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OPTIMALITY AND DUALITY FOR INVEX MULTI-TIME CONTROL PROBLEMS WITH MIXED CONSTRAINTS

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Abstract. The aim of this paper is to state and prove necessary optimality conditions and develop a duality for the multi-time control problem with mixed constraints (PCP). The duality uses the invexity notion, [1]÷[7].

1. INTRODUCTION

We consider a *multi-time control problem* based on a multiple integral cost functional and on some m-flow type PDE constraints given by

$$\text{(PCM)}\qquad \begin{cases} \underset{x,u}{\text{Maximize}}\ I(x,u)=\displaystyle\int_{\Omega} X(t,x(t),u(t))dv \\ \text{subject to} \\ \dfrac{\partial x^i}{\partial t^{\alpha}}=X^i_{\alpha}(t,x(t),u(t)),\ i=\overline{1,n},\ \alpha=\overline{1,m}, \\ u(t)\in U(t),\ \forall t\in\Omega;\ x(0)=x_0,\ x(t_0)=x_1. \end{cases}$$

In the problem given above $t=(t^{\alpha})\in\mathbb{R}^m_+$; $dv=dt^1\cdots dt^m$ is the volume element in $\mathbb{R}^m_+$; Ω is the hyper-parallelepiped in $\mathbb{R}^m_+$ defined by the closed interval $[0,t_0]=\{t\in\mathbb{R}^m_+|0\le t\le t_0\}$, where $0=(0,\dots,0)$ and $t_0=(t_0^1,\dots,t_0^m)$ are two points in $\mathbb{R}^m_+$; $x(t)=(x^i(t))$, $i=\overline{1,n}$, is a C^2-class state vector; $u(t)=(u^{\alpha}(t))$, $\alpha=\overline{1,k}$, is a continuous control vector; the *running cost* $X(t,x(t),u(t))$ is a C^1-class function; $X^i_{\alpha}(t,x(t),u(t))$, $\alpha=\overline{1,m}$, are C^1-class functions satisfying the complete integrability conditions (m-flow type problem).

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Recently, Constantin Udrişte (see [8] and [9]) stated a *multi-time maximum principle* for (PCM), that is he obtained for (PCM) necessary optimality conditions. Also, Udrişte established sufficient optimality conditions for this problem and pointed out a wide class of properties.

In the context of the problem (PCM), we consider the functions $Y_\beta(t, x(t), u(t))$, $\beta = \overline{1,m}$, of C^1-class, and the next scalar multi-time control problem of minimum with mixed constraints:

$$
\text{(PCP)} \qquad \begin{cases} \text{Minimize } I(x,u) = \displaystyle\int_\Omega X(t, x(t), u(t))dv \\ \text{subject to} \\ \dfrac{\partial x^i}{\partial t^\alpha} = X^i_\alpha(t, x(t), u(t)),\ i = \overline{1,n},\ \alpha = \overline{1,m}, \\ Y_\beta(t, x(t), u(t)) \leq 0,\ \beta = \overline{1,q}, \\ u(t) \in U(t),\ \forall t \in \Omega;\ x(0) = x_0,\ x(t_0) = x_1. \end{cases}
$$

In this paper, we establish necessary optimality conditions for the problem (PCP) and we develop a duality of Wolfe type through weak, direct and converse duality theorems. These theorems use hypotheses of invexity theory. It is defined the notion of normal optimal solution for (PCP) and there are stated direct and converse duality theorems for normal optimal solutions.

