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SOME CONVERGENCE RESULTS FOR NONEXPANSIVE AND QUASI-NONEXPANSIVE OPERATORS

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Abstract. In this paper, following the concepts in [7, 9], we shall establish some convergence results for nonexpansive and quasi-nonexpansive operators in a uniformly convex Banach space. For this purpose, we shall consider the two recently introduced Kirk-Ishikawa and Kirk-Mann type iterative algorithms of Olatinwo [11]. Our results improve, generalize and extend those of [2, 8, 9, 10, 12, 14].

1. INTRODUCTION

Suppose that $A = (a_{nk})$ is an infinite, lower triangular, regular row-stochastic matrix, E a closed convex subset of a Banach space and T a continuous mapping of E into itself and $x_1 \in E$. Then, the general Mann iterative process $M(x_1, A, T)$ which was introduced in Mann [10] is defined by

$$v_n = \sum_{k=1}^{n} a_{nk}x_k, \quad x_{n+1} = Tv_n, \quad n = 1, 2, \dots .$$

If A is the identity matrix, then each sequence of $M(x_1, A, T)$ becomes the sequence of Picard iterates of T at x_1. It was established in [10] that if either of the sequences $\{x_n\}$ and $\{v_n\}$ converges, then the other also converges to the same point, and their common limit is a fixed point of T.

In [7, 9], it is said that the matrix A is *segmenting* for the Mann process if

$$a_{n+1,k} = (1 - a_{n+1,n+1})a_{nk} \quad \text{for} \quad k \leq n.$$

In this case, v_{n+1} lies on the segment joining v_n and Tv_n:

$$v_{n+1} = (1 - d_n)v_n + d_nTv_n, \quad n = 1, 2, \dots, \tag{1.1}$$

where $d_n = a_{n+1,n+1}$.

A segmenting matrix is determined by its sequence of diagonal elements. Some authors including [2, 12, 14] have investigated the case $d_n = \lambda$, $0 < \lambda < 1$, while Mann [10] approximated the fixed points of continuous functions on a closed interval

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of the real line using the segmenting matrix determined by $d_n = \frac{1}{n}$, $\forall\, n$. Dotson [8] considered the case when d_n is bounded away from 0 and 1. Groetsch [9] generalized the results of [2, 8, 10, 12, 14] in a uniformly convex Banach space by employing (1.1) and assuming that A is a segmenting matrix for which $\sum_{n=1}^{\infty} d_n(1-d_n) = \infty$.

We shall give another definition of a segmenting matrix in the next section with a view to generalizing and extending Groetsch [9] and others mentioned earlier in this paper.

2. PRELIMINARIES

Throughout this paper, we shall assume that F_T is the fixed point set of T, while $\bigcap_{i=1}^{k} F_{T_i}$ denotes the set of common fixed points of operators T_i $(i = 1, 2, \dots, k)$.

We shall introduce and employ the following iterative algorithms: Let E be a Banach space, $T_i \colon E \to E$ $(i = 0, 1, \dots, k)$ selfmaps of E and $x_0 \in E$. Then, define the sequence $\{x_n\}_{n=0}^{\infty}$ by

$$\left.\begin{aligned} x_{n+1} &= \alpha_{n,0}\, x_n + \sum_{i=1}^{k} \alpha_{n,i}\, T_i y_n, \quad \sum_{i=0}^{k} \alpha_{n,i} = 1, \quad n = 0, 1, 2, \dots, \\ y_n &= \sum_{j=0}^{s} \beta_{n,j}\, T_j x_n, \quad \sum_{j=0}^{s} \beta_{n,j} = 1, \end{aligned}\right\} \tag{2.1}$$

$k \geq s$, $\alpha_{n,i} \geq 0$, $\alpha_{n,0} \neq 0$, $\beta_{n,j} \geq 0$, $\beta_{n,0} \neq 0$, $\alpha_{n,i}$, $\beta_{n,j} \in [0,1]$, where k and s are fixed integers and T_0 is an identity operator.

