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ON THE EIGENVALUES OF THE LAPLACIAN

CRISTINEL MORTICI AND ALIN POHOAŢĂ

ABSTRACT. We give in this paper a method for numerical computation of an orthogonal basis in the Sobolev space $H_0^1(D)$, where D is a quarter of a circle, using the method of bi-orthogonal sequences due to Bergmann and Vekua. The elements of the basis are the solutions of some eigenvalue boundary problem, which are calculated using polynomial approximations for small values of the arguments and asymptotic expansions of Bessel functions for larger values.

1. INTRODUCTION

Mortici has given in [4] and [5] a theoretical way for computing a basis in some Sobolev spaces, using the bi-orthogonal sequences method. This method was first introduced by Vekua in [8] and then extended by Bergmann in [2,3]. A similar approach can be found in [7].

In the last decades, a large number of authors are concentrated to develop performant numerical methods related with the problem of finding a basis in Hilbert spaces, because in practical applications of the theoretical results, a lot of difficulties appear.

One way to attack such problems is to use the method of bi-orthogonal sequences due to Bergmann and Vekua.

The functional background consists of two real, separable Hilbert spaces $(H,(\cdot,\cdot))$, $(V,<\cdot,\cdot>)$ and assume that H is compact imbedded in V. Then there is an increasing, unbounded sequence $(\lambda_n)_{n\geq 1}$ of positive reals and a sequence $(e_n)_{n\geq 1}\subset H$ which is orthogonal with respect to both inner products, *i.e.*,

$$(e_m,e_n)=\lambda_n\delta_{mn},\qquad <e_m,e_n>=\delta_{mn} \tag{1.1}$$

for all positive integers m, n. Moreover, $(e_n)_{n\geq 1}$ is complete in H. For proofs and other details, see [2]. Remark that from (1.1), we can derive the equalities

$$(e_m,e_n)=\lambda_n<e_m,e_n>$$

and because of the completness of the system $(e_n)_{n\geq 1}$, it follows that

$$(e_n,v)=\lambda_n<e_n,v>, \tag{1.2}$$

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for every $n \geq 1$ and $v \in H$. In consequence, the elements of the orthogonal basis $(e_n)_{n\geq 1}$ can be considered as the solutions of the eigenvalue problem (1.2).

If take $H = H_0^1(D)$ and $L = L^2(D)$ endowed with the usual norms, then we can rewrite (1.2) in the form

$$\int_D e_n v + \int_D \nabla e_n \nabla v = \lambda_n \int_D e_n v. \tag{1.3}$$

Because $v = 0$ on the boundary ∂D, we deduce that

$$\int_D (\Delta e_n + (\lambda_n - 1)e_n)v = 0,$$

for every $v \in H_0^1(D)$, so the elements of the basis $(e_n)_{n\geq 1}$ are the eigenfunctions of the problem

$$\begin{cases} -\Delta u(x) = (\lambda - 1)u(x) & \text{in } D \\ \quad u(x) = 0 & \text{on } \partial D. \end{cases} \tag{1.4}$$

2. THE NUMERICAL CONSTRUCTION

The one dimensional case $D = (a, b) \subset \mathbb{R}$ was threated in [5] using this method.

If the domain $D \subset \mathbb{R}^2$ is a corner of a circle of angle $\frac{\pi}{k}$, where $k \geq \frac{1}{2}$, then an advantageous formula for finding the eigenfunctions for the laplacian is the following formula due to Bergmann:

$$u(r, t) = \sum_{j=1}^{N} c_j J_{jk}(\sqrt{\lambda} r) \sin jkt. \tag{2.1}$$

Here, (r, t) are the polar coordinates, J_β is the Bessel function of order β, while the constants c_j and λ will be determined from the condition $u = 0$ on ∂D.

In fact, this condition is already satisfied on the part of the boundary from the coordinates axis. The difficulty appears now in imposing this condition on the remaining arc of the boundary.

Anyhow, this constraint on the boundary is just the main relation from which we will be able to find the coefficients c_j and the eigenvalues λ in (2.1).

In numerical computations, we impose the boundary conditions only at a finite number of points from the boundary and consequently, the obtained results are more exactly as we consider a greater number of points. On the other side, the computations become harder if the number of the considered points are too big. After a careful analysis, we have arrived at the conclusion that if around 9 points of the boundary are taken, then a good compromise is obtained.

In [6] it is given the numerical construction of the basis in case $k = 4$, *i.e.*, for the quaver of the circle, where the boundary condition at two points $\left(1, \frac{\pi}{12}\right)$ and $\left(1, \frac{\pi}{6}\right)$ was imposed.