2. NECESSARY OPTIMALITY CONDITIONS FOR (PCP)

The following result is true:

Theorem 2.1 (Necessary conditions). *Consider the multi-time control problem* (PCP) *in the framework presented by* §1 *and let the point* (x, u) *be an optimal solution of the problem* (PCP). *Then there are* $\theta(t) \in \mathbb{R}^p$, $\mu(t) \in \mathbb{R}^q$ *and* $\lambda^\alpha_i \in \mathbb{R}^n \times \mathbb{R}^m$, *all being piecewise smooth functions, which satisfy the following conditions:*

$$
\text{(MFJ)} \qquad \begin{cases} \theta(t)\dfrac{\partial X}{\partial x} + \lambda^\alpha_i(t)\dfrac{\partial X^i_\alpha}{\partial x} + \mu^\beta(t)\dfrac{\partial Y_\beta}{\partial x} + \dfrac{\partial \lambda^\alpha_t}{\partial t} = 0 \\ \theta(t)\dfrac{\partial X}{\partial u} + \lambda^\alpha_i(t)\dfrac{\partial X^i_\alpha}{\partial u} + \mu^\beta(t)\dfrac{\partial Y_\beta}{\partial u} = 0 \\ \mu^\beta(t) Y_\beta(t, x(t), u(t)) = 0 \\ \theta(t) \geq 0,\ \lambda^\alpha_i(t) \in \mathbb{R},\ \mu^\beta(t) \geq 0, \end{cases}
$$

where $\dfrac{\partial X}{\partial x} = \dfrac{\partial X}{\partial x}(t, x(t), u(t))$, $\dfrac{\partial X}{\partial u} = \dfrac{\partial X}{\partial u}(t, x(t), u(t))$, $\dfrac{\partial X^i_\alpha}{\partial x} = \dfrac{\partial X^i_\alpha}{\partial x}(t, x(t), u(t))$ *etc.*

Proof. Let $(\bar{x}, \bar{u})$ be a feasible solution of (PCP) and the vector functions $p(t) \in \mathbb{R}^n$ and $q(t) \in \mathbb{R}^k$, where $p, q \in C^1(\Omega)$, $p\big|_{\partial\Omega} = 0$. Let $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ be real numbers and consider for (x, u) the neighborhood

$$V_\varepsilon = \{(\bar{x}, \bar{u}) \,|\, \bar{x}(t) = x(t) + \varepsilon_1 p(t),\ \bar{u}(t) = u(t) + \varepsilon_2 q(t)\}.$$

If (x, u) is a minimum solution of (PCP), then $(0, 0)$ is a minimum solution to the following minimum problem:

$$\text{(PM1)} \begin{cases} \text{Minimize } f(\varepsilon_1, \varepsilon_2) = \displaystyle\int_\Omega X(t, x(t) + \varepsilon_1 p(t), u(t) + \varepsilon_2 q(t))\, dv \\ \text{subject to} \\ g^i_\alpha(\varepsilon_1, \varepsilon_2) = \displaystyle\int_\Omega \left[X^i_\alpha(t, x(t) + \varepsilon_1 p(t), u(t) + \varepsilon_2 q(t)) - \frac{\partial x^i}{\partial t^\alpha} - \varepsilon_1 \frac{\partial p}{\partial t^\alpha} \right] dv = 0 \\ h_\beta(\varepsilon_1, \varepsilon_2) = \displaystyle\int_\Omega Y_\beta(t, x(t) + \varepsilon_1 p(t), u(t) + \varepsilon_2 q(t)) dv \le 0 \\ \qquad t \in \Omega,\ u(t) \in U(t),\ p\big|_{\partial\Omega} = 0,\ q\big|_{\partial\Omega} = 0. \end{cases}$$

Since $(0, 0)$ is a minimum solution of (PM1), then there exist the real functions $\theta(x, u)$, $\lambda^\alpha_i(x, u)$ and $\mu^\beta(x, u)$ such that (PM1) satisfies at $(0, 0)$ the following Fritz John conditions:

$$\text{(FJ)} \begin{cases} \theta(x, u)\nabla f(0, 0) + \lambda^\alpha_i(x, u)\nabla g^i_\alpha(0, 0) + \mu^\beta(x, u)\nabla h_\beta(0, 0) = 0 \\ \mu^\beta(x, u) h_\beta(0, 0) = 0 \\ \theta(x, u) \ge 0,\ \mu^\beta(x, u) \ge 0. \end{cases}$$