If $s = 0$ in (2.1), we also obtain the following interesting iterative process in a Banach space:

$$x_{n+1} = \sum_{i=0}^{k} \alpha_{n,i}\, T_i x_n, \quad \sum_{i=0}^{k} \alpha_{n,i} = 1, \quad n = 0, 1, 2, \dots, \tag{2.2}$$

$\alpha_{n,i} \geq 0$, $\alpha_{n,0} \neq 0$, $\alpha_{n,i} \in [0,1]$, where k is a fixed integer and T_0 is an identity operator.

The iterative processes defined in (2.1) and (2.2) reduce to several others as special cases. See Olatinwo [11] for detail.

However, the following definitions and lemma shall be required in the sequel:

Definition 2.1 (Berinde [1]). A function $\psi \colon \mathbb{R}_+ \to \mathbb{R}_+$ is called a *comparison function* if:

(i) ψ is monotone increasing;

(ii) $\lim_{n\to\infty} \psi^n(t) = 0$, $\forall\, t \geq 0$.

Remark 2.1. Every comparison function satisfies the condition $\psi(0) = 0$. Also, both conditions (i) and (ii) of Definition 2.1 imply that $\psi(t) < t$, $\forall\, t > 0$.

Definition 2.2. A mapping $T\colon E \to E$ is called *nonexpansive* if

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall\, x, y \in E.$$

Lemma 2.1 (Groetsch [3, 9])**.** *Let X be a uniformly convex Banach space and let x, $y \in X$. If $\|x\| \le 1$, $\|y\| \le 1$ and $\|x - y\| \ge \epsilon > 0$, then*

$$\|\lambda x + (1-\lambda)y\| \le 1 - 2\lambda(1-\lambda)\delta(\epsilon) \quad \text{for } 0 \le \lambda < 1 \text{ and } \delta(\epsilon) > 0.$$

For a nonexpansive operator T, $F_T \neq \phi$ is not generally true. A generalization of a nonexpansive operator, with at least one fixed point, is that of the quasi-nonexpansive operators.

Definition 2.3. An operator $T\colon E \to E$ is said to be *quasi-nonexpansive* if T has at least one fixed point in E and, for each fixed point $p \in F_T$, we have

$$\|Tx - p\| \le \|x - p\|, \quad \forall\, x \in E.$$

See Berinde [1] for some examples of quasi-nonexpansive operators. Indeed, literature abounds with various type of contractive definitions for fixed point theorems. For some of these contractive definitions, we refer our readers to Berinde [1], Ciric [4, 5, 6] and Rhoades [13].

It is our purpose in this paper to establish some convergence results for nonexpansive and quasi-nonexpansive operators in a uniformly convex Banach space via the newly introduced iterative processes defined in (2.1) and (2.2). We shall assume that A is a segmenting matrix such that $\sum_{n=0}^{\infty} \alpha_{n,0}(1 - \alpha_{n,0}) = \infty$.

Our results improve, generalize and extend those of [2, 8, 9, 10, 12, 14].

In obtaining our results, we shall employ the following contractive definition: For quasi-nonexpansive operators $T_i\colon E \to E$ $(i = 1, 2, \ldots, k)$, there exist a sublinear comparison function $\psi\colon \mathbb{R}_+ \to \mathbb{R}_+$ and a monotone increasing function $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that $\varphi(0) = 0$ and $\forall\, x, y \in E$, we have

$$\|T_i x - T_i y\| \le \psi(\|x - y\|) + \varphi(\|x - T_i x\|). \tag{2.3}$$

Remark 2.2. If each T_i $(i = 1, 2, \ldots, k)$, is an operator satisfying (2.3), then T_i is a (strict) quasi-nonexpansive operator. That is, from (2.3) and the last part of the Remark 2.1, with $p \in \bigcap_{i=1}^{k} F_{T_i}$, we have

$$\|T_i x - p\| = \|T_i p - T_i x\| \le \psi(\|p - x\|) + \varphi(\|p - T_i p\|) < \|x - p\|. \tag{2.4}$$

