \tag{2.2}$$

We multiply (2.1) by $\Lambda_{\ell}^{10}>0$ and (2.2) by $\Lambda_{\ell}^{20}>0$. We make the sum and we obtain the following implications

$$\Big(\Lambda_{\ell}^{10}F^{\ell}(x^{\circ}(\cdot)))-\Lambda_{\ell}^{20}K^{\ell}(x^{\circ}(\cdot)) \leqq 0\Big)\Rightarrow\Big(b(x^{\circ}(\cdot),y(\cdot)) \int_{\gamma_{t_0,t_1}}\Big\{< \eta(t,x^{\circ}(t),x^{\circ}_{\gamma}(t),y(t),y_{\gamma}(t)),\ \Lambda_{\ell}^{10}\frac{\partial f^{\ell}_{\alpha}}{\partial y}(t,y(t),y_{\gamma}(t)) -\Lambda_{\ell}^{20}\frac{\partial k^{\ell}_{\alpha}}{\partial y}(t,y(t),y_{\gamma}(t)) >+< D_{\gamma}\eta(t,x^{\circ}(t),x^{\circ}_{\gamma}(t),y(t),y_{\gamma}(t)), \Lambda_{\ell}^{10}\frac{\partial f^{\ell}_{\alpha}}{\partial y_{\gamma}}(t,y(t),y_{\gamma}(t))-\Lambda_{\ell}^{20}\frac{\partial k^{\ell}_{\alpha}}{\partial y_{\gamma}}(t,y(t),y_{\gamma}(t)) >\Big\}dt^{\alpha} \leqq -b(x^{\circ}(\cdot),y(\cdot))\|\theta(x^{\circ}(\cdot),y(\cdot))\|^{2}(\rho'^{\ell}\Lambda_{\ell}^{10}+\rho''^{\ell}\Lambda_{\ell}^{20})\Big). \tag{2.3}$$

According to the hypothesis c), we have

$$\Big(\int_{\gamma_{t_0,t_1}} \big[< \mu_\alpha(t), g(t,x^\circ(t),x^\circ_\gamma(t)) > + < \nu_\alpha(t), h(t,x^\circ(t),x^\circ_\gamma(t)) >\big]\, dt^\alpha \leqq$$

$$\int_{\gamma_{t_0,t_1}} [< \mu_\alpha(t), g(t,y(t),y_\gamma(t)) > + < \nu_\alpha(t), h(t,y(t),y_\gamma(t)) >] dt^\alpha\Big) \Rightarrow$$

$$\Rightarrow \Big(b(x^\circ(\cdot),y(\cdot)) \int_{\gamma_{t_0,t_1}} < \eta(t,x^\circ(t),x^\circ_\gamma(t),y(t),y_\gamma(t)), < \mu_\alpha(t), \frac{\partial g}{\partial y}(t,y(t),y_\gamma(t)) >$$

$$+ < \nu_\alpha(t), \frac{\partial h}{\partial y}(t,y(t),y_\gamma(t)) >> + < D_\gamma \eta(t,x^\circ(t),x^\circ_\gamma(t),y(t),y_\gamma(t)),$$

$$< \mu_\alpha(t), \frac{\partial g}{\partial y_\gamma}(t,y(t),y_\gamma(t)) > + < \nu_\alpha(t), \frac{\partial h}{\partial y_\gamma}(t,y(t),y_\gamma(t)) >> dt^\alpha$$

$$\leqq -\rho''' b(x^\circ(\cdot),y(\cdot)) \|\theta(x^\circ(\cdot),y(\cdot))\|^2\Big). \tag{2.4}$$