But

$$\nabla f(\varepsilon_1, \varepsilon_2) = \begin{pmatrix} \dfrac{\partial f}{\partial \varepsilon_1} \\ \dfrac{\partial f}{\partial \varepsilon_2} \end{pmatrix} = \begin{pmatrix} \displaystyle\int_\Omega \frac{\partial X}{\partial x} p\, dv \\ \displaystyle\int_\Omega \frac{\partial X}{\partial u} q\, dv \end{pmatrix},$$

$$\nabla g^i_\alpha(0, 0) = \begin{pmatrix} \dfrac{\partial g^i_\alpha}{\partial \varepsilon_1} \\ \dfrac{\partial g^i_\alpha}{\partial \varepsilon_2} \end{pmatrix} = \begin{pmatrix} \displaystyle\int_\Omega \left(\frac{\partial X^i_\alpha}{\partial x} p - \frac{\partial p}{\partial t^\alpha} \right) dv \\ \displaystyle\int_\Omega \frac{\partial X^i_\alpha}{\partial u} q dv \end{pmatrix},$$

$$\nabla h_\beta(\varepsilon_1, \varepsilon_2) = \begin{pmatrix} \dfrac{\partial h_\beta}{\partial \varepsilon_1} \\ \dfrac{\partial h_\beta}{\partial \varepsilon_2} \end{pmatrix} = \begin{pmatrix} \displaystyle\int_\Omega \frac{\partial Y_\beta}{\partial x} p dv \\ \displaystyle\int_\Omega \frac{\partial Y_\beta}{\partial u} q dv \end{pmatrix},$$

where under all integrals, $p = p(t)$, $q = q(t)$, $\dfrac{\partial X}{\partial x} = \dfrac{\partial X}{\partial x}(t, x(t), u(t))$ etc.

Then the first conditions (FJ) becomes

$$\theta(x,u)\begin{pmatrix}\displaystyle\int_\Omega \frac{\partial X}{\partial x}pdv\\ \displaystyle\int_\Omega \frac{\partial X}{\partial u}qdv\end{pmatrix}+\lambda_i^\alpha(x,u)\begin{pmatrix}\displaystyle\int_\Omega\left(\frac{\partial X_\alpha^i}{\partial x}p-\frac{\partial p}{\partial t^\alpha}\right)dv\\ \displaystyle\int_\Omega \frac{\partial X^i}{\partial u}qdv\end{pmatrix}$$

$$+\mu^\beta(x,u)\begin{pmatrix}\displaystyle\int_\Omega \frac{\partial Y_\beta}{\partial x}pdv\\ \displaystyle\int_\Omega \frac{\partial Y_\beta}{\partial u}qdv\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix},$$

or componentwise,

$$\begin{cases}\theta(x,u)\displaystyle\int_\Omega \frac{\partial X}{\partial x}pdv+\lambda_i^\alpha(x,u)\int_\Omega\left(\frac{\partial X_\alpha^i}{\partial x}p-\frac{\partial p}{\partial t^\alpha}\right)dv+\mu^\beta(x,u)\int_\Omega\frac{\partial Y_\beta}{\partial x}pdv=0\\ \theta(x,u)\displaystyle\int_\Omega \frac{\partial X}{\partial u}qdv+\lambda_i^\alpha(x,u)\int_\Omega\frac{\partial X_\alpha^i}{\partial u}qdv+\mu^\beta(x,u)\int_\Omega\frac{\partial Y_\beta}{\partial u}qdv=0.\end{cases}\tag{2.1}$$

Relations (2.1) can be written in the form

$$\begin{cases}\displaystyle\int_\Omega\theta(x,u)\frac{\partial X}{\partial x}pdv+\int_\Omega\lambda_i^\alpha(x,u)\left(\frac{\partial X_\alpha^i}{\partial x}p-\frac{\partial p}{\partial t^\alpha}\right)dv+\int_\Omega\mu^\beta(x,u)\frac{\partial Y_\beta}{\partial x}pdv=0\\ \displaystyle\int_\Omega\theta(x,u)\frac{\partial X}{\partial u}qdv+\int_\Omega\lambda_i^\alpha(x,u)\frac{\partial X_\alpha^i}{\partial u}qdv+\mu^\beta(x,u)\int_\Omega\mu^\beta(x,u)\frac{\partial Y_\beta}{\partial u}qdv=0.\end{cases}\tag{2.2}$$

For the convergence of the integral in (2.2) it is sufficient that the functions θ, λ_i^α and μ^β should be measurable. Particularly, they can be piecewise smooth functions.