3. THE MAIN RESULTS

Theorem 3.1. *Let E be a convex subset of a uniformly convex Banach space X and $T_i\colon E \to E$ $(i = 1, 2, \ldots, k)$, quasi-nonexpansive mappings satisfying* (2.3) *with* *at least a common fixed point. Let* $\{v_n\}_{n=0}^{\infty}$ *be the sequence defined by* (2.1). *Suppose also that* $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ *is a monotone increasing function such that* $\varphi(0) = 0$ *and* $\psi\colon \mathbb{R}_+ \to \mathbb{R}_+$ *is a sublinear comparison function. Then, the sequence* $\left\{(I - T_i^j)v_n\right\}_{n=0}^{\infty}$, *for each* $j \in \mathbb{N}$, *converges to* $0 \in E$ *for each* i *such that* $\sum_{n=0}^{\infty} \alpha_{n,0}(1 - \alpha_{n,0}) = \infty$.

Proof. If $p \in \bigcap_{i=1}^{k} F_{T_i}$ for each i, then by using (2.3) and (2.4) we have

$$\begin{aligned}
\|v_{n+1} - p\| &= \left\| \alpha_{n,0} v_n + \sum_{i=1}^{k} \alpha_{n,i} T_i y_n - \sum_{i=0}^{k} \alpha_{n,i} T_i p \right\| \\
&\le \alpha_{n,0}\|v_n - p\| + \sum_{i=1}^{k} \alpha_{n,i}\|T_i y_n - T_i p\| \\
&< \alpha_{n,0}\|v_n - p\| + \sum_{i=1}^{k} \alpha_{n,i}\|y_n - p\| \\
&\le \alpha_{n,0}\|v_n - p\| + \sum_{i=1}^{k} \alpha_{n,i} \sum_{j=0}^{s} \beta_{n,j}\|T_j v_n - T_j p\| \\
&< \alpha_{n,0}\|v_n - p\| + \sum_{i=1}^{k} \alpha_{n,i} \sum_{j=0}^{s} \beta_{n,j}\|v_n - p\| \\
&= \alpha_{n,0}\|v_n - p\| + \sum_{i=1}^{k} \alpha_{n,i}\|v_n - p\| \\
&= \sum_{i=0}^{k} \alpha_{n,i}\|v_n - p\| = \|v_n - p\|.
\end{aligned}$$

Now,

$$\begin{aligned}
\|(I - T_i^j)v_n\| &= \|v_n - T_i^j v_n\| \le \|v_n - p\| + \|p - T_i^j v_n\| \\
&= \|v_n - p\| + \|T_i^j p - T_i^j v_n\| \le \|v_n - p\| + \|p - v_n\| = 2\|v_n - p\|.
\end{aligned}$$

Since $\|v_n - T_i^j v_n\| \le 2\|v_n - p\|$, if the sequence $\left\{(I - T^j)v_n\right\}_{n=0}^{\infty}$ does not converge to 0, then there exists $a > 0$ such that $\|v_n - p\| \ge a \;\forall n$. If $\left\{(I - T^j)v_n\right\}_{n=0}^{\infty}$ does not converge to 0, then there is an $\epsilon > 0$ such that $\|v_n - T_i^j v_n\| \ge \epsilon \;\forall n$. Let

$$b = 2\delta\left(\frac{\epsilon}{\|v_0 - p\|}\right), \quad x_n = \frac{v_n - p}{\|v_n - p\|} \quad \text{and} \quad z_n = \frac{\sum_{i=1}^{k} \alpha_{n,i}(T_i y_n - T_i p)}{(1 - \alpha_{n,0})\|v_n - p\|},$$

with $\|v_n - p\| \neq 0$. Then, we have

$$\|x_n\| = \left\| \left(\frac{v_n - p}{\|v_n - p\|} \right) \right\| \leq \frac{\|v_n - p\|}{\|v_n - p\|} = 1,$$