Making the sum of the implications (2.3) and (2.4) it follows

$$\Big(\Lambda^{10}_\ell F^\ell(x^\circ(\cdot)) - \Lambda^{20}_\ell K^\ell(x^\circ(\cdot)) + \int_{\gamma_{t_0,t_1}} \big[< \mu_\alpha(t), g(t,x^\circ(t),x^\circ_\gamma(t)) >$$

$$+ < \nu_\alpha(t), h(t,x^\circ(t),x^\circ_\gamma(t)) >] dt^\alpha - \int_{\gamma_{t_0,t_1}} [< \mu_\alpha(t), g(t,y(t),y_\gamma(t)) >$$

$$+ < \nu_\alpha(t), h(t,y(t),y_\gamma(t)) >] dt^\alpha \leqq 0\Big) \Rightarrow$$

$$\Rightarrow \Big(b(x^\circ(\cdot),y(\cdot)) \int_{\gamma_{t_0,t_1}} < \eta(t,x^\circ(t),x^\circ_\gamma(t),y(t),y_\gamma(t)), \Lambda^{10}_\ell \frac{\partial f^\ell_\alpha}{\partial y}(t,y(t),y_\gamma(t))$$

$$-\Lambda^{20}_\ell \frac{\partial k^\ell_\alpha}{\partial y}(t,y(t),y_\gamma(t)) + < \mu_\alpha(t), \frac{\partial g}{\partial y}(t,y(t),y_\gamma(t)) >$$

$$+ < \nu_\alpha(t), \frac{\partial h}{\partial y}(t,y(t),y_\gamma(t)) >> + < D_\gamma \eta(t,x^\circ(t),x^\circ_\gamma(t),y(t),y_\gamma(t)),$$

$$\Lambda^{10}_\ell \frac{\partial f^\ell_\alpha}{\partial y_\gamma}(t,y(t),y_\gamma(t)) - \Lambda^{20}_\ell \frac{\partial k^\ell_\alpha}{\partial y_\gamma}(t,y(t),y_\gamma(t))$$

$$+ < \mu_\alpha(t), \frac{\partial g}{\partial y_\gamma}(t,y(t),y_\gamma(t)) > + < \nu_\alpha(t), \frac{\partial h}{\partial y_\gamma}(t,y(t),y_\gamma(t)) >> dt^\alpha$$

$$< -b(x^\circ(\cdot),y(\cdot)) \|\theta(x^\circ(\cdot),y(\cdot))\|^2 \Big(\rho'^\ell \Lambda^{10}_\ell + \rho''^\ell \Lambda^{20}_\ell + \rho'''\Big)\Big). \tag{2.5}$$

Since $b(x^\circ(\cdot), y(\cdot)) > 0$, we obtain

$$\int_{\gamma_{t_0,t_1}} < \eta(t, x^\circ(t), x^\circ_\gamma(t), y(t), y_\gamma(t), \frac{\partial V_\alpha}{\partial y}(t, y(t), y_\gamma(t), \lambda, \mu(\cdot), \nu(\cdot) >$$

$$+ < D_\gamma \eta(t, x^\circ(t), x^\circ_\gamma(t), y(t), y_\gamma(t)), \frac{\partial V_\alpha}{\partial y_\gamma}(t, y(t), y_\gamma(t), \lambda, \mu(\cdot), \nu(\cdot)) > dt^\alpha$$

$$< -\|\theta(x^\circ(\cdot), y(\cdot))\|^2 \left(\rho'^\ell \Lambda^{10}_\ell + \rho''^\ell \Lambda^{20}_\ell + \rho''' \right), \tag{2.6}$$

where

$$V_\alpha(t, y(\cdot), y_\gamma(\cdot), \lambda, \mu(\cdot), \nu(\cdot)) = \Lambda^{10}_\ell f^\ell_\alpha(t, y(t), y_\gamma(t)) - \Lambda^{20}_\ell k^\ell_\alpha(t, y(t), y_\gamma(t))$$

$$+ < \mu_\alpha(y(t)), g(t, y(t), y_\gamma(t)) > + < \nu_\alpha(t), h(t, y(t), y_\gamma(t)) >, \; t \in \Omega_{t_0,t_1}, \; \alpha = \overline{1,p}.$$