We define now the functions $\theta(t):=\theta(x(t),u(t))$, $\lambda_i^\alpha(t):=\lambda_i^\alpha(x(t),u(t))$ and $\mu^\beta(t):=\mu^\beta(x(t),u(t))$ and the relations (2.2) become

$$\begin{cases}\displaystyle\int_\Omega\theta(t)\frac{\partial X}{\partial x}pdv+\int_\Omega\lambda_i^\alpha(t)\left(\frac{\partial X_\alpha^i}{\partial x}p-\frac{\partial p}{\partial t^\alpha}\right)dv+\int_\Omega\mu^\beta(t)\frac{\partial Y_\beta}{\partial x}pdv=0\\ \displaystyle\int_\Omega\theta(t)\frac{\partial X}{\partial u}qdv+\int_\Omega\lambda_i^\alpha(t)\frac{\partial X_\alpha^i}{\partial u}qdv+\mu^\beta(t)\int_\Omega\mu^\beta(t)\frac{\partial Y_\beta}{\partial u}qdv=0.\end{cases}\tag{2.3}$$

The first condition (2.3) becomes

$$\int_\Omega\left[\theta(t)\frac{\partial X}{\partial x}+\lambda_i^\alpha(t)\frac{\partial X_i}{\partial x}+\mu^\beta(t)\frac{\partial Y_\beta}{\partial x}\right]pdv=\int_\Omega\lambda_i^\alpha(t)\frac{\partial p}{\partial t^\alpha}dv.\tag{2.4}$$

In addition, we suppose λ_i^α of C^1-class. We have

$$\frac{\partial}{\partial t^\alpha}(\lambda_i^\alpha(t)p)=\frac{\partial\lambda_i^\alpha}{\partial t^\alpha}p+\lambda_i^\alpha\frac{\partial p}{\partial t^\alpha}$$

and then,

$$\int_\Omega \lambda_i^\alpha(t)\frac{\partial p}{\partial t^\alpha}dv = \int_\Omega \frac{\partial}{\partial t^\alpha}(\lambda_i^\alpha p)dv - \int_\Omega \frac{\partial \lambda_i^\alpha}{\partial t^\alpha}pdv.$$

According to Gauss-Ostrogradsky formula, we have

$$\int_\Omega \frac{\partial}{\partial t^\alpha}(\lambda_i^\alpha p)pdv = \int_{\partial\Omega}(\lambda_i^\alpha p)\vec{n}(t)dv = 0,$$

where $\vec{n}(t)$ is the normal unit vector to the boundary $\partial\Omega$ and $p\big|_{\partial\Omega} = 0$.

So, condition (2.4) becomes

$$\int_\Omega \left[\theta(t)\frac{\partial X}{\partial x} + \lambda_i^\alpha(t)\frac{\partial X_\alpha^i}{\partial x} + \mu^\beta(t)\frac{\partial Y_\beta}{\partial x} + \frac{\partial \lambda_t^\alpha}{\partial t}\right]pdv = 0. \tag{2.5}$$

According to a fundamental Lemma of the variational calculus, from (2.5) it results

$$\theta(t)\frac{\partial X}{\partial x} + \lambda_i^\alpha(t)\frac{\partial X_\alpha^i}{\partial x} + \mu^\beta(t)\frac{\partial Y_\beta}{\partial x} + \frac{\partial \lambda_t^\alpha}{\partial t} = 0$$

that is the first condition of (MFJ).

The second condition (2.3) can be written

$$\int_\Omega \left[\theta(t)\frac{\partial X}{\partial u} + \lambda_i^\alpha(t)\frac{\partial X_\alpha^i}{\partial u} + \mu^\beta(t)\frac{\partial Y_\beta}{\partial u}\right]qdv = 0.$$

Using the same fundamental Lemma, from this relation results the second condition of (MFJ).