and using (2.3) and (2.4) again, we have

$$\begin{aligned} \|z_n\| &= \left\| \frac{\sum_{i=1}^{k} \alpha_{n,i}(T_i y_n - T_i p)}{(1-\alpha_{n,0})\|v_n - p\|} \right\| \leq \frac{\sum_{i=1}^{k} \alpha_{n,i}\|T_i y_n - T_i p\|}{(1-\alpha_{n,0})\|v_n - p\|} < \frac{\sum_{i=1}^{k} \alpha_{n,i}\|y_n - p\|}{(1-\alpha_{n,0})\|v_n - p\|} \\ &\leq \frac{\sum_{i=1}^{k} \alpha_{n,i} \sum_{j=0}^{s} \beta_{n,j}\|T_j v_n - T_j p\|}{(1-\alpha_{n,0})\|v_n - p\|} < \frac{\sum_{i=1}^{k} \alpha_{n,i}\|v_n - p\|}{(1-\alpha_{n,0})\|v_n - p\|} = 1, \end{aligned}$$

since $\sum_{i=1}^{k} \alpha_{n,i} = 1 - \alpha_{n,0}$.

Hence, we have from (2.1) that

$$\begin{aligned} \|v_{n+1} - p\| &= \left\| \alpha_{n,0} v_n + \sum_{i=1}^{k} \alpha_{n,i} T_i y_n - \sum_{i=0}^{k} \alpha_{n,i} T_i p \right\| \\ &= \left\| \alpha_{n,0}(v_n - p) + \sum_{i=1}^{k} \alpha_{n,i}(T_i y_n - T_i p) \right\| \\ &= \left\| (\|v_n - p\|) \left[\alpha_{n,0} \frac{(v_n - p)}{\|v_n - p\|} + (1-\alpha_{n,0}) \frac{\sum_{i=1}^{k} \alpha_{n,i}(T_i y_n - T_i p)}{(1-\alpha_{n,0})\|v_n - p\|} \right] \right\| \\ &= \|(\|v_n - p\|)[\alpha_{n,0} x_n + (1-\alpha_{n,0}) z_n]\| \\ &\leq \|v_n - p\| \; \|\alpha_{n,0} x_n + (1-\alpha_{n,0}) z_n\|. \end{aligned} \tag{3.1}$$

Using Lemma 2.1 in (3.1) yields

$$\begin{aligned} \|v_{n+1} - p\| &\leq \left[1 - \alpha_{n,0}(1-\alpha_{n,0})b\right] \|v_n - p\| \\ &= \|v_n - p\| - b\alpha_{n,0}(1-\alpha_{n,0})\|v_n - p\| \\ &\leq \|v_{n-1} - p\| - b\alpha_{n-1,0}(1-\alpha_{n-1,0})\|v_{n-1} - p\| \\ &\quad - b\alpha_{n,0}(1-\alpha_{n,0})\|v_n - p\|. \end{aligned}$$

Repeating this process inductively leads to

$$a \leq \|v_{n+1} - p\| \leq \|v_0 - p\| - b[\alpha_{0,0}(1-\alpha_{0,0})\|v_0 - p\| + \alpha_{1,0}(1-\alpha_{1,0})\|v_1 - p\| + \cdots + \alpha_{n,0}(1-\alpha_{n,0})\|v_n - p\|] = \|v_0 - p\| - b\sum_{r=0}^{n}\alpha_{r,0}(1-\alpha_{r,0})\|v_r - p\|$$

$$\leq \|v_0 - p\| - ab\sum_{r=0}^{n}\alpha_{r,0}(1-\alpha_{r,0}).$$

Therefore, we obtain

$$a\left[1 + b\sum_{r=0}^{n}\alpha_{r,0}(1-\alpha_{r,0})\right] \leq \|v_0 - p\|,$$

from which it follows that

$$a \leq \frac{\|v_0 - p\|}{1 + b\sum_{r=0}^{n}\alpha_{r,0}(1-\alpha_{r,0})} \to 0 \text{ as n} \to \infty,$$

leading to a contradiction. Therefore, we have $a = 0$. Hence,

$$\lim_{n\to\infty}\|v_n - T_i^j v_n\| = 0,$$

for each i. □

Remark 3.1. Theorem 3.1 is a generalization of the results of [2, 8, 9, 10, 12, 14].

