

ON HERMITE-HADAMARD INTEGRAL INEQUALITIES FOR PRODUCT OF TWO NONCONVEX FUNCTIONS

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ABSTRACT. In this paper, we establish some new Hermite-Hadamard type integral inequalities for product of two log- φ -convex and φ -convex functions. Note that log- φ -convex and φ -convex functions are nonconvex functions and include the convex functions as special cases. As special cases, we obtain the well known results for the log-convex and convex functions.

1. INTRODUCTION

Convexity plays a central and fundamental role in mathematical finance, economics, engineering, management sciences and optimization theory. In recent years, several extensions and generalizations have been considered for classical convexity. A significant generalization of convex functions is that of φ -convex functions introduced by Noor [13]. Noor and Noor [17] and Noor [13] have studied the basic properties of the φ -convex functions. For their applications in optimization, variational inequalities and equilibrium problems, see [3-20] and the references therein. It is well-known that the φ -convex functions and φ -sets may not be convex functions and convex sets. Several refinements of the Hermite-Hadamard inequalities are being obtained for the convex functions and its variant forms, see [1, 2, 15, 21, 22] and the references therein. In this paper, we establish some Hermite-Hadamard type inequalities for involving two log- φ -convex functions which may be viewed as an improvement and extensions of Pachpatte's results [21]. We also derive some new Hadamard inequalities for the product of two φ -convex functions. This is the main motivation of this paper.

2. PRELIMINARIES

Let K be a nonempty closed set in R^n . We denote by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ be the inner product and norm respectively. Let $f, \varphi: K \rightarrow R$ be continuous functions. First of

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all, we recall the following well know results and concepts, which are mainly due to Noor and Noor [17] and Noor [13].

Definition 2.1 ([13,17]). Let $u \in K$. Then the set K is said to be φ -convex at u with respect to φ , if

$$u + te^{i\varphi}(v - u) \in K, \quad \forall u, v \in K, \quad t \in [0, 1].$$

Remark 2.1. We would like to mention that the definition 2.1 of a φ -convex set has a clear geometric interpretation. This definition essentially says that there is a path starting from a point u which is contained in K . We do not require that the point v should be one of the end points of the path. This observation plays an important role in our analysis. Note that, if we demand that v should be an end point of the path for every pair of points $u, v \in K$, then $e^{i\varphi}(v - u) = v - u$ if and only if, $\varphi = 0$, and consequently φ -convexity reduces to convexity. Thus, it is true that every convex set is also an φ -convex set, but the converse is not necessarily true, see [13,17] and the references therein.

Definition 2.2 ([13,17]). The function f on the φ -convex set K is said to be φ -convex with respect to φ , if

$$f(u + te^{i\varphi}(v - u)) \leq (1 - t)f(u) + tf(v), \quad \forall u, v \in K, \quad t \in [0, 1].$$

The function f is said to be φ -concave if and only if $-f$ is φ -convex. Note that every convex function is a φ -convex function, but the converse is not true.

Definition 2.3 ([15]). The differentiable function f on the φ -convex set K is said to be an φ -invex function with respect to φ , if

$$f(v) - f(u) \geq \langle f'_\varphi(u), \eta(v, u) \rangle, \quad \forall u, v \in K,$$

where $f'_\varphi(u)$ is the differential of f at u . The concepts of the φ -convex and φ -invex functions have played very important role in the development of generalized convex programming. From definitions 2.2 and 2.3, it is clear that the differentiable φ -convex function are φ -invex functions.

Definition 2.4 ([15]). The function f on the φ -convex set K is called quasi φ -convex with respect to φ , such that

$$f(u + te^{i\varphi}(v - u)) \leq \max\{f(u), f(v)\}, \quad \forall u, v \in K, \quad t \in [0, 1].$$

Definition 2.5 ([13,15,17]). The function f on the φ -convex set K is said to be logarithmic φ -convex with respect to φ , if

$$f(u + te^{i\varphi}(v - u)) \leq (f(u))^{1-t}(f(v))^t, \quad u, v \in K, \quad t \in [0, 1],$$

where $f(\cdot) > 0$.

