

STRONG CONVERGENCE OF SOME ITERATIVE ALGORITHMS FOR PSEUDO-CONTRACTIVE MAPPINGS

MENGQIN LI

ABSTRACT. In this paper, we suggest and analyze some new iterative algorithms for a strictly pseudo-contractive mapping T in the sense of Browder-Petryshyn in a real q -uniformly smooth Banach space. We prove that the proposed iterative algorithms converge strongly to some fixed point of T .

1. INTRODUCTION

Let E be an arbitrary real Banach space and let $J_q (q > 1)$ denote the generalized duality mapping from E into 2^{E^*} given by

$$J_q(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^q = \|x\| \|x^*\|\} \quad \text{for all } x \in E,$$

where E^* denote the dual space of E and $\langle \cdot, \cdot \rangle$ denotes the generated duality pairing between E and E^* . In particular, J_2 is called the normalized duality mapping and it is usually denoted by J . It is well known that $J_q(x) = \|x\|^{q-2}J(x)$ if $x \neq 0$ and that if E^* is strictly convex, then J_q is single-valued.

Let K be a nonempty closed convex subset of E . A mapping $f: K \rightarrow K$ is called contractive if there exists a constant $\alpha \in [0, 1)$ such that

$$\|f(x) - f(y)\| \leq \alpha \|x - y\|, \quad (1.1)$$

for all $x, y \in K$. Recall that a mapping $T: K \rightarrow K$ is called pseudo-contractive if for all $x, y \in K$, there exists $j_q(x - y) \in J_q(x - y)$ such that

$$\langle Tx - Ty, j_q(x - y) \rangle \leq \|x - y\|^q.$$

It is easy to see that (1.1) is equivalent to the following: there exists some $j(x - y) \in J(x - y)$ such that

$$\langle Tx - Ty, j(x - y) \rangle \leq \|x - y\|^2,$$

for all $x, y \in K$. Recall also that a mapping $T: K \rightarrow K$ is called strictly pseudo-contractive mapping (in the terminology of Browder-Petryshyn) [1] if there exists

Received: February 20, 2009.

2000 Mathematics Subject Classification: 47H09, 47H10, 47H17.

Key words and phrases: Fixed point, strictly pseudo-contractive mapping, iterative algorithm, variational inequality.

$\lambda > 0$ such that for all $x, y \in K$ there exists $j_q(x - y) \in J_q(x - y)$ satisfying

$$\langle (I - T)x - (I - T)y, j_q(x - y) \rangle \geq \lambda \|(I - T)x - (I - T)y\|^q, \quad (1.2)$$

where I denotes the identity operator. We denote by $F(T)$ the set of fixed points of T .

In the literature, many interesting and important results have been acquired to approximate the fixed points of pseudo-contractive mappings. We refer the reader to [1]-[12], [15], [17]-[20].

It is our purpose in this paper to introduce and analyze some new iterative algorithms for a strictly pseudo-contractive mapping T in the sense of Browder-Petryshyn in a real q -uniformly smooth Banach space. We prove that the proposed iterative algorithms converge strongly to some fixed point of T .

2. PRELIMINARIES

Let E be a real Banach space. The modulus of smoothness of E is defined as the function $\rho_E: [0, \infty) \rightarrow [0, \infty)$:

$$\rho_E(\tau) = \sup \left\{ \frac{1}{2}(\|x + y\| + \|x - y\|) - 1 : \|x\| \leq 1, \|y\| \leq \tau \right\}.$$

E is said to be uniformly smooth if and only if $\lim_{\tau \rightarrow 0^+} (\rho_E(\tau)/\tau) = 0$. Let $q > 1$.

The space E is said to be q -uniformly smooth (or to have a modulus of smoothness of power type $q > 1$), if there exists a constant $c_q > 0$ such that $\rho_E(\tau) \leq c_q \tau^q$. It is well known that Hilbert spaces, L_p and l_p spaces, $1 < p < \infty$, as well as the Sobolev spaces, W_m^p , $1 < p < \infty$, are q -uniformly smooth.

We shall require the following lemmas in the sequel.