Condition $\mu^\beta(x,u)h_\beta(0,0) = 0$ of (FJ) implies

$$\int_\Omega \mu^\beta(t)Y_\beta(t,x(t),u(t))dv = 0. \tag{2.6}$$

But condition $\mu^\beta(t)Y_\beta(t,x(t),u(t))dv = 0$ implies (2.6). Therefore, we get the third condition in (MFJ). Also we have $\theta(t) \geq 0$, $\mu^\beta(t) \geq 0$ for $t \in \Omega$. $\square$

Definition 2.1. The optimal solution (x^0, u^0) of (PCP) is called *normal* if $\theta(t) \neq 0$, for all $t \in \Omega$.

3. WOLFE'S DUALITY FOR (PCP)

The function

$$L(t,x(t),u(t),\lambda(t),\mu(t)) = X(t,x(t),u(t)) + \lambda_i^\alpha(t)\left[X_\alpha^i(t,x(t),u(t)) - \frac{\partial x^i}{\partial t^\alpha}\right] + \mu^\beta(t)Y_\beta(t,x(t),u(t))$$

is the *Lagrangian function* associated to (PCP).

We associate the next dual to the problem (PCP), [10]:

$$\text{(DCP)}\qquad \begin{cases} \text{Maximize } J(x,u,\lambda,\mu)=\displaystyle\int_\Omega L(t,x(t),u(t),\lambda(t),\mu(t))dv \\ \text{subject to} \\ \dfrac{\partial X}{\partial u}+\lambda_i^\alpha(t)\dfrac{\partial X_\alpha^i}{\partial u}+\mu_\beta(t)\dfrac{\partial Y_\beta}{\partial x}+\dfrac{\partial \lambda_i^\alpha}{\partial t^\alpha}=0, \\ \dfrac{\partial X}{\partial u}+\lambda_i^\alpha(t)\dfrac{\partial X_\alpha^i}{\partial u}+\mu_\beta(t)\dfrac{\partial Y_\beta}{\partial u}=0, \\ \lambda_i^\alpha(t)\in\mathbb{R},\ \mu^\beta(t)\geq 0,\quad x(0)=x_0,\ x(t_0)=x_1. \end{cases}$$

In the following, we develop (PCP) and (DCP), a duality of Wolfe type with weak, direct and converse duality theorems.

This duality uses the notion of invexity introduced right now.

Let the scalar function $f(t,x(t),x'(t),u(t))\in\mathbb{R}$ be of C^1- class, where $x'(t)$ is the derivative of $x(t)$ and $F(x,u)=\displaystyle\int_\Omega f(t,x(t),x'(t),u(t))dt$.

Definition 3.1. $F(x,u)$ is said to be *invex* at (x^*,u^*) if there exist the vector functions $\eta(t)\in\mathbb{R}^n$ of C^1 class, where $\eta|_{\partial\Omega}=0$ and $\xi(t)\in\mathbb{R}^k$ of C^0-class such that for every $\forall(x,u)[(x,u)\neq(x^*,u^*)]$,

$$F(x,u)-F(x^*,u^*)\geq\int_\Omega\left(\eta^t\frac{\partial f}{\partial x}(t,x^*,u^*)+D(\eta^t)\frac{\partial f}{\partial x'}(t,x^*,u^*)+\xi^t\frac{\partial f}{\partial u}(t,x^*,u^*)\right)dv.$$

We denote by $\mathcal{D}$ the domain of (PCP), by Δ the domain of (DCP) and by $(x,u,\lambda,\mu)=(x,u,\lambda_i^\alpha,\mu^\beta)$ the current point of Δ.

Now the above-mentioned duality theorems follow.