From the above definitions, we have

$$\begin{aligned} f(u + te^{i\varphi}(v - u)) &\leq (f(u))^{1-t}(f(v))^t \\ &\leq (1 - t)f(u) + tf(v) \\ &\leq \max \{f(u), f(v)\}. \end{aligned}$$

From definition 2.5, we have

$$\log f(u + te^{i\varphi}(v - u)) \leq (1 - t)\log(f(u)) + t\log(f(v)), \quad \forall u, v \in K, \quad t \in [0, 1].$$

In view of this fact, we have the following.

Definition 2.6. The differentiable function f on the φ -convex set K is said to be a log- φ -invex function with respect to φ , if

$$\log f(v) - \log f(u) \geq \left\langle \frac{f'_\varphi(u)}{f(u)}, v - u \right\rangle, \quad \forall u, v \in K.$$

It can be shown that every differentiable log- φ -convex functions is a log- φ -invex function, but the converse is not true. Note that for $e^{i\varphi}(v - u) = v - u$, the φ -convex set K becomes the convex set and consequently, φ -convex, φ -invex, and log- φ -invex functions reduce to convex and log-convex functions.

It is well known [1,2,21,22] that if f is a convex function on the interval $I = [a, b]$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}, \quad \forall a, b \in I, \tag{2.1}$$

which is known as the Hermite-Hadamard inequality for the convex functions. For some results related to this classical results, see [1,2, 21,22] and the references therein. Dragomir and Mond [1] proved the following Hermite-Hadamard type inequalities for the log-convex functions:

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \exp\left[\frac{1}{b-a} \int_a^b \ln[f(x)]dx\right] \\ &\leq \frac{1}{b-a} \int_a^b G(f(x), f(a+b-x))dx \\ &\leq \frac{1}{b-a} \int_a^b f(x)dx \\ &\leq L(f(a), f(b)) \leq \frac{f(a)+f(b)}{2}, \end{aligned} \tag{2.2}$$

where $G(p, q) = \sqrt{pq}$ is the geometric mean and $L(p, q) = \frac{p - q}{\ln p - \ln q}$ ($p \neq q$) is the logarithmic mean of the positive real numbers p, q (for $p = q$, we put $L(p, q) = p$).

Pachpatte [21] has also obtained some other refinement of the Hermite-Hadamard inequality for differentiable log-convex functions.

From now onward, we always assume that $K = [a, a + e^{i\varphi}(b - a)]$ and $0 \leq \varphi \leq \frac{\pi}{2}$, the interval, unless otherwise specified.

Note that, if $K = [a, a + e^{i\varphi}(b - a)]$ is an interval, then the φ -convex functions can be defined as

$$\begin{vmatrix} 1 & 1 & 1 \\ a & x & a + e^{i\varphi}(b - a) \\ f(a) & f(x) & f(a + e^{i\varphi}(b - a)) \end{vmatrix} \geq 0,$$

where $x = a + te^{i\varphi}(b - a) \in K$.

Using this definition, one can easily show that the φ -convex functions satisfy the inequality of the form:

$$f(x) \leq f(a) + \frac{f(b) - f(a)}{e^{i\varphi}(b - a)}(x - a), \quad (2.3)$$

using the fact that $f(a + e^{i\varphi}(b - a)) \leq f(b)$. This shows that function f is a φ -convex function on the φ -convex set K if and only if f satisfies (2.3).

Using this alternative equivalent definition of the φ -convex functions and technique of Pecaric et al [22], Noor [15] has obtained the following Hermite-Hadamard inequalities for the φ -convex functions and its variant forms. To convey the main idea and the technique, we include all the details.

