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ON A STUDY OF DISTINGUISHED STRUCTURES OF HESSIAN TYPE ON PSEUDO-RIEMANNIAN MANIFOLDS

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ABSTRACT. In [4], [10], [11] the Hessian Geometry is studied using a function defined on the Euclidean space $\mathbb{R}^n$. In our paper we replace the initial Euclidean space $\mathbb{R}^n$ with an arbitrary pseudo-Riemannian manifold (M, g) and consider a function $f\colon M \to \mathbb{R}$ whose Hessian with respect to the initial pseudo-Riemannian metric g is non-degenerate. Then we study properties of the pseudo-Riemannian manifold (M, h), $h = \nabla_g^2 f$, in terms of local computation. This allows us to solve and generalize an open problem stated by C. Udrişte in [12], [13].

The paper is organized as follows. In §1 we remind some notions on pseudo-Riemannian Geometry. In §2 we deduce the Christoffel symbols and the system of geodesics of pseudo-Riemannian manifold (M, h), $h = \nabla_g^2 f$. We determine the conditions for which the Christoffel symbols of (M, h) coincide with the Christoffel symbols of (M, g) and give some examples. We also consider the simplest non-homotetic deformations of a pseudo-Riemannian Hessian metric, namely the conformal ones. In §3 we establish a relation between the components of the curvature tensors field of (M, h) and (M, g). Then we discuss two examples of pseudo-Riemannian metrics which have constant sectional curvature. In §4 we consider remarkable metrics of Riemannian-Hessian metric type from Optimization Theory and Statistics and we study their geometries.

1. INTRODUCTION AND PRELIMINARIES

In [10], [11], H. Shima and K. Yagi studied the geometry of Hessian manifolds. The basic idea is that the Euclidean space $\mathbb{R}^n$ is endowed with a Riemannian metric $h_{ij} = \dfrac{\partial^2 \phi}{\partial x^i \partial x^j}$, where $\phi\colon \mathbb{R}^n \to \mathbb{R}$ is a C^∞-class function. In [7], Y. Nesterov and M. J. Todd considered the Riemannian geometry defined on a convex set (included in an n-dimensional real vector space) by the Hessian of a self-concordant barrier function and found its associated geodesics in several cases. In his paper [16], P. Wilson showed that for a Fermat form f of any degree d and any number of variables n, if the associated Hessian metric is of signature $(1, *)$ on a nonempty open subset of $\mathbb{R}^n$, then the associated Riemannian metric on $M = \{f = 1\}$ has constant sectional

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curvature $-\frac{d^2}{4}$. Hitchin [5] characterized the Hessian Riemannian structures in terms of Lagrangian submanifolds of the cotangent bundle. For other significant developments of these problems, see [1]÷[16] and the references therein.

To develop the theory in this paper it is necessary to recall some notions of pseudo-Riemannian manifolds, [9], [10].

Definition 1.1. A pseudo-Riemannian metric of signature (p,q) on a smooth manifold M of dimension $n=p+q$ is a smooth symmetric differentiable 2-form g on M such that, at each point x of M, g_x is non-degenerate on T_xM with the signature (p,q). We call (M,g) *a pseudo-Riemannian manifold.*

Theorem 1.1. *Given a pseudo-Riemannian manifold (M,g) there exists an unique linear connection ∇_g on M, called the Levi-Civita connection (of g), such that*[1]

a) ∇_g *is metric (i.e. $\nabla_g g=0$);*

b) ∇_g *is torsion-free (i.e. $T=0$).*

Definition 1.2. Given a pseudo-Riemannian manifold (M,g), the curvature tensor field R of the Levi-Civita connection is called *the Riemannian curvature of* (M,g).

Let $(U,x^1,\dots,x^n)$ be a coordinate chart on M. Then the Christoffel symbols Γ^k_{ij} of the Levi-Civita connection are related to the functions g_{ij} by the formulas

$$\Gamma^k_{ij}=\frac{1}{2}g^{k\ell}\left(\frac{\partial g_{\ell i}}{\partial x^j}+\frac{\partial g_{j\ell}}{\partial x^i}-\frac{\partial g_{ij}}{\partial x^\ell}\right).$$

Also, the curvature R has the components R^ℓ_{ijk} given by

$$R^\ell_{ijk}=\frac{\partial\Gamma^\ell_{ki}}{\partial x^j}-\frac{\partial\Gamma^\ell_{ji}}{\partial x^k}+\Gamma^r_{ki}\Gamma^\ell_{jr}-\Gamma^r_{ji}\Gamma^\ell_{kr}.$$

Note that in local coordinates a geodesic $\gamma(t)=(x^i(t))_{i=1,\dots,n}$ satisfies a system of n second order differential equations

$$\ddot{x}^i+\Gamma^i_{jk}\dot{x}^j\dot{x}^k=0,\qquad i=1,\dots,n.$$

If $f\colon M\to\mathbb{R}$ is a smooth function, then the second covariant derivative

$$\nabla^2_g f=\left(\frac{\partial^2 f}{\partial x^i\partial x^j}-\Gamma^k_{ij}\frac{\partial f}{\partial x^k}\right)dx^i\otimes dx^j$$

is called the Hessian of f.

Throughout this paper, we shall use the following notations:

$$f_{,i}=\frac{\partial f}{\partial x^i};\qquad f_{,ij}=\frac{\partial^2 f}{\partial x^i\partial x^j}-\Gamma^m_{ij}f_{,m};\qquad f_{,ijk}=\frac{\partial f_{,ij}}{\partial x^k}-\Gamma^\ell_{ki}f_{,\ell j}-\Gamma^\ell_{kj}f_{,\ell i}.$$

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[1] Fundamental theorem of pseudo-Riemannian Geometry

2. CHRISTOFFEL SYMBOLS OF PSEUDO-RIEMANNIAN MANIFOLD (M,h), $h = \nabla^2_g f$

Suppose that the Hessian $h = \nabla^2_g f$ is non-degenerate. Then h is a pseudo-Riemannian metric which produces the Levi-Civita connection ∇_h and the Chrisoffel symbols $\bar{\Gamma}^k_{ij}$.