Lemma 2.1 ([13]). *Let $q > 1$ and E be a real smooth Banach space. Then the following are equivalent:*

- (1) E is q -uniformly smooth;
- (2) There exists a constant $c_q > 0$ such that for all $x, y \in E$

$$\|x + y\|^q \leq \|x\|^q + q\langle y, j_q(x) \rangle + c_q \|y\|^q.$$

Lemma 2.2 ([14]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space E such that $x_{n+1} = \sigma_n x_n + (1 - \sigma_n)y_n$, $n \geq 0$ where $\{\sigma_n\}$ is a sequence in $[0, 1]$ such that $0 < \liminf_{n \rightarrow \infty} \sigma_n \leq \limsup_{n \rightarrow \infty} \sigma_n < 1$. Assume $\limsup_{n \rightarrow \infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then $\lim_{n \rightarrow \infty} \|y_n - x_n\| = 0$.*

It is easy to check that the following result holds.

Lemma 2.3 ([8], [9]). *Let K be a nonempty closed convex subset of a q -uniformly smooth Banach space E . Let $T: K \rightarrow K$ be a strictly pseudo-contractive mapping such that $F(T) \neq \emptyset$. Let f be a fixed Lipschitzian strongly pseudo-contractive mapping from K to K . For any $t \in (0, 1)$, let x_t be the unique fixed point of*

$tf + (1 - t)T$. Then as $t \rightarrow 0+$, $\{x_t\}$ converges to a fixed point $p \in F(T)$ which is the unique solution in $F(T)$ to the variational inequality:

$$\langle f(p) - p, j(z - p) \rangle \leq 0 \quad \text{for all } z \in F(T).$$

Lemma 2.4 ([16]). Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that $a_{n+1} \leq (1 - b_n)a_n + c_n, n \geq 0$, where $\{b_n\}$ is a sequence in $(0, 1)$ and $\{c_n\}$ is a sequence in \mathbb{R} such that

$$(i) \sum_{n=0}^{\infty} b_n = \infty; \quad (ii) \limsup_{n \rightarrow \infty} \frac{c_n}{b_n} \leq 0 \quad \text{or} \quad \sum_{n=0}^{\infty} |c_n| < \infty.$$

Then $\lim_{n \rightarrow \infty} a_n = 0$.

3. MAIN RESULTS

Let K be a nonempty closed convex subset of a real q -uniformly smooth Banach space E . Let $f: K \rightarrow K$ be a contraction with coefficient $\alpha \in [0, 1)$ and $T: K \rightarrow K$ be a strictly pseudo-contractive mapping. First we introduce the following iterative algorithm.

Algorithm 3.1. Let $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ be real sequences in $[0, 1]$ such that $\alpha_n + \beta_n + \gamma_n = 1, n \geq 0$. Let the sequence $\{x_n\}$ be generated from an arbitrary $x_0 \in K$ by the following iterative scheme:

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n ((1 - \delta)x_n + \delta T x_n), \quad n \geq 0, \quad (3.1)$$

where $\delta \in \left(0, \min \left\{1, \left(\frac{q\lambda}{c_q}\right)^{\frac{1}{q-1}}\right\}\right)$ is a constant.

Now we state and prove our main results as follows.

Theorem 3.1. Let K be a nonempty closed convex subset of a q -uniformly smooth Banach space E , and $T: K \rightarrow K$ be a strictly pseudo-contractive mapping such that $F(T) \neq \emptyset$. Let $f: K \rightarrow K$ be a contraction. Let $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ be real sequences in $[0, 1]$ satisfying the following conditions:

- (C1) $\lim_{n \rightarrow \infty} \alpha_n = 0$;
- (C2) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
- (C3) $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$.