Theorem 2.1 ([15]). *Let $f: K = [a, a + e^{i\varphi}(b - a)] \rightarrow (0, \infty)$ be a φ -convex function on the interval of real numbers K° (the interior of K) and $a, b \in K^\circ$ with $a < a + e^{i\varphi}(b - a)$ and $0 \leq \varphi \leq \frac{\pi}{2}$. Then*

$$f\left(\frac{2a + e^{i\varphi}(b - a)}{2}\right) \leq \frac{1}{e^{i\varphi}(b - a)} \int_a^{a + e^{i\varphi}(b - a)} f(x) dx \leq \frac{f(a) + f(b)}{2}. \quad (2.4)$$

Proof. Let f be a φ -convex function. Then

$$\begin{aligned} \int_a^{a + e^{i\varphi}(b - a)} f(x) dx &= e^{i\varphi}(b - a) \int_0^1 f((1 - t)f(a) + tf(b)) dt \\ &\leq e^{i\varphi}(b - a) \frac{f(a) + f(b)}{2}, \end{aligned}$$

from which one can obtain the right hand of (2.4). Essentially using the technique and ideas of Pecaric et al [22, Theorem 5.11, pp. 143-144], one can obtain the left hand side of (2.4). This completes the proof. \square

For differentiable log- φ -convex functions, we have the following inequality, which is gain due to Noor [15].

Theorem 2.2 ([15]). *Let f be a log- φ -convex function on the interval*

$$[a, a + e^{i\varphi}(b - a)] \text{ and } 0 \leq \varphi \leq \frac{\pi}{2}.$$

Then

$$\frac{1}{e^{i\varphi}(b - a)} \int_a^{a + e^{i\varphi}(b - a)} f(x) \leq \frac{f(b) - f(a)}{\log f(b) - \log f(a)} = L(f(b), f(a)) \leq \frac{f(a) + f(b)}{2}, \quad (2.5)$$

where $L(\cdot, \cdot)$ is the logarithmic mean.

Proof. Let f be a log- φ -convex function. Then

$$\begin{aligned} \int_a^{a+e^{i\varphi}(b-a)} f(x) &= e^{i\varphi}(b-a) \int_0^1 f(a+te^{i\varphi}(b-a)) dt \\ &\leq e^{i\varphi}(b-a) \int_0^1 (f(a))^{1-t} (f(b))^t dt \\ &= e^{i\varphi}(b-a) f(a) \int_0^1 \left(\frac{f(b)}{f(a)}\right)^t dt \\ &= e^{i\varphi}(b-a) f(a) \left[\frac{\left[\frac{f(b)}{f(a)}\right]^t}{\log\left[\frac{f(b)}{f(a)}\right]} \right]_0^1 \\ &= e^{i\varphi}(b-a) f(a) \frac{1}{\log f(b) - \log f(a)} \left[\frac{f(b)}{f(a)} - 1 \right] \\ &= e^{i\varphi}(b-a) \frac{f(b) - f(a)}{\log f(b) - \log f(a)} = e^{i\varphi}(b-a) L(f(b), f(a)) \\ &\leq e^{i\varphi}(b-a) \frac{f(a) + f(b)}{2}, \end{aligned}$$

from which the required result follows. □

3. MAIN RESULTS

The main purpose of this paper is to establish new inequalities involving product of two log- φ -invex functions using essentially the technique of Pachpatte [21], which are extensions and improvement of Pachpatte’s results [21]. See also [14] for preinvex functions.

Theorem 3.1. *Let $f, g: K = [a, a + e^{i\varphi}(b - a)] \rightarrow (0, \infty)$ be log- φ -convex functions on the interval of real numbers K° (the interior of K) and $a, b \in K^\circ$ with $a < a + e^{i\varphi}(b - a)$ and $0 \leq \varphi \leq \frac{\pi}{2}$. Then*

$$\begin{aligned} \frac{1}{e^{i\varphi}(b-a)} \int_a^{a+e^{i\varphi}(b-a)} f(x)g(x)dx &\leq L(f(b)g(b), f(a)g(a)) \\ &\leq \frac{1}{4} [f(a) + f(b)] L(f(a), f(b)) \\ &\quad + \frac{1}{4} [g(a) + g(b)] L(g(a), g(b)). \end{aligned} \tag{3.1}$$