Theorem 2.1. *Let* $f,^{pk}$ *be the contravariant components of the pseudo-Riemannian metric* $h_{pk} = f_{,pk}$ *and* R^m_{ijk} *be the components of the curvature tensor field produced by the pseudo-Riemannian metric* g_{ij}. *Then the components of Levi-Civita connection* ∇_h *are given by the following formula*

$$\bar{\Gamma}^p_{ij} = \Gamma^p_{ij} + \frac{1}{2} f,^{kp}\,[f_{,ijk} + (R^m_{ikj} + R^m_{jki}) f_{,m}].$$

Proof. We use the fomulas $h_{ij} = f_{,ij}$ and

$$2h_{pk}\bar{\Gamma}^p_{ij} = \frac{\partial h_{jk}}{\partial x^i} + \frac{\partial h_{ik}}{\partial x^j} - \frac{\partial h_{ij}}{\partial x^k}.$$

Then

$$\frac{\partial h_{ij}}{\partial x^k} = \frac{\partial^3 f}{\partial x^i \partial x^j \partial x^k} - \frac{\partial \Gamma^m_{ij}}{\partial x^k} f_{,m} - \Gamma^m_{ij} f_{,km} - \Gamma^m_{ij}\Gamma^\ell_{km} f_{,\ell} \tag{2.1}$$

and similar formulas for $\dfrac{\partial h_{jk}}{\partial x^i}$ and $\dfrac{\partial h_{ik}}{\partial x^j}$. We may write

$$\begin{aligned} 2f_{,pk}\bar{\Gamma}^p_{ij} \;=\; & \frac{\partial^3 f}{\partial x^i \partial x^j \partial x^k} + \left(\frac{\partial \Gamma^\ell_{ij}}{\partial x^k} - \frac{\partial \Gamma^\ell_{ik}}{\partial x^j} - \frac{\partial \Gamma^\ell_{jk}}{\partial x^i} + \Gamma^m_{ij}\Gamma^\ell_{km} - \Gamma^m_{ik}\Gamma^\ell_{jm} - \Gamma^m_{jk}\Gamma^\ell_{im} \right) f_{,\ell} \\ & + \Gamma^m_{ij} f_{,km} - \Gamma^m_{ik} f_{,jm} - \Gamma^m_{jk} f_{,im}. \end{aligned} \tag{2.2}$$

Using the formula for $f_{,ijk}$ the equality (2.1) becomes

$$\frac{\partial^3 f}{\partial x^i \partial x^j \partial x^k} = f_{,ijk} + \Gamma^\ell_{ki} f_{,\ell j} + \Gamma^\ell_{kj} f_{,\ell i} + \Gamma^\ell_{ij} f_{,k\ell} + \frac{\partial \Gamma^\ell_{ij}}{\partial x^k} f_{,\ell} + \Gamma^m_{ij}\Gamma^\ell_{km} f_{,\ell}. \tag{2.3}$$

Substituting (2.3) in (2.2), we obtain

$$\begin{aligned} 2f_{,pk}\bar{\Gamma}^p_{ij} = & f_{,ijk} + \Gamma^\ell_{ki} f_{,\ell j} + \Gamma^\ell_{kj} f_{,\ell i} + \Gamma^\ell_{ij} f_{,k\ell} + \frac{\partial \Gamma^\ell_{ij}}{\partial x^k} f_{,\ell} + \Gamma^m_{ij}\Gamma^\ell_{km} f_{,\ell} \\ & + \left(\frac{\partial \Gamma^\ell_{ij}}{\partial x^k} - \frac{\partial \Gamma^\ell_{ik}}{\partial x^j} - \frac{\partial \Gamma^\ell_{jk}}{\partial x^i} + \Gamma^m_{ij}\Gamma^\ell_{km} - \Gamma^m_{ik}\Gamma^\ell_{jm} - \Gamma^m_{jk}\Gamma^\ell_{im} \right) f_{,\ell} \\ & + \Gamma^m_{ij} f_{,km} - \Gamma^m_{ik} f_{,jm} - \Gamma^m_{jk} f_{,im}. \end{aligned}$$

We reduce the terms $\Gamma^{\ell}_{ki} f_{,\ell j}$ with $-\Gamma^{m}_{ik} f_{,jm}$ and $\Gamma^{\ell}_{kj} f_{,\ell i}$ with $-\Gamma^{m}_{jk} f_{,im}$ and we have

$$\begin{aligned} 2f_{,pk}\bar{\Gamma}^{p}_{ij} &= f_{,ijk} + 2\Gamma^{\ell}_{ij} f_{,k\ell} + \left(\frac{\partial \Gamma^{\ell}_{ij}}{\partial x^k} + \frac{\partial \Gamma^{\ell}_{ij}}{\partial x^k} - \frac{\partial \Gamma^{\ell}_{ki}}{\partial x^j} - \frac{\partial \Gamma^{\ell}_{jk}}{\partial x^i} + \Gamma^{m}_{ij}\Gamma^{\ell}_{km} + \Gamma^{m}_{ij}\Gamma^{\ell}_{km} \right. \\ &\quad \left. -\Gamma^{m}_{ik}\Gamma^{\ell}_{jm} - \Gamma^{m}_{jk}\Gamma^{\ell}_{im} \right) f_{,\ell} \\ &= f_{,ijk} + 2\Gamma^{\ell}_{ij} f_{,k\ell} + (R^{\ell}_{ikj} + R^{\ell}_{jki}) f_{,\ell}. \end{aligned}$$

Finally the expression of $\bar{\Gamma}^{p}_{ij}$ is $\bar{\Gamma}^{p}_{ij} = \Gamma^{p}_{ij} + \frac{1}{2} f,^{pk} [f_{,ijk} + (R^{\ell}_{ikj} + R^{\ell}_{jki}) f_{,\ell}]$. $\square$

Remark 2.1. If the initial Riemannian manifold (M, g) is the Euclidean space, then $\Gamma^{p}_{ij} = 0$, $R^{h}_{ijk} = 0$, $f_{,ij} = \dfrac{\partial^2 f}{\partial x^i \partial x^j}$ and $f_{,ijk} = \dfrac{\partial^3 f}{\partial x^i \partial x^j \partial x^k}$. Therefore the formula for $\bar{\Gamma}^{p}_{ij}$ takes the form $\bar{\Gamma}^{p}_{ij} = \dfrac{1}{2} f,^{pk} \cdot \dfrac{\partial^3 f}{\partial x^i \partial x^j \partial x^k}$ or, for the Christoffel symbols of the first kind, $\bar{\Gamma}_{ijp} = \dfrac{1}{2} \cdot \dfrac{\partial^3 f}{\partial x^i \partial x^j \partial x^p}$, a well-known formula.

Corollary 2.1. *The differential system of geodesics is*

$$\ddot{x}^p + \left[\Gamma^{p}_{ij} + f,^{pk} \left(\frac{1}{2} f_{,ijk} + R^{\ell}_{ikj} f_{,\ell} \right) \right] \dot{x}^i \dot{x}^j = 0, \quad p = 1, \ldots, n.$$

Remark 2.2. Corollary 2.1 is the generalization of Theorem 2.1 from [7] in the pseudo-Riemannian case.

Corollary 2.2. $\bar{\Gamma}^{p}_{ij} = \Gamma^{p}_{ij}$, *for all* $i, j, p = 1, \ldots, n$, *iff* $f_{,ijk} + (R^{\ell}_{ikj} + R^{\ell}_{jki}) f_{,\ell} = 0$, *for all* $i, j, k = 1, \ldots, n$.