More recently, it is studied in [4] the case $k = 2$, *i.e.*, for a quarter of the circle, where for the boundary condition, the points $\left(1, \frac{\pi}{6}\right)$ and $\left(1, \frac{\pi}{3}\right)$ was taken.

Now we consider here the domain D as the quarter of the circle ($k = 2$) with radius $r = 1$. As we have already explained, we will impose the boundary conditions at 9 distinct points $P_i(1, t_i)$, with $t_i \in \left(0, \frac{\pi}{2}\right)$, for $1 \leq i \leq 9$ (in polar coordinates).

In this case, $N = 9$ and the solution is given by the formula

$$u(r,t) = \sum_{j=1}^{9} c_j J_{2j}(\sqrt{\lambda}) \sin 2jt. \tag{2.2}$$

The boundary condition $u = 0$ on ∂D, written in extend form, becomes a linear, omogen system:

$$\begin{cases} c_1 J_2(\sqrt{\lambda}) \sin(2t_1) + c_2 J_4(\sqrt{\lambda}) \sin(4t_1) + \cdots + c_9 J_{18}(\sqrt{\lambda}) \sin(18t_1) = 0 \\ c_1 J_2(\sqrt{\lambda}) \sin(2t_2) + c_2 J_4(\sqrt{\lambda}) \sin(4t_2) + \cdots + c_9 J_{18}(\sqrt{\lambda}) \sin(18t_2) = 0 \\ \vdots \\ c_1 J_2(\sqrt{\lambda}) \sin(2t_9) + c_2 J_4(\sqrt{\lambda}) \sin(4t_9) + \cdots + c_9 J_{18}(\sqrt{\lambda}) \sin(18t_9) = 0. \end{cases}$$

We are interested in finding nontrivial solutions $(c_1, c_2, \ldots, c_9)$ of this system, so the attached determinant must vanishes,

$$\begin{vmatrix} J_2(\sqrt{\lambda}) \sin(4t_1) & J_4(\sqrt{\lambda}) \sin(4t_1) & \ldots & J_{18}(\sqrt{\lambda}) \sin(18t_1) \\ J_2(\sqrt{\lambda}) \sin(4t_2) & J_4(\sqrt{\lambda}) \sin(4t_2) & \ldots & J_{18}(\sqrt{\lambda}) \sin(18t_2) \\ \vdots & \vdots & & \vdots \\ J_2(\sqrt{\lambda}) \sin(4t_9) & J_4(\sqrt{\lambda}) \sin(4t_9) & \ldots & J_{18}(\sqrt{\lambda}) \sin(18t_9) \end{vmatrix} = 0. \tag{2.3}$$

Hence we must have

$$\begin{vmatrix} \sin(4t_1) & \sin(4t_1) & \ldots & \sin(18t_1) \\ \sin(4t_2) & \sin(4t_2) & \ldots & \sin(18t_2) \\ \vdots & \vdots & & \vdots \\ \sin(4t_9) & \sin(4t_9) & \ldots & \sin(18t_9) \end{vmatrix} \cdot \prod_{i=1}^{9} J_{2i}(\sqrt{\lambda}) = 0$$

and if

$$\prod_{i=1}^{9} J_{2i}(\sqrt{\lambda}) = 0, \tag{2.4}$$

then the system (2.3) admits indeed non-trivial solutions. We also must be careful to choose $t_1, t_2, \ldots, t_9$ so that the previous determinant is non-zero.

We will compute the solutions using numerical approximations.

For the estimation of the solutions of (2.4), we use the ideas from [6,10] of asymptotic expansions of the Bessel functions. For $\nu \in \mathbb{R}$ and $p \in \mathbb{N}$ with $\nu - p \leq \frac{1}{2}$, we approximate $J_\nu(x) \simeq J_{\nu,p}(x)$, where

$$J_{\nu,p}(x) = \frac{1}{\sqrt{2\pi x}} \sum_{i=1}^{2} e^{\sigma_k i \varphi_\nu(x)} P_k(x, \nu, p),$$

with $\varphi_\nu(x) = x - \dfrac{\nu\pi}{2} - \dfrac{\pi}{4}$ and $P_k(x,\nu,p) = \displaystyle\sum_{m=0}^{m-1} \frac{a_{\nu,m}}{(2\sigma_k i x)^m}$.