Proof. Let f, g be log- φ -convex functions. Then

$$\begin{aligned} f(a + te^{i\varphi}(b - a)) &\leq [f(a)]^{1-t} [f(b)]^t \\ g(a + te^{i\varphi}(b - a)) &\leq [g(a)]^{1-t} [g(b)]^t \end{aligned}$$

Consider

$$\begin{aligned}
\int_a^{a+e^{i\varphi}(b-a)} f(x)g(x)dx &= e^{i\varphi}(b-a) \int_0^1 f(a+te^{i\varphi}(b-a))g(a+e^{i\varphi}(b-a))dt \\
&\leq e^{i\varphi}(b-a) \int_0^1 (f(a)g(a))^{1-t} (f(b)g(b))^t dt \\
&= e^{i\varphi}(b-a) f(a)g(a) \int_0^1 \left[\frac{f(b)g(b)}{f(a)g(a)} \right]^t dt \\
&= e^{i\varphi}(b-a) \left\{ f(a)g(a) \left[\frac{\left[\frac{f(b)g(b)}{f(a)g(a)} \right]^t}{\log \frac{f(b)g(b)}{f(a)g(a)}} \right]_0^1 \right\} \\
&= e^{i\varphi}(b-a) \left[\frac{f(b)g(b) - f(a)g(a)}{\log f(b)g(b) - \log f(a)g(a)} \right] \\
&= e^{i\varphi}(b-a) L(f(b)g(b), f(a)g(a)) \\
&\leq \frac{e^{i\varphi}(b-a)}{2} \int_0^1 \left[\{f(a+te^{i\varphi}(b-a))\}^2 + \{g(a+te^{i\varphi}(b-a))\}^2 \right] dt \\
&\leq \frac{e^{i\varphi}(b-a)}{2} \int_0^1 \left[[f(a)]^{1-t} [f(b)]^t \right]^2 + [g(a)]^{1-t} [g(b)]^t \right]^2 dt \\
&= \frac{e^{i\varphi}(b-a)}{2} \left\{ [f(a)]^2 \int_0^1 \left[\frac{f(b)}{f(a)} \right]^{2t} dt + [g(a)]^2 \int_0^1 \left[\frac{g(b)}{g(a)} \right]^{2t} dt \right\} \\
&= \frac{e^{i\varphi}(b-a)}{4} \left\{ [f(b)]^2 \int_0^2 \left[\frac{f(b)}{f(a)} \right]^w dw + [g(a)]^2 \int_0^2 \left[\frac{g(b)}{g(a)} \right]^w dw \right\} \\
&= \frac{e^{i\varphi}(b-a)}{4} \left\{ [f(a)]^2 \left[\frac{\left[\frac{f(b)}{f(a)} \right]^w}{\log \frac{f(b)}{f(a)}} \right]_0^2 + [g(a)]^2 \left[\frac{\left[\frac{g(b)}{g(a)} \right]^w}{\log \frac{g(b)}{g(a)}} \right]_0^2 \right\} \\
&= \frac{e^{i\varphi}(b-a)}{4} \left\{ \frac{[f(a)+f(b)][f(b)-f(a)]}{\log f(b)-\log f(a)} + \frac{[g(a)+g(b)][g(b)-g(a)]}{\log g(b)-\log g(a)} \right\} \\
&= \frac{e^{i\varphi}(b-a)}{4} \{ [f(b)+f(a)] L(f(b), f(a)) + [g(b)+g(a)] L(g(b), g(a)) \},
\end{aligned}$$

which is the required (3.1). This completes the proof. \square

Using the technique of Theorem 3.1, we have:

$$\begin{aligned}
\frac{1}{e^{i\varphi}(b-a)} \int_a^{a+e^{i\varphi}(b-a)} f(x)dx &\leq L(f(b), f(a)) \\
&\leq \frac{1}{4} [(f(b) + f(a)) L(f(b) + f(a)) + 1] \\
&\leq \frac{1}{8} [f(a) + f(b)]^2 + \frac{1}{4}.
\end{aligned}$$

For the differentiable log- φ -invex functions, we have the following result.