Theorem 3.2. *Let E be a convex subset of a uniformly convex Banach space X and $T_i\colon E \to E$ $(i = 1, 2, \ldots, k)$, quasi-nonexpansive mappings satisfying* (2.3) *with at least a common fixed point. Suppose also that $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a monotone increasing function such that $\varphi(0) = 0$ and $\psi\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a sublinear comparison function. Let $\{v_n\}_{n=0}^{\infty}$ be the sequence defined by* (2.2)*. Then, the sequence $\left\{(I - T_i^j)v_n\right\}_{n=0}^{\infty}$, for each $j \in \mathbb{N}$, converges to $0 \in E$ for each i such that*

$$\sum_{n=0}^{\infty}\alpha_{n,0}(1-\alpha_{n,0}) = \infty.$$

Theorem 3.3. *Let E be a convex subset of a uniformly convex Banach space X and $T_i\colon E \to E$ $(i = 1, 2, \ldots, k)$ nonexpansive mappings with at least a common fixed point. Let $\{v_n\}_{n=0}^{\infty}$ be the sequence defined by* (2.1)*. Then, the sequence $\left\{(I - T_i^j)v_n\right\}_{n=0}^{\infty}$, for each $j \in \mathbb{N}$, $1 \leq j \leq k$, converges to $0 \in E$ for each i such that*

$$\sum_{n=0}^{\infty}\alpha_{n,0}(1-\alpha_{n,0}) = \infty.$$

Theorem 3.4. *Let E be a convex subset of a uniformly convex Banach space X and $T_i\colon E \to E$ $(i = 1, 2, \ldots, k)$ nonexpansive mappings with at least a common fixed point. Let $\{v_n\}_{n=0}^{\infty}$ be the sequence defined by (2.2). Then, the sequence $\{(I - T_i^j)v_n\}_{n=0}^{\infty}$, for each $j \in \mathbb{N}$, $1 \le j \le k$, converges to $0 \in E$ for each i such that*

$$\sum_{n=0}^{\infty} \alpha_{n,0}(1 - \alpha_{n,0}) = \infty.$$

Remark 3.2. The proof of each of Theorem 3.2, Theorem 3.3 and Theorem 3.4 is analogous to that of Theorem 3.1.

Remark 3.3. Theorem 3.2, Theorem 3.3 and Theorem 3.4 are also generalizations and extensions of the results of [2, 8, 9, 10, 12, 14].

Example 3.1. When $T_i = T^i$, $i = 1, 2, \ldots$, in (2.2), we obtain an iteration process of the form

$$x_{n+1} = \sum_{i=0}^{k} \alpha_{n,i}\, T^i x_n, \quad \sum_{i=0}^{k} \alpha_{n,i} = 1, \quad n = 0, 1, 2, \ldots, \tag{3.2}$$

which has been used to approximate the fixed point for nonexpansive mappings. For instance, the fixed point of the nonexpansive mapping $T\colon [0,1] \to [0,1]$ defined by

$$Tx = 1 - x,\ x \in [0,1],$$

has been obtained by both Kirk's iteration process and the iteration process defined in (3.2) for the same choice of initial point $x_0 \in [0,1]$. The fixed point of T is given by $F_T = \left\{\frac{1}{2}\right\}$. Specifically, Kirk's iteration process converges to the fixed point of T after seven iterations when $x_0 = 0.6$, $k = 2$ and $\alpha_0 = \alpha_1 = \alpha_2 = \frac{1}{3}$, whereas our iteration process defined in (3.2) converges to the same fixed point of T after the second iteration when $x_0 = 0.6$, $k = 2$ and $\alpha_{n,0} = 1 - \frac{1}{n+1} - \frac{1}{n+1}$, $\alpha_{n,1} = \alpha_{n,2} = \frac{1}{n+1}$. Hence, our iteration process converges faster to the fixed point of T than the Kirk's iteration process.

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