Example 2.1. Let us consider $M = \mathbb{R}^2$ endowed with the metric

$$g(x^1, x^2) = \begin{pmatrix} 1 + 4(x^1)^2 & -2x^1 \\ -2x^1 & 1 \end{pmatrix}.$$

It is known [3] that the Christoffel symbols are given by

$$\Gamma^1_{11} = \Gamma^1_{12} = \Gamma^1_{21} = \Gamma^2_{22} = \Gamma^2_{12} = \Gamma^2_{21} = \Gamma^2_{22} = 0 \text{ and } \Gamma^2_{11} = -2.$$

Also, $(\mathbb{R}^2, g)$ has zero sectional curvature, $K = 0$, therefore $R^{\ell}_{ijk} = 0$, for all indices $i, j, k, \ell = \overline{1, 2}$. Taking the Rosenbrock's banana function

$$f\colon \mathbb{R}^2 \to \mathbb{R}, \quad f(x^1, x^2) = 100 \left[x^2 - (x^1)^2 \right] + (1 - x^1)^2,$$

we deduce

$$f_{,11} = \frac{\partial^2 f}{\partial (x^1)^2} - (\Gamma^1_{11} f_{,1} + \Gamma^2_{11} f_{,2}) = 800(x^1)^2 + 2,$$

$$f_{,12} = f_{,21} = \frac{\partial^2 f}{\partial x^1 \partial x^2} = -400x^1 \quad \text{and} \quad f_{,22} = \frac{\partial^2 f}{\partial (x^2)^2} = 200.$$

The Riemannian Hessian $\nabla_g^2 f(x^1, x^2) = \begin{pmatrix} 800(x^1)^2 + 2 & -400x^1 \\ -400x^1 & 200 \end{pmatrix}$ is positive definite, hence $(\mathbb{R}^2, \nabla_g^2 f)$ is a Riemannian manifold. It is a straightforward computation to show that $f_{,ijk} = 0$ for all $i, j, k = \overline{1,2}$. Therefore, $\bar{\Gamma}^p_{ij} = \Gamma^p_{ij}$ for all $i, j, p = \overline{1,2}$.

Proposition 2.1. *Let k_{ij} be the components of the pseudo-Riemannian metric $\nabla_h^2 f$, where $h = \nabla_g^2 f$ and let h_{ij} be the components of the pseudo-Riemannian metric $\nabla_g^2 f$. Then*

$$k_{ij} = h_{ij} - \frac{1}{2} f,^{pk} \left[f_{,ijk} + (R^\ell_{ikj} + R^\ell_{jki}) f_{,\ell} \right] f_{,p}.$$

Proof. Indeed,

$$\begin{aligned} k_{ij} = (\nabla_h^2 f)_{ij} &= \frac{\partial^2 f}{\partial x^i \partial x^j} - \bar{\Gamma}^p_{ij} f_{,p} \\ &= \frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma^p_{ij} f_{,p} - \frac{1}{2} f,^{pk} [f_{,ijk} + (R^\ell_{ikj} + R^\ell_{jki}) f_{,\ell}] f_{,p} \\ &= h_{ij} - \frac{1}{2} f,^{pk} [f_{,ijk} + (R^\ell_{ikj} + R^\ell_{jki}) f_{,\ell}] f_{,p}. \end{aligned}$$

□

As an application of the Theorem 2.1, we consider the simplest non-homothetic deformations of a pseudo-Riemannian Hessian metric, namely the conformal ones. More precisely, we shall deduce the necessary conditions for f such that f belongs to the set $\mathcal{F}_g = \{f \in \mathcal{F}(M), \nabla_g^2 f = e^{2u} g\}$, where $u \in \mathcal{F}(M)$. Firstly, we remark that $\nabla_g^2 f = e^{2u} g$ is equivalent to $h_{ij} = e^{2u} g_{ij}$. Then $h^{ij} = f,^{ij} = e^{-2u} g^{ij}$.

Since h and g are conformal, we may write

$$\bar{\Gamma}^p_{ij} = \Gamma^p_{ij} + \delta^p_i u_{,j} + \delta^p_j u_{,i} - g_{ij} g^{pk} u_{,k}.$$

Substituting $\bar{\Gamma}^p_{ij}$ from the Theorem 2.1, we obtain

$$f_{,ijk} + (R^\ell_{ikj} + R^\ell_{jki}) f_{,\ell} = 2e^{2u} [g_{pk}(\delta^p_i u_{,j} + \delta^p_j u_{,i}) - g_{ij} u_{,k}]. \tag{2.4}$$

Hence, if $f \in \mathcal{F}_g$, then f satisfies the system (2.4).

Remark 2.3. a) In (2.4) we have a system of partial differential equations on manifolds.

b) If $f \in \mathcal{F}_g$, then we call f *a metric potential*, because it *generates* the metric.

The simplest example is a graph hypersurface f in the Euclidean space, with its first fundamental form g.

The following two items are particular cases of (2.4).

1) If the pseudo-Riemannian manifold (M, g) has the constant sectional curvature K, then $R^\ell_{ikj} = K(\delta^\ell_k g_{ij} - \delta^\ell_j g_{ik})$. Therefore the system (2.4) takes the form

$$f_{,ijk} + K(2\delta^\ell_k g_{ij} - \delta^\ell_j g_{ik} - \delta^\ell_i g_{jk}) f_{,\ell} = 2e^{2u} [g_{pk}(\delta^p_i u_{,j} + \delta^p_j u_{,i}) - g_{ij} u_{,k}].$$

2) If $u \in \mathcal{F}(M)$ is a constant function, then $u_{,j} = 0$ for all $j = 1, \dots, n$. The system (2.4) takes the form

$$f_{,ijk} + (R^{\ell}_{ikj} + R^{\ell}_{jki}) f_{,\ell} = 0.$$

Example 2.2 (conformal metric, [14]). Let us take $M = \mathbb{R}^n_+$ be the positive orthant endowed with the metric $g = (g_{ij})$, where

$$g_{ij}(x^1, \dots, x^n) = \begin{pmatrix} \dfrac{1}{(x^1)^{\frac{2(a+1)}{a}}} & & 0 \\ & \ddots & \\ 0 & & \dfrac{1}{(x^n)^{\frac{2(a+1)}{a}}} \end{pmatrix}, \quad a > 0.$$

It is known that $(\mathbb{R}^n_+, g)$ is a noncomplete Riemannian manifold with sectional curvature $K = 0$, having the Christoffel components of ∇_g: $\Gamma^p_{pp} = \dfrac{-\frac{a+1}{a}}{x^p}$ for all $p = \overline{1,n}$ and 0 otherwise.

We choose the function

$$f \colon \mathbb{R}^n_+ \to \mathbb{R}, \quad f(x^1, \dots, x^n) = \frac{a^2}{2(a+1)} (x^1)^{-\frac{2}{a}} + \dots + \frac{a^2}{2(a+1)} (x^n)^{-\frac{2}{a}}.$$

Let us compute $h = \nabla^2_g f$. We find

$$h_{ij} = \frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma^m_{ij} f_{,m} = \begin{cases} \dfrac{1}{a+2} \cdot \dfrac{1}{(x^i)^{\frac{2(a+1)}{a}}} & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases}$$

Therefore $(\mathbb{R}^n_+, h)$ is a Riemannian Hessian manifold and $h = \dfrac{1}{a+2} g$. Hence $f \in \mathcal{F}_g$.

3. CURVATURE TENSOR OF PSEUDO-RIEMANNIAN MANIFOLD (M, h), $h = \nabla^2_g f$

Suppose again that the Hessian $h = \nabla^2_g f$ is non-degenerate.