Theorem 3.2. Let $f, g: [a, a + e^{i\varphi}(b - a)] \rightarrow (0, \infty)$ be differentiable log- φ -invex functions with $a < a + e^{i\varphi}(b - a)$ and $0 \leq \varphi \leq \frac{\pi}{2}$. Then

$$\begin{aligned} & \frac{2}{e^{i\varphi}(b - a)} \int_a^{a+e^{i\varphi}(b-a)} f(x)g(x)dx \geq \\ & \geq \frac{1}{e^{i\varphi}(b - a)} f\left(\frac{2a + e^{i\varphi}(b - a)}{2}\right) \times \\ & \times \int_a^{a+e^{i\varphi}(b-a)} g(x) \exp \left[\left\langle \frac{f'_\varphi\left(\frac{2a+e^{i\varphi}(b-a)}{2}\right)}{f\left(\frac{2a+e^{i\varphi}(b-a)}{2}\right)}, x - \frac{2a + e^{i\varphi}(b - a)}{2} \right\rangle \right] dx \\ & + \frac{1}{e^{i\varphi}(b - a)} g\left(\frac{2a + e^{i\varphi}(b - a)}{2}\right) \times \\ & \times \int_a^{a+e^{i\varphi}(b-a)} f(x) \exp \left[\left\langle \frac{g'_\varphi\left(\frac{2a+e^{i\varphi}(b-a)}{2}\right)}{g\left(\frac{2a+e^{i\varphi}(b-a)}{2}\right)}, x - \frac{2a + e^{i\varphi}(b - a)}{2} \right\rangle \right] dx. \end{aligned} \tag{3.2}$$

Proof. Let f, g be differentiable log- φ -invex functions. Then

$$\begin{aligned} \log f(x) - \log f(y) & \geq \left\langle \frac{f'_\varphi(y)}{f(y)}, x - y \right\rangle, \\ \log g(x) - \log g(y) & \geq \left\langle \frac{g'_\varphi(y)}{g(y)}, x - y \right\rangle, \quad \forall x, y \in K, \end{aligned}$$

which implies that

$$\log \frac{f(x)}{f(y)} \geq \left\langle \frac{f'_\varphi(y)}{f(y)}, x - y \right\rangle.$$

That is,

$$f(x) \geq f(y) \exp \left[\left\langle \frac{f'_\varphi(y)}{f(y)}, x - y \right\rangle \right] \tag{3.3}$$

$$g(x) \geq g(y) \exp \left[\left\langle \frac{g'_\varphi(y)}{g(y)}, x - y \right\rangle \right]. \tag{3.4}$$

Multiplying both sides of (3.3) and (3.4) by $g(x)$ and $f(x)$ respectively and adding the resultant, we have

$$2f(x)g(x) \geq g(x)f(x) \exp \left[\left\langle \frac{f'_\varphi(y)}{f(y)}, x - y \right\rangle \right] + f(x)g(x) \exp \left[\left\langle \frac{g'_\varphi(y)}{g(y)}, x - y \right\rangle \right]. \tag{3.5}$$

Taking $y = \frac{2a + e^{i\varphi}(b-a)}{2}$ in (3.5), we have

$$2g(x)f(x) \geq g(x)f\left(\frac{2a + e^{i\varphi}(b-a)}{2}\right) \exp\left[\left\langle \frac{f'_\varphi\left(\frac{2a + e^{i\varphi}(b-a)}{2}\right)}{f\left(\frac{2a + e^{i\varphi}(b-a)}{2}\right)}, x - \frac{2a + e^{i\varphi}(b-a)}{2} \right\rangle\right] \\ + f(x)g\left(\frac{2a + e^{i\varphi}(b-a)}{2}\right) \exp\left[\left\langle \frac{g'_\varphi\left(\frac{2a + e^{i\varphi}(b-a)}{2}\right)}{g\left(\frac{2a + e^{i\varphi}(b-a)}{2}\right)}, x - \frac{2a + e^{i\varphi}(b-a)}{2} \right\rangle\right], \\ x \in [a, a + e^{i\varphi}(b-a)].$$