The following relation holds

$$< D_\gamma \eta(t, x^\circ(t), x^\circ_\gamma(t), y(t), y_\gamma(t)), \frac{\partial V_\alpha}{\partial y_\gamma}(t, y(t), y_\gamma(t), \lambda, \mu(t), \nu(t)) >$$

$$= D_\gamma < \eta(t, x^\circ(t), x^\circ_\gamma(t), y(t), y_\gamma(t)), \frac{\partial V_\alpha}{\partial y_\gamma}(t, y(t), y_\gamma(t), \lambda, \mu(t), \nu(t)) >$$

$$- < \eta(t, x^\circ(t), x^\circ_\gamma(t), y(t), y_\gamma(t)), D_\gamma \left(\frac{\partial V_\alpha}{\partial y_\gamma} \right)(t, y(t), y_\gamma(t), \lambda, \mu(t), \nu(t)) >. \tag{2.7}$$

By replacing the relations (2.7) and by using Euler-Lagrange PDE, the relation (2.6) becomes

$$\int_{\gamma_{t_0,t_1}} D_\gamma < \eta(t, x^\circ(t), x^\circ_\gamma(t), y(t), y_\gamma(t)), \frac{\partial V_\alpha}{\partial y_\gamma}(t, y(t), y_\gamma(t), \lambda, \mu(t), \nu(t)) > dt^\alpha$$

$$< -\|\theta(x^\circ(\cdot), y(\cdot))\|^2 \left(\rho'^\ell \Lambda^{10}_\ell + \rho''^\ell \Lambda^{20}_\ell + \rho''' \right). \tag{2.8}$$

Let

$$Q^\gamma_\alpha(t) = < \eta(t, x^\circ(t), x^\circ_\gamma(t), y(t), y_\gamma(t)), \frac{\partial V_\alpha}{\partial y_\gamma}(t, y(t), y_\gamma(t), \lambda, \mu(t), \nu(t)) >, \quad \alpha, \gamma = \overline{1,p},$$

and

$$I = \int_{\gamma_{t_0,t_1}} D_\gamma Q^\gamma_\alpha(t)\, dt^\alpha.$$

According to [14], §9, we have the following

Lemma 2.1. *A total divergence is equal to a total derivative.*

Consequently, there exists $Q(t)$, with $Q(t_0) = 0$ and $Q(t_1) = 0$ such that $D_\gamma Q^\gamma_\alpha(t) = D_\alpha Q(t)$ and

$$I = \int_{\gamma_{t_0,t_1}} D_\alpha Q(t)\, dt^\alpha = Q(t_1) - Q(t_0) = 0.$$

Replacing into the inequality (2.8), it follows that

$$0 < -\|\theta(x^\circ(\cdot), y(\cdot))\|^2 \left(\rho'^\ell \Lambda^{10}_\ell + \rho''^\ell \Lambda^{20}_\ell + \rho'''\right).$$

From hypothesis e), the previous relation becomes $0 < 0$, that is false. From relation (2.5), it follows

$$0 \leqq \Lambda^{10}_\ell F^\ell(x^\circ(\cdot)) - \Lambda^{20}_\ell K^\ell(x^\circ(\cdot)) + \int_{\gamma_{t_0,t_1}} \big[< \mu_\alpha(t), g(t, x^\circ(t), x^\circ_\gamma(t)) >$$

$$+ < \nu_\alpha(t), h(t, x^\circ(t), x^\circ_\gamma(t)) >]dt^\alpha - \int_{\gamma_{t_0,t_1}} [< \mu_\alpha(t), g(t, y(t), y_\gamma(t)) >$$

$$+ < \nu_\alpha(t), h(t, y(t), y_\gamma(t)) >]dt^\alpha.$$

Taking into account the inequality

$$\int_{\gamma_{t_0,t_1}} [< \mu_\alpha(t), g(t, y(t), y_\gamma(t)) > + < \nu_\alpha(t), h(t, y(t), y_\gamma(t)) >]dt^\alpha \geqq 0,$$

the above-mentioned relation becomes

$$\Lambda^{10}_\ell F^\ell(x^\circ(\cdot)) - \Lambda^{20}_\ell K^\ell(x^\circ(\cdot)) > 0,$$

that is

$$K^\ell(x^\circ(\cdot))K^\ell(y(\cdot)) \left[\frac{F^\ell(x^\circ(\cdot))}{K^\ell(x^\circ(\cdot))} - \frac{F^\ell(y(\cdot))}{K^\ell(y(\cdot))}\right] > 0.$$