Theorem 3.1. *The curvature tensor field $\bar{R}$ of the pseudo-Riemannian manifold (M, h), $h = \nabla^2_g f$, has the components:*

$$\bar{R}^h_{ijk} = R^h_{ijk} + \frac{1}{2} \frac{\partial f,^{hp}}{\partial x^j} [f_{,kip} + (R^m_{kpi} + R^m_{ipk}) f_{,m}] - \frac{1}{2} \frac{\partial f,^{hp}}{\partial x^k} [f_{,jip} + (R^m_{jpi} + R^m_{ipj}) f_{,m}]$$

$$+ \frac{1}{2} f,^{hp} \Big\{ (\Gamma^r_{ki} f_{,jrp} - \Gamma^r_{ji} f_{,krp}) + \Big[\frac{\partial}{\partial x^j} (R^s_{kpi} - \Gamma^r_{pi} \Gamma^s_{kr}) - \frac{\partial}{\partial x^k} (R^s_{jpi} - \Gamma^r_{pi} \Gamma^s_{jr})$$

$$+\left(R^m_{ipk}+R^m_{kpi}+\frac{\partial\Gamma^m_{ki}}{\partial x^p}+\Gamma^r_{ki}\Gamma^m_{pr}\right)\Gamma^s_{jm}-\left(R^m_{ipj}+R^m_{jpi}+\frac{\partial\Gamma^m_{ji}}{\partial x^p}+\Gamma^r_{ji}\Gamma^m_{pr}\right)\Gamma^s_{km}$$

$$+\Gamma^r_{ki}(R^s_{jpr}+R^s_{rpj})-\Gamma^r_{ji}(R^s_{kpr}+R^s_{rpk})\Big]f_{,s}$$

$$+\left(R^s_{ipk}+R^s_{kpi}+\frac{\partial\Gamma^s_{ki}}{\partial x^p}+\Gamma^r_{ki}\Gamma^s_{pr}+\frac{\partial\Gamma^s_{pi}}{\partial x^k}\right)f_{,js}$$

$$-\left(R^s_{ipj}+R^s_{jpi}+\frac{\partial\Gamma^s_{ji}}{\partial x^p}+\Gamma^r_{ji}\Gamma^s_{pr}+\frac{\partial\Gamma^s_{pi}}{\partial x^j}\right)f_{,ks}$$

$$+\left(\frac{\partial\Gamma^s_{ji}}{\partial x^k}-\frac{\partial\Gamma^s_{ki}}{\partial x^j}\right)f_{,sp}+\left(\frac{\partial\Gamma^s_{pj}}{\partial x^k}-\frac{\partial\Gamma^s_{pk}}{\partial x^j}\right)f_{,si}$$

$$+\Gamma^s_{ji}\frac{\partial f_{,sp}}{\partial x^k}+\Gamma^s_{pj}\frac{\partial f_{,si}}{\partial x^k}+\Gamma^s_{pi}\frac{\partial f_{,sj}}{\partial x^k}-\Gamma^s_{ki}\frac{\partial f_{,sp}}{\partial x^j}-\Gamma^s_{pk}\frac{\partial f_{,si}}{\partial x^j}-\Gamma^s_{pi}\frac{\partial f_{,sk}}{\partial x^j}\Bigg\}$$

$$+\frac{1}{2}f,^{rp}\left\{[f_{,kip}+(R^m_{kpi}+R^m_{ipk})f_{,m}]\Gamma^h_{jr}-[f_{,jip}+(R^m_{jpi}+R^m_{ipj})f_{,m}]\Gamma^h_{kr}\right\}$$

$$+\frac{1}{4}f,^{rp}f,^{h\ell}\{[f_{,kip}+(R^m_{kpi}+R^m_{ipk})f_{,m}][f_{,jr\ell}+(R^m_{j\ell r}+R^m_{r\ell j})f_{,m}]$$

$$-[f_{,jip}+(R^m_{jpi}+R^m_{ipj})f_{,m}][f_{,kr\ell}+(R^m_{k\ell r}+R^m_{r\ell k})f_{,m}]\}\,.$$

Proof. By denoting

$$A^\ell_{ij}=\frac{1}{2}f,^{\ell p}\,[f_{,ijp}+(R^m_{ipj}+R^m_{jpi})f_{,m}],$$

the relation from the Theorem 2.1 becomes $\bar{\Gamma}^\ell_{ij}=\Gamma^\ell_{ij}+A^\ell_{ij}$. Then

$$\begin{aligned}\bar{R}^h_{ijk}&=\frac{\partial\bar{\Gamma}^h_{ki}}{\partial x^j}-\frac{\partial\bar{\Gamma}^h_{ji}}{\partial x^k}+\bar{\Gamma}^r_{ki}\bar{\Gamma}^h_{jr}-\bar{\Gamma}^r_{ji}\bar{\Gamma}^h_{kr}\\&=\frac{\partial\Gamma^h_{ki}}{\partial x^j}+\frac{\partial A^h_{ki}}{\partial x^j}-\frac{\partial\Gamma^h_{ji}}{\partial x^k}-\frac{\partial A^h_{ji}}{\partial x^k}+\Gamma^r_{ki}\Gamma^h_{jr}+\Gamma^r_{ki}A^h_{jr}+A^r_{ki}\Gamma^h_{jr}+A^r_{ki}A^h_{jr}\\&\quad-\Gamma^r_{ji}\Gamma^h_{kr}-\Gamma^r_{ji}A^h_{kr}-A^r_{ji}\Gamma^h_{kr}-A^r_{ji}A^h_{kr}\\&=R^h_{ijk}+\left(\frac{\partial A^h_{ki}}{\partial x^j}-\frac{\partial A^h_{ji}}{\partial x^k}\right)+(\Gamma^r_{ki}A^h_{jr}-\Gamma^r_{ji}A^h_{kr})\\&\quad+(A^r_{ki}\Gamma^h_{jr}+A^r_{ki}A^h_{jr}-A^r_{ji}\Gamma^h_{kr}-A^r_{ji}A^h_{kr}).\end{aligned}\tag{3.1}$$

By a direct computation,

$$\begin{aligned}\frac{\partial A^h_{ki}}{\partial x^j} - \frac{\partial A^h_{ji}}{\partial x^k} = &\frac{1}{2}\frac{\partial f,^{hp}}{\partial x^j}[f_{,kip} + (R^m_{kpi} + R^m_{ipk})f_{,m}] \\ &-\frac{1}{2}\frac{\partial f,^{hp}}{\partial x^k}[f_{,jip}+(R^m_{jpi}+R^m_{ipj})f_{,m}]+\frac{1}{2}f,^{hp}\left[\left(\frac{\partial f_{,kpi}}{\partial x^j}-\frac{\partial f_{jip}}{\partial x^k}\right)\right. \\ &+\left(\frac{\partial R^m_{kpi}}{\partial x^j} + \frac{\partial R^m_{ipk}}{\partial x^j} - \frac{\partial R^m_{jpi}}{\partial x^k} - \frac{\partial R^m_{ipj}}{\partial x^k}\right) f_{,m} \\ &\left.+\left(R^m_{kpi} + R^m_{ipk}\right)\frac{\partial^2 f}{\partial x^j \partial x^m} - \left(R^m_{jpi} + R^m_{ipj}\right)\frac{\partial^2 f}{\partial x^k \partial x^m}\right].\end{aligned} \tag{3.2}$$