Integrating the above inequality with respect to x on $[a, a + e^{i\varphi}(b-a)]$, and dividing both sides of the resultant inequality by $e^{i\varphi}(b-a)$, we can obtain the desired inequality (3.2). \square

We would like to mention that Theorem 3.2 can be formulated for y . We leave this for the reader.

Theorem 3.3. *Let $f, g: K = [a, a + e^{i\varphi}(b-a)] \rightarrow (0, \infty)$ be φ -convex functions on the interval of real numbers K° (the interior of K) and $a, b \in K^\circ$ with $a < a + e^{i\varphi}(b-a)$ and $0 \leq \varphi \leq \frac{\pi}{2}$. Then*

$$\frac{6}{e^{i\varphi}(b-a)} \int_a^{a+e^{i\varphi}(b-a)} f(x)g(x)dx \leq 2M(a,b)N(a,b) - [f(b)g(a) + f(a)g(b)] \\ \leq [M(a,b)]^2 + [N(a,b)]^2 - f(a)f(b) - g(a)g(b) \quad (3.6)$$

where

$$M(a,b) = f(a) + f(b) \quad (3.7)$$

$$N(a,b) = g(a) + g(b) \quad (3.8)$$

Proof. Let f, g be two φ -convex functions. Then

$$f(a + te^{i\varphi}(b-a)) \leq (1-t)f(a) + tf(b) \\ g(a + te^{i\varphi}(b-a)) \leq (1-t)g(a) + tg(b).$$

Consider

$$\begin{aligned}
 \int_a^{a+e^{i\varphi}(b-a)} f(x)g(x)dx &= e^{i\varphi}(b-a) \int_0^1 f(a+te^{i\varphi}(b-a))g(a+e^{i\varphi}(b-a))dt \\
 &\leq e^{i\varphi}(b-a) \int_0^1 [(1-t)f(a)+tf(b)][(1-t)g(a)+tg(b)]dt \\
 &= e^{i\varphi}(b-a) \int_0^1 [(1-t)^2f(a)g(a)+t^2f(b)g(b) \\
 &\quad +t(1-t)[f(a)g(b)+f(b)g(a)]]dt \\
 &= e^{i\varphi}(b-a) \left[\frac{1}{3}f(a)g(b) + \frac{1}{3}f(b)g(a) + \frac{1}{6}f(a)g(b) + \frac{1}{6}f(b)g(a) \right] \\
 &= e^{i\varphi}(b-a) \left\{ \frac{1}{3}[f(a)+f(b)][g(a)+g(b)] - \frac{1}{6}[f(a)g(b)+f(b)g(a)] \right\} \\
 &= \frac{e^{i\varphi}(b-a)}{6} [2M(a,b)N(a,b) - [f(a)g(b)+f(b)g(a)]] \\
 &\leq \frac{e^{i\varphi}(b-a)}{2} \int_0^1 \{ [(1-t)f(a)+tf(b)]^2 + [(1-t)g(a)+tg(b)]^2 \} \\
 &\leq \frac{e^{i\varphi}(b-a)}{6} \{ (f(a)+f(b))^2 + (g(a)+g(b))^2 \} \\
 &\quad - \frac{e^{i\varphi}(b-a)}{6} \{ f(a)f(b) + g(a)g(b) \} \\
 &= \frac{e^{i\varphi}(b-a)}{6} \{ [M(a,b)]^2 + [N(a,b)]^2 - f(a)f(b) - g(a)g(b) \},
 \end{aligned}$$

from which the required (3.6) follows. □

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