For initial guess $x_0 \in K$, then the sequence $\{x_n\}$ defined by (3.1) converges strongly to $p \in F(T)$ which solves the variational inequality:

$$\langle f(p) - p, j(z - p) \rangle \leq 0 \quad \text{for all } z \in F(T).$$

Proof. We first show that $\{x_n\}$ is bounded. Indeed, take a point $p \in F(T)$ to get

$$\begin{aligned} \|x_{n+1} - p\| &= \|\alpha_n(f(x_n) - p) + \beta_n(x_n - p) \\ &\quad + \gamma_n[(1 - \delta)(x_n - p) + \delta(Tx_n - p)]\| \\ &\leq \alpha_n\|f(x_n) - f(p)\| + \alpha_n\|f(p) - p\| + \beta_n\|x_n - p\| \\ &\quad + \gamma_n\|(x_n - p) + \delta(Tx_n - x_n)\|. \end{aligned} \quad (3.2)$$

From Lemma 2.1 and (1.2), we obtain

$$\begin{aligned} \|(x_n - p) + \delta(Tx_n - x_n)\|^q &\leq \|x_n - p\|^q + q\delta\langle Tx_n - x_n, j_q(x_n - p) \rangle \\ &\quad + c_q\delta^q\|Tx_n - x_n\|^q \\ &\leq \|x_n - p\|^q - q\delta\lambda\|Tx_n - x_n\|^q \\ &\quad + c_q\delta^q\|Tx_n - x_n\|^q \\ &= \|x_n - p\|^q + (c_q\delta^q - q\delta\lambda)\|Tx_n - x_n\|^q \\ &\leq \|x_n - p\|^q, \end{aligned}$$

which implies that

$$\|(x_n - p) + \delta(Tx_n - x_n)\| \leq \|x_n - p\|. \quad (3.3)$$

By using (3.2) and (3.3), we get

$$\begin{aligned} \|x_{n+1} - p\| &\leq \alpha_n\alpha\|x_n - p\| + \alpha_n\|f(p) - p\| + \beta_n\|x_n - p\| \\ &\quad + \gamma_n\|x_n - p\| \\ &= [1 - (1 - \alpha)\alpha_n]\|x_n - p\| + (1 - \alpha)\alpha_n\frac{\|f(p) - p\|}{1 - \alpha}. \end{aligned}$$

By induction, we obtain, for all $n \geq 0$,

$$\|x_n - p\| \leq \max \left\{ \|x_0 - p\|, \frac{\|f(p) - p\|}{1 - \alpha} \right\}.$$

Define $x_{n+1} = \beta_n x_n + (1 - \beta_n)y_n, n \geq 0$. It follows that

$$\begin{aligned} y_{n+1} - y_n &= \frac{x_{n+2} - \beta_{n+1}x_{n+1}}{1 - \beta_{n+1}} - \frac{x_{n+1} - \beta_n x_n}{1 - \beta_n} \\ &= \frac{\alpha_{n+1}f(x_{n+1}) + \gamma_{n+1}((1 - \delta)x_{n+1} + \delta Tx_{n+1})}{1 - \beta_{n+1}} \\ &\quad - \frac{\alpha_n f(x_n) + \gamma_n((1 - \delta)x_n + \delta Tx_n)}{1 - \beta_n} \\ &= \frac{\alpha_{n+1}f(x_{n+1})}{1 - \beta_{n+1}} - \frac{\alpha_n f(x_n)}{1 - \beta_n} \\ &\quad + \frac{\gamma_{n+1}((1 - \delta)(x_{n+1} - x_n) + \delta(Tx_{n+1} - Tx_n))}{1 - \beta_{n+1}} \\ &\quad + \left(\frac{\gamma_{n+1}}{1 - \beta_{n+1}} - \frac{\gamma_n}{1 - \beta_n} \right) ((1 - \delta)x_n + \delta Tx_n). \end{aligned} \quad (3.4)$$