Because $K^\ell(x^\circ(\cdot))K^\ell(y(\cdot)) > 0$, $\ell = \overline{1, r}$, we conclude that

$$\left((\frac{F^1(x^\circ(\cdot))}{K^1(x^\circ(\cdot))} - \frac{F^1(y(\cdot))}{K^1(y(\cdot))}, \dots, \frac{F^r(x^\circ(\cdot))}{K^r(x^\circ(\cdot))} - \frac{F^r(y(\cdot))}{K^r(y(\cdot))}\right) \not\leq (0, \dots, 0)))$$

or

$$\left(\frac{F^1(x^\circ(\cdot))}{K^1(x^\circ(\cdot))}, \dots, \frac{F^r(x^\circ(\cdot))}{K^r(x^\circ(\cdot))}\right) \not\leq \left((\frac{F^1(y(\cdot))}{K^1(y(\cdot))}, \dots, \frac{F^r(y(\cdot))}{K^r(y(\cdot))}\right).$$

Therefore, the relation $\pi(x^\circ(\cdot)) \leq \delta(y(\cdot), y_\gamma(\cdot), \Lambda^{10}, \Lambda^{20}, \mu(\cdot), \nu(\cdot))$ is not satisfied. □

We shall finish our considerations by giving the statements of two results on direct duality and converse duality, respectively. For proofs, see [8].

Theorem 2.2 (Direct duality). *Let* $x^\circ(\cdot)$ *be an efficient solution of* (MFP) *and suppose that the hypotheses of Theorem* 2.1 *are satisfied. Then there are the scalars* $\Lambda^{10}, \Lambda^{20} \in \mathbb{R}^r$ *and the smooth functions* $\mu^\circ\colon \Omega_{t_0,t_1} \to \mathbb{R}^{msp}$, $\nu^\circ\colon \Omega_{t_0,t_1} \to \mathbb{R}^{qsp}$ *such* *that* $(x^\circ(\cdot), x^\circ_\gamma(\cdot), \Lambda^{10}, \Lambda^{20}, \mu^\circ(\cdot), \nu^\circ(\cdot))$ *is an efficient solution of the dual program* (MFD) *and* $\pi(x^\circ(\cdot)) = \delta(x^\circ(\cdot), x^\circ_\gamma(\cdot), \Lambda^{10}, \Lambda^{20}, \mu^\circ(\cdot), \nu^\circ(\cdot))$.

We shall present now a theorem concerning the converse duality, by changing some of the hypotheses.

Theorem 2.3 (Converse duality). *Let* $(x^\circ(\cdot), x^\circ_\gamma(\cdot), \Lambda^{10}, \Lambda^{20}, \mu^\circ(\cdot), \nu^\circ(\cdot))$ *be an efficient solution of the dual problem* (MFD) *and suppose that the following conditions are fulfilled:*

a) $\bar{x}(\cdot)$ *is an efficient solution of the primal problem* (MFP)*;*

b) *for any* $\ell=\overline{1,r}$*, we have*

$$F^\ell(x^\circ(\cdot)) > 0, \quad K^\ell(x^\circ(\cdot)) > 0, \quad \Lambda^{10}_\ell F^\ell(x^\circ(\cdot)) - \Lambda^{20}_\ell K^\ell(x^\circ(\cdot)) = 0;$$

c) *for any* $\ell = \overline{1,r}$*, the functional* $F^\ell(x(\cdot))$ *is* (ρ'^ℓ, b)*-quasiinvex at the point* $x^\circ(\cdot)$ *and the functional* $-K^\ell(x(\cdot))$ *is* (ρ''^ℓ, b)*-quasiinvex at the point* $x^\circ(\cdot)$*, with respect to* η *and* θ*;*

d) $$\int_{\gamma_{t_0,t_1}} [<\mu_\alpha(t), g(t, x(t), x_\gamma(t))> + <\nu_\alpha(t), h(t, x(t), x_\gamma(t))>]\, dt^\alpha$$

is (ρ''', b)*-quasiinvex at the point* $x^\circ(\cdot)$ *with respect to* η *and* θ*;*

e) *one of the functionals of* c), d) *is strictly* (ρ'^ℓ, b)*,* (ρ''^ℓ, b) *or* (ρ''', b)*-quasiinvex with respect to* η *and* θ*, respectively;*

f) $\rho'^\ell \Lambda^{10}_\ell + \rho''^\ell \Lambda^{20}_\ell + \rho''' \geqq 0$.