In the following, we calculate $\frac{\partial f_{,kip}}{\partial x^j} - \frac{\partial f_{,jip}}{\partial x^k}$. The relation (2.3) from Theorem 2.1 becomes

$$f_{,kip} = \frac{\partial^3 f}{\partial x^k \partial x^i \partial x^p} - \left(\frac{\partial \Gamma^\ell_{ki}}{\partial x^p} + \Gamma^m_{ki}\Gamma^\ell_{pm}\right) f_{,\ell} - (\Gamma^\ell_{ki}f_{,\ell p} + \Gamma^\ell_{pk}f_{,\ell i} + \Gamma^\ell_{pi}f_{,\ell k})$$

or equivalent

$$f_{,kip} = \frac{\partial^3 f}{\partial x^k \partial x^i \partial x^p} - \left(R^\ell_{ipk} + \frac{\partial \Gamma^\ell_{pi}}{\partial x^k} + \Gamma^m_{pi}\Gamma^\ell_{km}\right) f_{,\ell} - (\Gamma^\ell_{ki}f_{,\ell p} + \Gamma^\ell_{pk}f_{,\ell i} + \Gamma^\ell_{pi}f_{,\ell k}).$$

Then we obtain

$$\begin{aligned}\frac{\partial f_{,kip}}{\partial x^j} = &\frac{\partial^4 f}{\partial x^j \partial x^k \partial x^i \partial x^p} - \left(\frac{\partial R^\ell_{ipk}}{\partial x^j} + \frac{\partial^2 \Gamma^\ell_{pi}}{\partial x^j \partial x^k} + \frac{\partial \Gamma^m_{pi}}{\partial x^j}\Gamma^\ell_{km} + \Gamma^m_{pi}\frac{\partial \Gamma^\ell_{km}}{\partial x^j}\right) f_{,\ell} + \\ &+\left(R^\ell_{ipk} + \frac{\partial \Gamma^\ell_{pi}}{\partial x^k} + \Gamma^m_{pi}\Gamma^\ell_{km}\right)\frac{\partial^2 f}{\partial x^j \partial x^\ell} \\ &-\left(\frac{\partial \Gamma^\ell_{ki}}{\partial x^j}f_{,\ell p} + \Gamma^\ell_{ki}\frac{\partial f_{,\ell p}}{\partial x^j} + \frac{\partial \Gamma^\ell_{pk}}{\partial x^j}f_{,\ell i} + \Gamma^\ell_{pk}\frac{\partial f_{,\ell i}}{\partial x^j} + \frac{\partial \Gamma^\ell_{pi}}{\partial x^j}f_{,\ell k} + \Gamma^\ell_{pi}\frac{\partial f_{,\ell k}}{\partial x^j}\right)\end{aligned}$$

and a similar formula for $\frac{\partial f_{,jip}}{\partial x^k}$. Therefore, the relation (3.2) is equivalent to

$$\begin{aligned}\frac{\partial A^h_{ki}}{\partial x^j} - \frac{\partial A^h_{ji}}{\partial x^k} = &\frac{1}{2}\frac{\partial f,^{hp}}{\partial x^j}[f_{,kip}+(R^m_{kpi} + R^m_{ipk})f_{,m}] - \frac{1}{2}\frac{\partial f,^{hp}}{\partial x^k}[f_{,jip} + (R^m_{jpi} + R^m_{ipj})f_{,m}] \\ &+\frac{1}{2}f,^{hp}\left\{\left[\frac{\partial}{\partial x^j}(R^s_{kpi} - \Gamma^r_{pi}\Gamma^s_{kr}) - \frac{\partial}{\partial x^k}(R^s_{jpi} - \Gamma^r_{pi}\Gamma^s_{jr}) + \right.\right. \\ &+\left(R^m_{ipk} + R^m_{kpi} + \frac{\partial \Gamma^m_{ki}}{\partial x^p} + \Gamma^r_{ki}\Gamma^m_{pr}\right)\Gamma^s_{jm}\end{aligned}$$

$$
\left. - \left(R^m_{ipj} + R^m_{jpi} + \frac{\partial \Gamma^m_{ji}}{\partial x^p} + \Gamma^r_{ji}\Gamma^m_{pr} \right) \Gamma^s_{km} \right] f_{,s}
$$

$$
+ \left(R^s_{ipk} + R^s_{kpi} + \frac{\partial \Gamma^s_{ki}}{\partial x^p} + \Gamma^r_{ki}\Gamma^s_{pr} + \frac{\partial \Gamma^s_{pi}}{\partial x^k} \right) f_{,js}
$$

$$
- \left(R^s_{ipj} + R^s_{jpi} + \frac{\partial \Gamma^s_{ji}}{\partial x^p} + \Gamma^r_{ji}\Gamma^s_{pr} + \frac{\partial \Gamma^s_{pi}}{\partial x^j} \right) f_{,ks}
$$

$$
+ \left(\frac{\partial \Gamma^s_{ji}}{\partial x^k} - \frac{\partial \Gamma^s_{ki}}{\partial x^j} \right) f_{,sp} + \left(\frac{\partial \Gamma^s_{pj}}{\partial x^k} - \frac{\partial \Gamma^s_{pk}}{\partial x^j} \right) f_{,si} + \Gamma^s_{ji} \frac{\partial f_{,sp}}{\partial x^k}
$$

$$
\left. + \Gamma^s_{pj} \frac{\partial f_{,si}}{\partial x^k} + \frac{\partial f_{,sj}}{\partial x^k} \Gamma^s_{pi} - \Gamma^s_{ki} \frac{\partial f_{,sp}}{\partial x^j} - \Gamma^s_{pk} \frac{\partial f_{,si}}{\partial x^j} - \frac{\partial f_{,sk}}{\partial x^j} \Gamma^s_{pi} \right\} .
$$

The expression $\Gamma^r_{ki}A^h_{jr} - \Gamma^r_{ji}A^h_{kr}$ from the relation (3.1) has the equivalent form

$$
\frac{1}{2} f,^{hp} \left\{ (\Gamma^r_{ki} f_{,jrp} - \Gamma^r_{ji} f_{,krp}) + [\Gamma^r_{ki}(R^s_{jpr} + R^s_{rpj}) - \Gamma^r_{ji}(R^s_{kpr} + R^s_{rpk})] f_{,s} \right\}
$$

and also, the expression $A^r_{ki}\Gamma^h_{jr} + A^r_{ki}A^h_{jr} - A^r_{ji}\Gamma^h_{kr} - A^r_{ji}A^h_{kr}$ from the relation (3.1) takes the equivalent form

$$
\frac{1}{2} f,^{rp} \left\{ [f_{,kip} + (R^m_{kpi} + R^m_{ipk}) f_{,m}]\Gamma^h_{jr} - [f_{,jip} + (R^m_{jpi} + R^m_{ipj}) f_{,m}]\Gamma^h_{kr} \right\} +
$$

$$
+ \frac{1}{4} f,^{rp} f,^{h\ell} \left\{ [f_{,kip} + (R^m_{kpi} + R^m_{ipk}) f_{,m}][f_{,jr\ell} + (R^m_{j\ell r} + R^m_{r\ell j}) f_{,m}] - \right.
$$

$$
\left. - [f_{,jip} + (R^m_{jpi} + R^m_{ipj}) f_{,m}][f_{,kr\ell} + (R^m_{k\ell r} + R^m_{r\ell k}) f_{,m}] \right\} .
$$

Then the relation (3.1) becomes the relation from the Theorem 3.1. □

Remark 3.1. The formula for the components of the curvature tensor $\bar{R}^h_{ijk}$ involves only first, second and third order derivatives of the function f, and no fourth order ones as one would a priori expected.