Again from Lemma 2.1 and (1.2), we have

$$\begin{aligned}
 & \| (1 - \delta)(x_{n+1} - x_n) + \delta(Tx_{n+1} - Tx_n) \|^q \\
 &= \| (x_{n+1} - x_n) - \delta[x_{n+1} - Tx_{n+1} - (x_n - Tx_n)] \|^q \\
 &\leq \| x_{n+1} - x_n \|^q - q\delta \langle x_{n+1} - Tx_{n+1} - (x_n - Tx_n), j_q(x_{n+1} - x_n) \rangle \\
 &\quad + c_q \delta^q \| x_{n+1} - Tx_{n+1} - (x_n - Tx_n) \|^q \\
 &\leq \| x_{n+1} - x_n \|^q - q\delta\lambda \| x_{n+1} - Tx_{n+1} - (x_n - Tx_n) \|^q \\
 &\quad + c_q \delta^q \| x_{n+1} - Tx_{n+1} - (x_n - Tx_n) \|^q \\
 &= \| x_{n+1} - x_n \|^q + (c_q \delta^q - q\delta\lambda) \| x_{n+1} - Tx_{n+1} - (x_n - Tx_n) \|^q \\
 &\leq \| x_{n+1} - x_n \|^q,
 \end{aligned}$$

that is,

$$\| (1 - \delta)(x_{n+1} - x_n) + \delta(Tx_{n+1} - Tx_n) \| \leq \| x_{n+1} - x_n \|. \tag{3.5}$$

Combining (3.4) and (3.5) yields

$$\begin{aligned}
 \| y_{n+1} - y_n \| &\leq \frac{\alpha_{n+1}}{1 - \beta_{n+1}} (\| f(x_{n+1}) \| + \| (1 - \delta)x_n + \delta Tx_n \|) \\
 &\quad + \frac{\alpha_n}{1 - \beta_n} (\| f(x_n) \| + \| (1 - \delta)x_n + \delta Tx_n \|) \\
 &\quad + \frac{\gamma_{n+1}}{1 - \beta_{n+1}} \| x_{n+1} - x_n \|.
 \end{aligned}$$

Since $\alpha_n \rightarrow 0$, $\gamma_{n+1} \leq 1 - \beta_{n+1}$, the last inequality implies

$$\limsup_{n \rightarrow \infty} (\| y_{n+1} - y_n \| - \| x_{n+1} - x_n \|) \leq 0.$$

Apply Lemma 2.2 to get

$$\lim_{n \rightarrow \infty} \| y_n - x_n \| = 0.$$

Consequently,

$$\lim_{n \rightarrow \infty} \| x_{n+1} - x_n \| = \lim_{n \rightarrow \infty} (1 - \beta_n) \| y_n - x_n \| = 0.$$

Note that

$$\begin{aligned}
 \| x_n - Tx_n \| &\leq \| x_{n+1} - x_n \| + \| x_{n+1} - Tx_n \| \\
 &\leq \| x_{n+1} - x_n \| + \alpha_n \| f(x_n) - Tx_n \| \\
 &\quad + \beta_n \| x_n - Tx_n \| + \gamma_n (1 - \delta) \| x_n - Tx_n \|,
 \end{aligned}$$

that is,

$$\| x_n - Tx_n \| \leq \frac{1}{\alpha_n + \gamma_n \delta} \{ \| x_{n+1} - x_n \| + \alpha_n \| f(x_n) - Tx_n \| \} \rightarrow 0.$$

Next we prove that

$$\limsup_{n \rightarrow \infty} \langle f(p) - p, j(x_n - p) \rangle \leq 0, \quad p \in F(T).$$

Let z_t be the unique solution to the equation

$$z_t = tf(z_t) + (1-t)Tz_t.$$

From Lemma 2.3, we know that $z_t \rightarrow p \in F(T)$ which solves the variational inequality

$$\langle f(p) - p, j(z - p) \rangle \leq 0 \quad \text{for all } z \in F(T).$$

We can write

$$z_t - x_n = (1-t)(Tz_t - x_n) + t(f(z_t) - x_n).$$

It follows that

$$\begin{aligned} \|z_t - x_n\|^2 &= (1-t)\langle Tz_t - x_n, j(z_t - x_n) \rangle + t\langle f(z_t) - x_n, j(z_t - x_n) \rangle \\ &= (1-t)(\langle Tz_t - Tx_n, j(z_t - x_n) \rangle + \langle Tx_n - x_n, j(z_t - x_n) \rangle) \\ &\quad + t\langle f(z_t) - z_t, j(z_t - x_n) \rangle + t\langle z_t - x_n, j(z_t - x_n) \rangle \\ &\leq (1-t)\|z_t - x_n\|^2 + (1-t)\|Tx_n - x_n\|\|z_t - x_n\| \\ &\quad + t\langle f(z_t) - z_t, j(z_t - x_n) \rangle + t\|z_t - x_n\|^2 \\ &= \|z_t - x_n\|^2 + (1-t)\|Tx_n - x_n\|\|z_t - x_n\| \\ &\quad + t\langle f(z_t) - z_t, j(z_t - x_n) \rangle, \end{aligned}$$

and hence

$$\langle f(z_t) - z_t, j(x_n - z_t) \rangle \leq \frac{1-t}{t}\|Tx_n - x_n\|\|z_t - x_n\|.$$