With $\sigma_1 = 1$, $\sigma_2 = -1$ and

$$a_{\nu,m} = \frac{\left(\frac{1}{2} - \nu\right)_m \left(\frac{1}{2} + \nu\right)_m}{m!},$$

with $(x)_m = x(x+1)(x+2)\cdots(x+m+1)$, by denoting

$$\varepsilon_{\nu,p}(x) = J_\nu(x) - J_{\nu,p}(x)$$

we have, for $x > 0$,

$$|\varepsilon_{\nu,p}(x)| \leq \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} \frac{|\, a_{\nu,p} \,|}{(2x)^p}.$$

According to [9], for

$$\varepsilon(x) = J_\mu(x) J_\nu(x) - J_{\mu,q}(x) J_{\nu,p}(x),$$

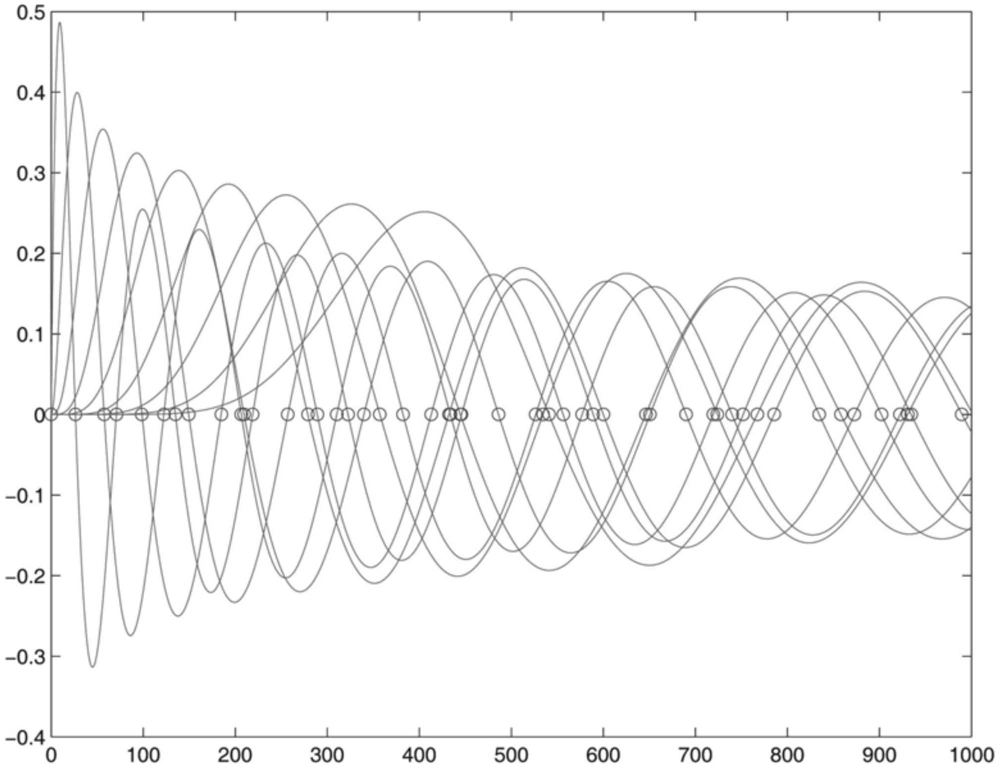
$\nu - p \leq \dfrac{1}{2}$ and $\mu - q \leq \dfrac{1}{2}$, we have the following estimation:

$$|\varepsilon(x)| = O(x^{-\min(p,q)}) \quad \text{as } x \longrightarrow \infty$$

It is also stated in [9] that for $p \geq 6$ and $x \geq 5\pi$ we have $|\, \varepsilon(x) \,| \leq 10^{-5}$.

Below are given the roots between 0 and 1000 calculated with a precision of 10^{-4}, then the corresponding graph is drawn.

0 26.3746 57.5829 70.8500 98.7263 122.4278 135.0207 149.4529 184.6688 206.5698
209.5401 218.9202 257.2102 278.8316 289.1299 310.3223 322.5551 339.7926
357.2099 382.3799 412.9345 432.2202 433.7611 444.5834 445.9276 486.0696
526.4810 534.3392 540.8776 556.3034 576.9133 589.0384 599.9994 646.0249
650.7321 689.9456 719.3206 724.0065 739.7904 751.8879 767.1736 785.4965
834.4804 857.9551 872.9460 902.0245 922.3981 930.6130 934.4763 989.7291



Finally we mention that we used the bisection method applied to the approximated functions for the numerical computation of the previous roots.

In order to obtain better approximations, remark that on the interval $[0, 5\pi]$, the method of partial sums of the Taylor expansions is favourable, while for higher values, we are forced to use the more complicated $J_{\nu,n}$ approximation functions.

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Valahia University of Târgovişte
Department of Mathematics
Bd. Unirii 18, 130082*, Târgovişte, Romania*
E-mail address: `cmortici@valahia.ro`

Valahia University of Târgovişte
Department of Mathematics
Bd. Unirii 18, 130082*, Târgovişte, Romania*
E-mail address: `alin@valahia.ro`