Then $\bar{x}(\cdot) = x^\circ(\cdot)$ *and moreover,* $\pi(x^\circ(\cdot)) = \delta(x^\circ(\cdot), x^\circ_\gamma(\cdot), \Lambda^{10}, \Lambda^{20}, \mu^\circ(\cdot), \nu^\circ(\cdot))$.

Remark 2.1. If we would like to make a computer aided study of PDE and/or PDI optimization problems we can try to perform symbolic computations. In this respect, we recommend the Maple software package [1], [11].

REFERENCES

[1] Maria Teresa Calapso and C. Udrişte: *Isothermic surfaces as solutions of Calapso PDE*, Balkan J. Geom. Appl., **13**(2008), No. 1, 20-26.

[2] M. Ferarra and Şt. Mititelu: *Mond-Weir duality in vector programming with generalized invex functions on differentiable manifolds*, Balkan J. Geom. Appl., **11**(2006), No. 1, 80-87.

[3] Şt. Mititelu: *Extensions in invexity theory*, J. Adv. Math. Studies, **1**(2008), No. 1-2, 63-70.

[4] Şt. Mititelu: *Optimality and duality for invex multi-time control problems with mixed constraints*, J. Adv. Math. Studies, **2**(2009), No. 1, 25-34.

[5] Şt. Mititelu and I. M. Stancu-Minasian: *Invexity at a point: generalization and clasifications*, Bull. Austral. Math. Soc., **48**(1993), 117-126.

[6] B. Mond and I. Husain: *Sufficient optimality criteria and duality for variational problems with generalised invexity*, J. Austral. Math. Soc., Series B, **31**(1989), 106-121.

[7] Ariana Pitea: *Integral Geometry and PDE Constrained Optimization Problems*, Ph. D Thesis, "Politehnica" University of Bucharest, 2008.

[8] Ariana Pitea, C. Udrişte and Şt. Mititelu: *PDI & PDE-constrained optimization problems with curvilinear functional quotients as objective vectors*, Balkan J. Geom. Appl., **14**(2009), No. 2, 75-88.

[9] V. Preda: *On Mond-Weir duality for variational problems*, Rev. Roumaine Math. Appl., **28**(1993), No. 2, 155-164.

[10] V. Preda and Sorina Gramatovici: *Some sufficient optimlity conditions for a class of multiobjective variational problems*, An. Univ. Bucureşti, Matematică-Informatică, **61**(2002), No. 1, 33-43.

[11] C. Udrişte: *Tzitzeica theory - opportunity for reflection in Mathematics*, Balkan J. Geom. Appl., **10**(2005), No. 1, 110-120.

[12] C. Udrişte and I. Ţevy: *Multi-time Euler-Lagrange-Hamilton theory*, WSEAS Trans. Math., **6**(2007), No. 6, 701-709.

[13] C. Udrişte and I. Ţevy: *Multi-time Euler-Lagrange dynamics*, Proc. 5-th WSEAS Int. Conf. Systems Theory and Sci. Comp., Athens, Greece, August 24-26 (2007), 66-71.

[14] C. Udrişte, O. Dogaru and I. Ţevy: *Null Lagrangian forms and Euler-Lagrange PDEs*, J. Adv. Math. Studies, **1**(2008), No. 1-2, 143-156.

[15] C. Udrişte, P. Popescu and Marcela Popescu: *Generalized multi-time Lagrangians and Hamiltonians*, WSEAS Trans. Math., **7**(2008), 66-72.

[16] F. A. Valentine: *The problem of Lagrange with differentiable inequality as added side conditions*, in "Contributions to the Calculus of Variations, 1933-1937", Univ. of Chicago Press, 1937, 407-448.

[17] T. Weir and B. Mond: *Generalized convexity and duality in multiobjective programming*, Bull. Austral. Math. Soc., **39**(1989), 287-299.

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