Since $\|x_n - Tx_n\| \rightarrow 0$ and $\{z_t\}$ and $\{x_n\}$ are bounded, using the standard proof, it is easy to obtain

$$\limsup_{n \rightarrow \infty} \langle f(p) - p, j(x_n - p) \rangle \leq 0, \quad p \in F(T). \quad (3.6)$$

Finally we claim that $x_n \rightarrow p$ in norm. From (3.1) and (3.3), we have

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \alpha_n \langle f(x_n) - p, j(x_{n+1} - p) \rangle + \beta_n \langle x_n - p, j(x_{n+1} - p) \rangle \\ &\quad + \gamma_n \langle (1-\delta)(x_n - p) + \delta(Tx_n - p), j(x_{n+1} - p) \rangle \\ &\leq \alpha_n \langle f(x_n) - p, j(x_{n+1} - p) \rangle + \beta_n \|x_n - p\| \|x_{n+1} - p\| \\ &\quad + \gamma_n \|x_n - p + \delta(Tx_n - x_n)\| \|x_{n+1} - p\| \\ &\leq \alpha_n \langle f(x_n) - f(p), j(x_{n+1} - p) \rangle + \beta_n \|x_n - p\| \|x_{n+1} - p\| \\ &\quad + \alpha_n \langle f(p) - p, j(x_{n+1} - p) \rangle + \gamma_n \|x_n - p\| \|x_{n+1} - p\| \\ &\leq \alpha_n \alpha \|x_n - p\| \|x_{n+1} - p\| + (1 - \alpha_n) \|x_n - p\| \|x_{n+1} - p\| \\ &\quad + \alpha_n \langle f(p) - p, j(x_{n+1} - p) \rangle \\ &\leq [1 - (1 - \alpha)\alpha_n] \frac{1}{2} (\|x_n - p\|^2 + \|x_{n+1} - p\|^2) \\ &\quad + \alpha_n \langle f(p) - p, j(x_{n+1} - p) \rangle, \end{aligned}$$

that is,

$$[1 + (1 - \alpha)\alpha_n] \|x_{n+1} - p\|^2 \leq [1 - (1 - \alpha)\alpha_n] \|x_n - p\|^2 + 2\alpha_n \langle f(p) - p, j(x_{n+1} - p) \rangle,$$

which implies that

$$\|x_{n+1} - p\|^2 \leq [1 - (1 - \alpha)\alpha_n]\|x_n - p\|^2 + 2\alpha_n\langle f(p) - p, j(x_{n+1} - p) \rangle. \quad (3.7)$$

From (3.6), (3.7) and Lemma , we conclude that $\|x_n - p\| \rightarrow 0$. This completes the proof. \square

As direct consequence of Theorem 3.1, we obtain the following result.

Corollary 3.1. *Let $K, C, T, \{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ be as in Theorem 3.1. For initial guess $x_0 \in K$, let the sequence $\{x_n\}$ be defined by*

$$x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n((1 - \delta)x_n + \delta T x_n), \quad n \geq 0,$$

where $\delta \in \left(0, \min \left\{1, \left(\frac{q\lambda}{c_q}\right)^{\frac{1}{q-1}}\right\}\right)$ is a constant. Then $\{x_n\}$ converges strongly to $p \in F(T)$.

Remark 3.1. It is clear that the importance of well-known Halpern iterative algorithm lies in the fact that the strong convergence is achieved for nonexpansive mappings. Our algorithms studied in this paper can be reviewed as significant extensions of the Halpern iterative algorithm. At the same time we obtain strong convergence results concerning strictly pseudo-contractive mappings under very weak restrictions on parameters in the setting of Banach spaces.