Theorem 3.1 (Weak duality). *Let* $(x^*,u^*)\in\mathcal{D}$ *and* $(x,u,\lambda,\mu)\in\Delta$ *be two feasible solutions to* (PCP) *and* (DCP). *Consider the functions* $\lambda_i^\alpha(t)$ *and* $\mu^\beta(t)$ *as in Theorem* 2.1. *Moreover, suppose the next conditions are fulfilled:*

a) $\displaystyle\int_\Omega X(t,x(t),u(t))dv$ *is invex at* (x,u)*;*

b) $\displaystyle\int_\Omega \lambda_i^\alpha(t)\left[X_\alpha^i(t,x(t),u(t))-\frac{\partial x^i}{\partial t^\alpha}\right]dv$ *is invex at* (x,u)*;*

c) $\displaystyle\int_\Omega \mu^\beta(t)Y_\beta(t,x(t),u(t))dv$ *is invex at* (x,u)*, with respect to* η *and* ξ*, as in Definition* 3.1.

Then $I(x^*,u^*)\geq J(x,u,\lambda,\mu)$.

Proof. According to a) and Definition 3.1, we have

$$\int_\Omega [X(t,x^*,u^*) - X(t,x,u)]dv \geq \int_\Omega \left[\eta^t \frac{\partial X}{\partial x} + \xi^t \frac{\partial X}{\partial u}\right] dv. \tag{3.1}$$

Using the constraints of (DCP), (3.1) becomes

$$\begin{aligned}\int_\Omega [X(t,x^*,u^*) - X(t,x,u)]dv \geq & \int_\Omega \left\{\eta^t \left[-\lambda_i^\alpha(t)\frac{\partial X_\alpha^i}{\partial x} - \mu^\beta(t)\frac{\partial Y_\beta}{\partial x} - \frac{\partial \lambda_i^\alpha}{\partial t^\alpha}\right]\right. \\ & \left. +\xi^t \left[-\lambda_i^\alpha(t)\frac{\partial X_\alpha^i}{\partial u} - \mu^\beta(t)\frac{\partial Y_\beta}{\partial u}\right]\right\} dv \qquad (3.2) \\ = & -\int_\Omega \eta^t \frac{\partial \lambda_i^\alpha}{\partial t^\alpha} dv - \int_\Omega \left[\eta^t \lambda_i^\alpha(t)\frac{\partial X_\alpha^i}{\partial x} + \xi^t \lambda_i^\alpha(t) \frac{\partial X_\alpha^i}{\partial u}\right] dv \\ & - \int_\Omega \left[\eta^t \mu^\beta(t) \frac{\partial Y_\beta}{\partial x} + \xi^t \mu^\beta(t)\frac{\partial Y_\beta}{\partial u}\right] dv.\end{aligned}$$

But

$$\frac{\partial}{\partial t^\alpha}(\eta^t \lambda_i^\alpha) = \frac{\partial \eta^t}{\partial t^\alpha}\lambda_i^\alpha + \eta^t \frac{\partial \lambda_i^\alpha}{\partial t^\alpha}$$

and then,

$$\int_\Omega \eta^t \frac{\partial \lambda_i^\alpha}{\partial t^\alpha} dv = \int_\Omega \frac{\partial}{\partial t^\alpha}(\eta^t \lambda_i^\alpha) dv - \int_\Omega \frac{\partial \eta^t}{\partial t^\alpha}\lambda_i^\alpha dv.$$

Using the flow-divergence formula, we have

$$\int_\Omega \frac{\partial}{\partial t^\alpha}(\eta^t \lambda_i^\alpha) dv = \int_{\partial\Omega} (\eta^t \lambda_i^\alpha)\vec{n}(t) d\sigma = 0,$$

where $\vec{n}(t)$ is the normal unit vector to the boundary $\partial\Omega$ and $\eta|_{\partial\Omega} = 0$.