Particular case. If the initial Riemannian manifold (M, g) is the Euclidean space, then $\Gamma^k_{ij} = 0$, $R^h_{ijk} = 0$, $f_{,ij} = \dfrac{\partial^2 f}{\partial x^i \partial x^j}$, $f_{,ijk} = \dfrac{\partial^3 f}{\partial x^i \partial x^j \partial x^k}$ and the formula for $\bar{R}^h_{ijk}$ becomes

$$
\bar{R}^h_{ijk} = \frac{1}{2} \left(\frac{\partial f,^{hp}}{\partial x^j} f_{,kip} - \frac{\partial f,^{hp}}{\partial x^k} f_{,jip} \right) + \frac{1}{4} f,^{rp} f,^{h\ell} (f_{,kip} f_{,jr\ell} - f_{,jip} f_{,kr\ell}).
$$

Then

$$
\begin{aligned}
\bar{R}_{aijk} &= f_{,ah} \bar{R}^h_{ijk} = \frac{1}{2} f_{,ah} \left(\frac{\partial f,^{hp}}{\partial x^j} f_{,kip} - \frac{\partial f,^{hp}}{\partial x^k} f_{,jip} \right) \\
&\quad + \frac{1}{4} f,^{rp} (f_{,kip} f_{,jra} - f_{,jip} f_{,kra}).
\end{aligned}
$$

On the other hand, we have $f,^{ph} f_{,ha} = \delta^p_a$. If we make the derivative with respect to x^j, by multiplying with $f_{,kip}$ and summing for $p = \overline{1,n}$ we obtain

$$\frac{\partial f,^{ph}}{\partial x^j} f_{,ha} f_{,kip} + f,^{ph} f_{,jha} f_{,kip} = 0.$$

Also we obtain

$$\frac{\partial f,^{ph}}{\partial x^k} f_{,ha} f_{,jip} + f,^{ph} f_{,kha} f_{,jip} = 0.$$

Hence

$$f_{,ha}\left(\frac{\partial f,^{ph}}{\partial x^j} f_{,kip} - \frac{\partial f,^{ph}}{\partial x^k} f_{,jip}\right) = -f_,^{pr}(f_{,jra} f_{,kip} - f_{,kra} f_{,jip}).$$

Therefore, we obtained the well-known formula

$$\begin{aligned}\bar{R}_{aijk} &= -\frac{1}{2} f,^{pr} \left(f_{,jra} f_{,kip} - f_{,kra} f_{,jip}\right) + \frac{1}{4} f,^{pr} \left(f_{,kip} f_{,jra} - f_{,jip} f_{,kra}\right) \\ &= -\frac{1}{4} f,^{pr} \left(f_{,kip} f_{,jra} - f_{,jip} f_{,kra}\right).\end{aligned}$$

In the following, we give some examples of pseudo-Riemannian manifolds whose curvatures vanish identically.

Example 3.1. Consider the Riemannian manifold $(\mathbb{R}^2, g)$, where $g_{ij}(x) = \delta_{ij}$ and the subset $A = \{(x^1, x^2) \in \mathbb{R}^2 : (x^1)^2 - (x^2)^2 > 0\}$. We also consider the function $f \colon A \to \mathbb{R}$, $f(x^1, x^2) = \ln((x^1)^2 - (x^2)^2)$. The Hessian of the function f has the form

$$\nabla^2_\delta f = \begin{pmatrix} \dfrac{-2((x^1)^2 + (x^2)^2)}{((x^1)^2 - (x^2)^2)^2} & \dfrac{4x^1x^2}{((x^1)^2 - (x^2)^2)^2} \\ \dfrac{4x^1x^2}{((x^1)^2 - (x^2)^2)^2} & \dfrac{-2((x^1)^2 + (x^2)^2)}{((x^1)^2 - (x^2)^2)^2} \end{pmatrix}.$$

Then $\det(\nabla^2_\delta f) = \dfrac{4}{((x^1)^2 - (x^2)^2)^2} \neq 0$ and $\nabla^2_\delta f$ has the constant signature $(0, 2)$. Hence $(A, \nabla^2_\delta f)$ is a pseudo-Riemannian manifold. Moreover

$$(\nabla^2_\delta f)^{-1} = \begin{pmatrix} -\dfrac{(x^1)^2 + (x^2)^2}{2} & -x^1x^2 \\ -x^1x^2 & -\dfrac{(x^1)^2 + (x^2)^2}{2} \end{pmatrix}.$$

Since $\Gamma^k_{ij} = 0$, $R^\ell_{ijk} = 0$ for all $i, j, k, \ell = \overline{1,2}$ we have

$$f_{,ijk} = \frac{\partial^3 f}{\partial x^i \partial x^j \partial x^k} \quad \text{and} \quad \bar{\Gamma}^p_{ij} = \frac{1}{2} f,^{pk} \frac{\partial^3 f}{\partial x^i \partial x^j \partial x^k}.$$

Therefore $f_{,121} = f_{,211} = f_{,112} = \dfrac{-4(x^2)^3 - 12(x^1)^2x^2}{((x^1)^2 - (x^2)^2)^3}$, $f_{,221} = f_{,212} = f_{,122} = \dfrac{4(x^1)^3 + 12x^1(x^2)^2}{((x^1)^2 - (x^2)^2)^3}$, $f_{,111} = \dfrac{4(x^1)^3 + 12x^1(x^2)^2}{((x^1)^2 - (x^2)^2)^3}$, $f_{,222} = \dfrac{-4(x^2)^3 - 12(x^1)^2x^2}{((x^1)^2 - (x^2)^2)^3}$. From

Theorem 3.1, we deduce that

$$\bar{R}_{aijk} = -\frac{1}{4} f,^{pr} \left(f_{,kip} f_{,jra} - f_{,jip} f_{,kra} \right) = 0.$$

Hence the sectional curvature is constant $\bar{K} = \dfrac{\bar{R}_{2121}}{\det(\nabla_\delta^2 f)} = 0.$

Example 3.2. Consider the Riemannian manifold $(\mathbb{R}^2, g)$, where $g_{ij}(x) = \delta_{ij}$ and the subset

$$A = \{(x^1, x^2) \in \mathbb{R}^2 : (x^1, x^2) \neq (0,0)\}.$$

Also, we consider the function $f\colon A \to \mathbb{R}$, $f(x^1, x^2) = \ln((x^1)^2 + (x^2)^2)$. The Hessian of f has the form

$$\nabla_\delta^2 f = \begin{pmatrix} \dfrac{2((x^2)^2 - (x^1)^2)}{((x^1)^2 + (x^2)^2)^2} & \dfrac{-4x^1 x^2}{((x^1)^2 + (x^2)^2)^2} \\ \dfrac{-4x^1 x^2}{((x^1)^2 + (x^2)^2)^2} & \dfrac{2((x^1)^2 - (x^2)^2)}{((x^1)^2 + (x^2)^2)^2} \end{pmatrix}.$$

Then $\det(\nabla_\delta^2 f) = \dfrac{-4}{((x^1)^2 + (x^2)^2)^2} \neq 0$ and $\nabla_\delta^2 f$ has the constant signature (1,1). This means that $\nabla_\delta^2 f$ is Lorentzian.