REFERENCES

- [1] F. E. Browder and W. V. Petryshyn: *Construction of fixed points of nonlinear mappings in Hilbert spaces*, J. Math. Anal. Appl., **20**(1967), 197-228.
- [2] M. O. Osilike and A. Udomene: *Demiclosedness principle and convergence theorems for strictly pseudo-contractive mappings of Browder-Petryshyn type*, J. Math. Anal. Appl., **256**(2001), 431-445.
- [3] C. E. Chidume and M. Chika: *Fixed point iteration for pseudo-contractive maps*, Proc. Amer. Math. Soc., **127**(1999), 1163-1170.
- [4] R. Chen, P. K. Lin and Y. Song: *An approximation method for strictly pseudocontractive mappings*, Nonlinear Anal., **64**(2006), 2527-2535.
- [5] Y. Song and R. Chen: *Convergence theorems of iterative algorithms for continuous pseudo-contractive mappings*, Nonlinear Anal., **67**(2007), 486-497.
- [6] C. E. Chidume: *Iterative approximation of fixed points of Lipschitz pseudo-contractive maps*, Proc. Amer. Math. Soc., **129**(2001), 2245-2251.
- [7] C. E. Chidume and H. Zegeye: *Approximate fixed point sequences and convergence theorems for Lipschitz pseudo-contractive maps*, Proc. Amer. Math. Soc., **132**(2003), 831-840.
- [8] C. H. Morales and J. S. Jung: *Convergence of paths for pseudo-contractive mappings in Banach spaces*, Proc. Amer. Math. Soc., **128**(2000), 3411-3419.
- [9] J. Schu: *Approximating fixed points of Lipschitzian pseudo-contractive mappings*, Houston J. Math., **19**(1993), 107-115.
- [10] A. Udomene: *Path convergence, approximation of fixed points and variational solutions of Lipschitz pseudo-contractions in Banach spaces*, Nonlinear Anal., (in printing).

- [11] C. H. Morales: *On the fixed-point theory for local k -pseudo-contractions*, Proc. Amer. Math. Soc., **81**(1981), 71-74.
- [12] C. E. Chidume and S. A. Mutangadura: *An example on the Mann iteration method for Lipschitz pseudo-contractions*, Proc. Amer. Math. Soc., **129**(2001), 2359-2363.
- [13] H. K. Xu: *Inequalities in Banach spaces with applications*, Nonlinear Anal., **16**(1991), 1127-1138.
- [14] T. Suzuki: *A sufficient and necessary condition for Halpern-type strong convergence to fixed points of nonexpansive mappings*, Proc. Amer. Math. Soc., **135**(2007), 99-106.
- [15] G. Marino and H. K. Xu: *Weak and strong convergence theorems for strictly pseudo-contractions in Hilbert spaces*, J. Math. Anal. Appl., **329**(2007), 336-349.
- [16] H. K. Xu: *Iterative algorithms for nonlinear operators*, J. London Math. Soc., **2**(2002), 240-256.
- [17] L. C. Zeng, N. C. Wong and J. C. Yao: *Strong convergence theorems for strictly pseudo-contractive mappings of Browder-Petryshyn type*, Taiwanese J. Math., **10**(2006), 837-849.
- [18] Y. Yao, Y. C. Liou and R. Chen: *Strong convergence of an iterative algorithm for pseudo-contractive mapping in Banach spaces*, Nonlinear Anal., **67**(2007), 3311-3317.
- [19] M. O. Olatinwo: *Some stability results for two hybrid fixed point iterative algorithms of Kirk-Ishikawa and Kirk-Mann type*, J. Adv. Math. Studies, **1**(2008), 87-96.
- [20] I. Altun and H. Simsek: *Some fixed point theorems on dualistic partial metric spaces*, J. Adv. Math. Studies, **1**(2008), 01-08.

Tianjin Polytechnic University
Department of Mathematics
Tianjin 300160, P. R. China
E-mail address: mengqinli@yahoo.cn