Then (3.2) becomes

$$\begin{aligned}\int_\Omega [X(t,x^*,u^*) - X(t,x,u)]dv \geq & \int_\Omega \lambda_i^\alpha(t)\left[\eta^t \frac{\partial X_\alpha^i}{\partial x} + \xi^t \frac{\partial X_\alpha^i}{\partial u} - \frac{\partial \lambda_i^\alpha}{\partial t^\alpha}\right] dv \\ & - \int_\Omega \mu^\beta(t)\left[\eta^t \frac{\partial Y_\beta}{\partial x} + \xi^t \frac{\partial Y_\beta}{\partial u}\right] dv. \qquad (3.3)\end{aligned}$$

Taking into account hypotheses b) and c), respectively, we have

$$\begin{aligned}\int_\Omega \lambda_i^\alpha(t) & \left[X_\alpha^i(t,x^*,u^*) - \frac{\partial x^{*i}}{\partial t^\alpha} - X_\alpha^i(t,x,u) + \frac{\partial x^i}{\partial t^\alpha}\right] dv \\ & \geq \int_\Omega \lambda_i^\alpha(t)\left[\eta^t \frac{\partial X_\alpha^i}{\partial x} + \xi^t \frac{\partial X_\alpha^i}{\partial u} - \frac{\partial \lambda_i^\alpha}{\partial t^\alpha}\right] dv, \qquad (3.4)\end{aligned}$$

$$\int_\Omega \mu^\beta(t)[Y_\beta(t,x^*,u^*) - Y_\beta(t,x,u)]dv \geq \int_\Omega \mu^\beta(t)\left[\eta^t \frac{\partial Y_\beta}{\partial x} + \xi^t \frac{\partial Y_\beta}{\partial t^\alpha}\right] dv. \tag{3.5}$$

Having $X^i_\alpha(t,x^*,u^*)-\dfrac{\partial x^{*i}}{\partial t^\alpha}=0$ and $-\mu^\beta(t)Y_\beta(t,x^*,u^*)\geq 0$ and summing, side by side (3.4) and (3.5) multiplied by -1, we obtain

$$-\int_\Omega \lambda^\alpha_i(t)\left[\frac{\partial X^i_\alpha}{\partial x}+\xi^t\frac{\partial X^i_\alpha}{\partial u}-\frac{\partial \eta^t}{\partial t^\alpha}\right]dv-\int_\Omega \mu^\beta(t)\left[\eta^t\frac{\partial Y_\beta}{\partial x}+\xi^t\frac{\partial Y_\beta}{\partial t^\alpha}\right]dv$$

$$\geq -\int_\Omega \lambda^\alpha_i(t)\left[-X^i_\alpha(t,x,u)+\frac{\partial x^i}{\partial t^\alpha}\right]dv+\int_\Omega \mu^\beta(t)Y_\beta(t,x,u)dv. \tag{3.6}$$

Then, from (3.3) and (3.6), successively, it results

$$\int_\Omega [X(t,x^0,u^0)-X(t,x,u)]dv\geq -\int_\Omega \lambda^\alpha_i(t)\left[-X^i_\alpha(t,x,u)+\frac{\partial x^i}{\partial t^\alpha}\right]dv$$
$$+\int_\Omega \mu^\beta(t)Y_\beta(t,x,u)dv$$

or

$$\int_\Omega X(t,x^0,u^0)dv\geq\int_\Omega\left(X(t,x,u)+\lambda^\alpha_i(t)\left[X^i_\alpha(t,x,u)-\frac{\partial x^i}{\partial t^\alpha}\right]+\mu^\beta(t)Y_\beta(t,x,u)\right)dv,$$

$$\int_\Omega X(t,x^*,u^*)dv\geq\int_\Omega L(t,x,u,\lambda,\mu)dv,\quad \text{that is}\quad I(x^*,u^*)\geq J(x,u,\lambda,\mu).$$

□

Theorem 3.2 (Direct duality). *Let* (x^0,u^0) *be a normal optimal solution of the problem* (PCP). *Suppose that the conditions of Theorem* 2.1 *are satisfied. Then there exist the functions* $(\lambda^\alpha_i)^0(t)$ *and* $(\mu^\beta)^0(t)$ *such that* $(x^0,u^0,(\lambda^\alpha_i)^0,(\mu^\beta)^0)$ *is an optimal solution of the dual* (DCP). *Moreover,* $I(x^0,u^0)=J(x^0,u^0,(\lambda^\alpha_i)^0,(\mu^\beta)^0)$, *that is* $\min(\text{PCP})=\max\ (\text{DCP})$.