Using a similar technique, we find that the pseudo-Riemannian manifold $(\mathbb{R}^2, \nabla_\delta^2 f)$ has the sectional curvature $\bar{K} = 0$.

4. REMARKABLE METRICS OF RIEMANNIAN HESSIAN TYPE

The aim of this section is to study remarkable examples of Riemannian Hessian metrics and their geometries.

Example 4.1. Let us consider the Riemannian manifold $(\mathbb{R}^2, g)$, where $g_{ij}(x) = \delta_{ij}$ and the subset $A = \{(x^1, x^2) \in \mathbb{R}^2 : x^2 > a(x^1)^2 + k,\ a > 0,\ k \in \mathbb{R}\}$. We also take the function $f\colon A \to \mathbb{R}$, $f(x^1, x^2) = -\ln(x^2 - a(x^1)^2 - k)$. The Hessian of f has the form

$$\nabla_\delta^2 f = \begin{pmatrix} \dfrac{2ax^2 + 2a^2(x^1)^2 - 2ak}{(x^2 - a(x^1)^2 - k)^2} & \dfrac{-2ax^1}{(x^2 - a(x^1)^2 - k)^2} \\ \dfrac{-2ax^1}{(x^2 - a(x^1)^2 - k)^2} & \dfrac{1}{(x^2 - a(x^1)^2 - k)^2} \end{pmatrix}.$$

Then $\det(\nabla_\delta^2 f) = \dfrac{2a}{(x^2 - a(x^1)^2 - k)^3} > 0$ for $(x^1, x^2) \in A$ and $\nabla_\delta^2 f$ has the constant signature $(2,0)$. Hence $\nabla_\delta^2 f$ is a family of Riemannian metrics on A. We proved in [14] that the manifold $(A, \nabla_\delta^2 f)$ has the sectional curvature $\bar{K} = -\dfrac{1}{4}$. We also found the family of geodesic curves:

Proposition 4.1. 1) *For any points* $(x_0^1, x_0^2) \neq (x_1^1, x_1^2)$ *from* A*, the coresponding geodesic has the form* $x^2 = \frac{1}{2}a(x^1)^2 + mx^1 + n$ *for some constants* m *and* n*.*

2) *If* $(x_0^1, x_0^2) = (x_1^1, x_1^2)$*, then the unique geodesic is reduced to a straight line.*

Remark 4.1. We consider the set A in the form $A = \{(x^1, x^2) : x^2 > (x^1)^2\}$ and $f(x^1, x^2) = -\ln(x^2 - (x^1)^2)$. Since the set A is the interior of the parabola $x^2 = (x^1)^2$ and $(A, \nabla_\delta^2 f)$ has the sectional curvature $\bar{K} = -\frac{1}{4}$ we consider the diffeomorphism

$$\phi\colon \mathbb{R}^2 \to \mathbb{R}^2, \quad \phi(x^1, x^2) = (\phi^1(x^1, x^2), \phi^2(x^1, x^2)) = (x^1, x^2 + (x^1)^2).$$

We have the Jacobian $J_\phi = 1 \neq 0$ and the inverse $\phi^{-1}\colon \mathbb{R}^2 \to \mathbb{R}^2$ is given by the formula $\phi^{-1}(\bar{x}^1, \bar{x}^2) = (\bar{x}^1, \bar{x}^2 - (\bar{x}^1)^2)$. We note that ϕ transforms the set $H = \{(x^1, x^2) : x^2 > 0\}$ into the interior of the parabola $\{(x^1, x^2) : x^2 > (x^1)^2\}$.

We recall that two Riemannian manifolds (M, g) and (M', g') are equivalent if there exists a diffeomorphism $\phi\colon M \to M'$ such that

$$g_{ij}(x) = g'_{rs}(\phi(x)) \cdot \left.\frac{\partial \phi^r}{\partial x^i}\right|_x \cdot \left.\frac{\partial \phi^s}{\partial x^j}\right|_x,$$

where $\phi^r = x'^r \circ \phi$.

Therefore, if the set $H = \{(x^1, x^2) : x^2 > 0\}$ is endowed with the Poincaré metric $g = \begin{pmatrix} \frac{1}{(x^2)^2} & 0 \\ 0 & \frac{1}{(x^2)^2} \end{pmatrix}$, then we determine the metric g' (defined on the set A) as: $g' = \begin{pmatrix} \frac{1 + 4(x^1)^2}{(x^2)^2} & \frac{-2x^1}{(x^2)^2} \\ \frac{-2x^1}{(x^2)^2} & \frac{1}{(x^2)^2} \end{pmatrix}$. Hence (A, g') is izometric to the Poincaré plane (H, g). We also note that g' is conformal to Rosenbrock's banana metric $g'' = \begin{pmatrix} 1 + 4(x^1)^2 & -2x^1 \\ -2x^1 & 1 \end{pmatrix}$.

Example 4.2. Let us take $M = \mathbb{R}^n$ endowed with the Euclidean metric $g = (g_{ij})$, where $g_{ij}(x) = \delta_{ij}$. Choosing the function

$$f\colon M \to \mathbb{R}, \quad f(x^1, \ldots, x^n) = \frac{1}{4c^2}e^{-2cx^1} + \ldots + \frac{1}{4c^2}e^{-2cx^n}, \ c > 0,$$

we deduce $f_{,i} = -\frac{1}{2c} e^{-2cx^i}$, $f_{,ii} = e^{-2cx^i}$ for all $i = \overline{1,n}$ and $f_{,ij} = 0$ for $i \neq j$. Then the Hessian

$$h = \nabla_\delta^2 f(x^1, \dots, x^n) = \begin{pmatrix} e^{-2cx^1} & 0 & 0 & \dots & 0 \\ 0 & e^{-2cx^2} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & e^{-2cx^n} \end{pmatrix}$$

is positive definite, hence $(\mathbb{R}^n, \nabla_\delta^2 f)$ is a Riemannian manifold.

From Theorem 2.1, we have $\bar{\Gamma}_{pp}^p = -c$ for all $p = \overline{1,n}$ and 0 otherwise. From Corollary 2.1, we get the system of differential equations of geodesics $\ddot{x}^p - c(\dot{x}^p)^2 = 0$ for all $p = \overline{1,n}$. Then we obtain $x^p = -\frac{1}{c} \ln(a_p t + b_p)$, where a_p and b_p are real constants. Since the definition domain of the geodesics must satisfy the positivity of the values $a_p t + b_p$, $p = \overline{1,n}$, it follows that $(\mathbb{R}^n, \nabla_\delta^2 f)$ is a manifold, which is not complete. From Theorem 3.1, we deduce that the sectional curvature $\bar{K}$ of $(\mathbb{R}^n, \nabla_\delta^2 f)$ is null.