Proof. According to Theorem 2.1, there exist the functions $(\lambda^\alpha_i)^0(t)$ and $(\mu^\beta)^0(t)$ such that the conditions (MFJ) are satisfied at (x^0,u^0) with $\theta(t)=1$. Then, it follows that $(x^0,u^0,(\lambda^\alpha_i)^0,(\mu^\beta)^0)\in\Delta$ and moreover, $I(x^0,u^0)=J(x^0,u^0,(\lambda^\alpha_i)^0,(\mu^\beta)^0)$. Using the hypotheses of Theorem 3.1, it follows that the point $(x^0,u^0,(\lambda^\alpha_i)^0,(\mu^\beta)^0)$ is optimal to dual problem (DCP). □

Theorem 3.3 (Converse duality). *Let* $(x^0,u^0,(\lambda^\alpha_i)^0,(\mu^\beta)^0)$ *be an optimal solution of the dual problem* (DCP) *and suppose that the following conditions are satisfied:*

i) $(\bar{x},\bar{u})$ *is a normal solution to primal* (PCP).

a) $\displaystyle\int_\Omega X(t,x(t),u(t))dv$ *is invex at* (x^0,u^0)*;*

b) $\displaystyle\int_\Omega \lambda^\alpha_i\left[X^i_\alpha(t,x(t),u(t))-\frac{\partial x^i}{\partial t^\alpha}\right]dv$ *is invex at* (x^0,u^0)*;*

c) $\displaystyle\int_\Omega \mu^\beta(t)Y_\beta(t,x(t),u(t))dv$ *is invex at* (x^0,u^0),

all with respect to η *and* ξ, *as in Definition* 3.1.

Then $(x^0,u^0)=(\bar{x},\bar{u})$. *Moreover,* $I(x^0,u^0)=J(x^0,u^0,(\lambda_i^\alpha)^0,(\mu^\beta)^0)$, *that is* $\min(\text{PCP})=\max(\text{DCP})$.

Proof. On the contrary, we suppose that $(\bar{x},\bar{u})\neq(x^0,u^0)$ and we shall find a contradiction. Because $(\bar{x},\bar{u})$ is a normal optimal solution to (PCP), there are the scalar continuous functions $\bar{\lambda}_i^\alpha(t)$ and $\bar{\mu}^\beta(t)$ of C^1-class which satisfy conditions (MFJ) at the point $(\bar{x},\bar{u})$ with $\theta(t)=1$. It follows $(\bar{x},\bar{u},\bar{\lambda}_i^\alpha,\bar{\mu}^\beta)\in\Delta$. Moreover, $I(\bar{x},\bar{u})=J(\bar{x},\bar{u},\bar{\lambda}_i^\alpha,\bar{\mu}^\beta)$.

Taking into account the hypotheses a), b), c) of this theorem and according to Theorem 3.1, we have $I(\bar{x},\bar{u})\geq J(x^0,u^0,(\lambda_i^\alpha)^0,(\mu^\beta)^0)$. This is equivalent to the inequality $J(\bar{x},\bar{u},\bar{\lambda}_i^\alpha,\bar{\mu}^\beta)\geq J(x^0,u^0,(\lambda_i^\alpha)^0,(\mu^\beta)^0)$. The equality is excluded because $(\bar{x},\bar{u})\neq(x^0,u^0)$. Then, $J(\bar{x},\bar{u},\bar{\lambda}_i^\alpha,\bar{\mu}^\beta)>J(x^0,u^0,(\lambda_i^\alpha)^0,(\mu^\beta)^0)$. But this inequality contradicts the optimality of $(x^0,u^0,(\lambda_i^\alpha)^0,(\mu^\beta)^0)$ in the dual (DCP). Therefore, we have $(\bar{x},\bar{u})=(x^0,u^0)$ and

$$I(x^0,u^0)=I(\bar{x},\bar{u})=J(\bar{x},\bar{u},\bar{\lambda}_i^\alpha,\bar{\mu}^\beta)=J(x^0,u^0,(\lambda_i^\alpha)^0,(\mu^\beta)^0).$$

□

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