In the following, let us compute $\nabla_h^2 f$: $k_{ij} = \frac{\partial^2 f}{\partial x^i \partial x^j} - \bar{\Gamma}_{ij}^m f_{,m} = 0$ if $i \neq j$ and $k_{ii} = \frac{1}{2} e^{-2cx^i}$ for all $i = \overline{1,n}$. Therefore $k = \nabla_h^2 f = \frac{1}{2} h$, hence k produces the same connection as h. It appears a constant sequence

$$\delta \to h = \nabla_h^2 f \to k = \nabla_h^2 f = \frac{1}{2} h.$$

Remark 4.2. 1) As in example of conformal metric, we may write that $f \in \mathcal{F}_h$.

2) The Riemannian metric h, introduced in Example 4.2, has been studied by Antonelli [2], in relation with population ecosystems.

3) In [8], T. Rapcsák and T. Csendes use the metric h in order to discuss nonlinear coordinate transformations.

We remark that the study of the metric h is equivalent to a nonlinear coordinate transformation. These transformations are discussed in order to clarify some structural properties of global unconstrained optimization problems.

Example 4.3. We present a Hessian metric type which appears in Statistics [1].

A family $S = \{p(x, \theta)\}$ of distributions is said to be an exponential family when the density function has the form $p(x, \theta) = \exp(\theta^i x_i - \psi(\theta))$ with respect to some common dominating measure P, by taking an appropriate parametrization $\theta = (\theta^i)$ and appropriate random variables $x = (x_i)$. The parameter θ is called the natural parameter of the exponential family.

In the following, we study the geometrical structures of the manifold S of an exponential family in terms of the natural coordinate system θ.

From the relation $l(x, \theta) = \ln p(x, \theta)$ (equivalent to $l(x, \theta) = \theta^i x_i - \psi(\theta)$), it follows that $l_{,i}(x, \theta) = x_i - \psi_{,i}(\theta)$ and $l_{,ij}(x, \theta) = -\psi_{,ij}(\theta)$. Therefore the normalization factor $\psi(\theta)$ defined by $\psi(\theta) = \ln \int \exp(\theta^i x_i) \mathrm{d}P$ plays a fundamental role. It is a potential function.

Directly from the definition, $E[f(x)] = \int f(x)p(x,\theta)\mathrm{d}P$, the expectation, covariance and third-order central moments of x_i are given by

$$\begin{aligned} &E[x_i] = \psi_{,i}(\theta), \qquad \operatorname{cov}[x_i, x_j] = \psi_{,ij}(\theta), \\ &E[(x_i, -\psi_{,i})(x_j - \psi_{,j})(x_k - \psi_{,k})] = \psi_{,ijk}(\theta) \end{aligned}$$

respectively.

The next theorem states that the metric tensor is of the Hessian type.

Theorem 4.1 ([1]). *The metric tensor and the α-connection of an exponential family are given by $g_{ij} = \psi_{,ij}(\theta)$, $\Gamma^{(\alpha)}_{ijk}(\theta) = \frac{1-\alpha}{2}\psi_{,ijk}(\theta)$ in the natural coordinate system.*

The proof results from the relations $g_{ij}(\theta) = -E[\ell_{,ij}(x,\theta)]$ and $\ell_{,i} = x_i - \psi_{,i}$. Also the α-connection $\Gamma^{(\alpha)}_{ijk}(\theta)$ is obtained from the following relations:

$$\begin{aligned} &E[\ell_{,ijk}] = E[-(\psi_{,ij})\ell_{,k}(x,\theta)] = 0, \\ &\Gamma^{(\alpha)}_{ijk}(\theta) = E\left[\left\{\ell_{,ij}(x,\theta) + \frac{1-\alpha}{2}\cdot \ell_{,i}(x,\theta)\ell_{,j}(x,\theta)\right\}\ell_{,k}(x,\theta)\right], \end{aligned}$$

or equivalent $\Gamma^{(\alpha)}_{ijk} = \Gamma^{(1)}_{ijk} + \frac{1-\alpha}{2}T_{ijk}$, where $T_{ijk}(\theta) = E[\ell_{,i}\ell_{,j}\ell_{,k}] = \psi_{,ijk}(\theta)$.

Moreover, the Riemannian-Christoffel curvature tensor has the form given by Theorem 3.1

$$R_{ijkh}(\theta) = \frac{1-\alpha^2}{4} g^{mn}(T_{kmi}T_{jhn} - T_{kmj}T_{ihn}),$$

which vanishes for $\alpha = \pm 1$.

For example, the family $S = \{N(\mu,\sigma^2)\}$ of normal distributions is of the exponential type. Indeed, the density function of $N(\mu,\sigma^2)$ is

$$p(x,\mu,\sigma^2) = \exp\left\{\left(\frac{\mu}{\sigma^2}\right)x - \left(\frac{1}{2\sigma^2}\right)x^2 - \left(\frac{\mu^2}{2\sigma^2}\right) - \ln(\sqrt{2\pi}\cdot\sigma)\right\}.$$

Consider a new two-dimensional parameter $\theta = (\theta^1, \theta^2)$ defined by $\theta^1 = \frac{\mu}{\sigma^2}$, $\theta^2 = -\frac{1}{(2\sigma^2)}$ and a new two-dimensional random variable $x = (x_1, x_2)$ by $x_1 = x$, $x_2 = x^2$. The density function becomes $p(x,\theta) = \exp(\theta^i x_i - \psi(\theta))$, where

$$\psi(\theta) = -\frac{(\theta^1)^2}{4\theta^2} - \frac{1}{2}\ln(-\theta^2) + \frac{1}{2}\ln\pi.$$

Then S is an exponential family with the natural parameter $\theta = (\theta^1, \theta^2)$. The random variables x_1 and x_2 are related by $x_2 = (x_1)^2$, so that the dominating measure $P(x)$ is concentrated on the parabola $x_2 = (x_1)^2$ in the (x_1, x_2)-plane. The Hessian metric tensor is given by

$$g_{ij}(\theta) = \begin{pmatrix} \sigma^2 & 2\mu\sigma^2 \\ 2\mu\sigma^2 & 4\mu^2\sigma^2 + 2\sigma^4 \end{pmatrix},$$

where μ and σ^2 are considered as function of the parameter θ.

The tensor T_{ijk} has the components $T_{111} = 0$, $T_{112} = 2\sigma^2$, $T_{122} = 8\mu\sigma^4$, $T_{222} = 24\mu^2\sigma^4 + 8\sigma^6$ and the α-connection is $\Gamma^{(\alpha)}_{ijk} = \dfrac{1-\alpha}{2} \cdot T_{ijk}$.

The Riemannian-Christoffel curvature R_{ijkm} is given by $R_{1212} = (1 - \alpha^2)\sigma^6$.

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