J. Adv. Math. Studies
Vol. **2**(2009), No. 1, 107-112
`http://journal.fairpartners.ro`

HELICES OF THE 3-DIMENSIONAL FINSLER MANIFOLDS

MÜNEVVER YILDIRIM YILMAZ AND MEHMET BEKTAŞ

ABSTRACT. Simply a helix is a twisted, three-dimensional shape. There has been growing interest for analysis of helices. This growing trends due to the following features. It has a large importance scale from mathematics to biology. Natural scientists have long held a fascination, some bordering on mystical obsession for helical structure in nature. They arise nanosprings, α-helices, DNA double and collagen triple helix, [5, 9]. Helices are also can be used in the fields of computer aided design and computer graphics [10].

T. Ikawa obtained in [7] the following characteristic ordinary differential equation

$$\nabla_X \nabla_X \nabla_X X - K \nabla_X X = 0, \quad K = k^2 - \tau^2$$

for the circular helix which corresponds to the case that the curvatures k and τ of a time-like curve α on the Lorentzian manifold M are constant.

N. Ekmekçi and H. H. Hacisalihoglu generalized in [6] T. Ikawa's this result, i.e. k and τ are variable, but $\frac{k}{\tau}$ is constant.

In [2] H. Balgetir, M. Bektaş and M. Ergüt obtained a geometric characterization of null Frenet curve with constant ratio of curvature and torsion (called null general helix).

Our study is inspired by the recent studies indicated above. In this paper after a short description of Finsler manifolds, we give two characterizations for a curve with respect to the Frenet frame of the 3-dimensional Finsler manifold $\mathbb{F}^3$.

1. PRELIMINARIES

Finsler geometry is the most natural generalization of Riemannian geometry. It started in 1918 when P. Finsler wrote his thesis on curves and surfaces in what he called generalized metric spaces. Due to its importance it has a huge research field from geometry to biology, physics and also engineering and computer sciences, [1], [4], [8]. The following part of the study is on the basic concepts of the Finsler manifolds

Definition 1.1. Let M be a real m-dimensional smooth manifold and TM be the tangent bundle of M. Denote by Π the *canonical projection of* TM *on* M.

Received: December 12, 2008.
2000 *Mathematics Subject Classification*: 53B40,58B20.
Key words and phrases: Finsler manifold, general helix.

Let $M^{'}$ be an non-empty open submanifold of TM such that $\Pi\big(M^{'}\big) = M$ and $\theta(M) \cap M^{'} = \emptyset$, where θ is the zero section of TM.

We now consider a smooth function $F\colon M^{'} \to (0,\infty)$ and take $F^{*} = F^{2}$. Then suppose that for any coordinate system $\left\{\big(u^{'},\Phi^{'}\big); x^{i}, y^{i}\right\}$ in $M^{'}$, the following conditions are fulfilled:

(F_1) F is positively homogeneus of degree one with respect to $\big(y^{1},\ldots,y^{m}\big)$ i.e. we have

$$F\left(x^{1},\ldots,x^{m},ky^{1},\ldots,ky^{m}\right) = kF\left(x^{1},\ldots,x^{m},y^{1},\ldots,y^{m}\right)$$

for any $(x,y) \in \Phi^{'}\big(U^{'}\big)$ and any $k > 0$.

(F_2) At any point $(x,y) \in \Phi^{'}\big(U^{'}\big)$

$$g_{ij}\left(X,Y\right) = \frac{1}{2}\,\frac{\partial^{2}F^{*}}{\partial y^{i}\partial y^{j}}\left(X,Y\right), \quad i,j \in \{1,\ldots,m\}$$

are the components of a positive definite quadratic form on $\mathbb{R}^{m}$, [3].

We say that the triple $\mathbb{F}^{m} = \big(M,\, M^{'}, F\big)$ with satisfying (F_1) and (F_2) is a Finsler manifold and F is the fundamental function of $\mathbb{F}^{m}$.

Definition 1.2. Let $\mathbb{F}^{m+1} = \big(M, M^{'}, F\big)$ be a Finsler manifold and $\mathbb{F}^{'} = \big(C, C^{'}, F\big)$ be a 1-dimensional Finsler submanifold of $\mathbb{F}^{m+1}$, where C is a smooth curve in M given locally by the equations

$$x^{i} = x^{i}\left(s\right), \quad i \in \left\{1,\ldots,m+n\right\},\ s \in (a,b)$$

s being the arclength parameter on C. Denote by (s,v) the coordinates on $C^{'}$. Then we have

$$y^{i}\left(s,v\right) = v\frac{dx^{i}}{ds}, \quad i \in \{0,\ldots,m\}$$

Moreover $\left\{\dfrac{\partial}{\partial s}, \dfrac{\partial}{\partial v}\right\}$ is a natural field of frames on C where $\dfrac{\partial}{\partial v}$ is a unit Finsler vector field, [3].

Definition 1.3. Let $\mathbb{F}^{3} = \big(M,\, M^{'}, F\big)$ be a 3-dimensional Finsler manifold and C be a smooth curve in M given locally by the parametric equations

$$x^{i} = x^{i}\left(s\right); \quad \left(x^{'1}\left(s\right),\ x^{'2},\ x^{'3}\left(s\right)\right) \neq (0,0,0)$$

where s is the arclength parameter on C.

Then we have

$$\begin{aligned}\nabla^{*}_{\frac{\partial}{\partial s}}\frac{\partial}{\partial v} &= kn,\\ \nabla^{*}_{\frac{\partial}{\partial s}} n &= -k\frac{\partial}{\partial v} + \tau b\\ \nabla^{*}_{\frac{\partial}{\partial s}} b &= \tau n,\end{aligned} \tag{1.1}$$

where n and b called principal normal Finsler vector field and binormal Finsler vector field on C respestively. We are entiteled to call $\left\{\frac{\partial}{\partial v}, n, b\right\}$ be the Frenet frame for the curve C in $\mathbb{F}^3$. As in the Riemannian case we call k the curvature and τ the torsion of C respesctively, [3].

2. CHARACTERIZATIONS OF A HELIX IN $\mathbb{F}^3$

Definition 2.1. Let C be a curve of a Finsler manifold $\mathbb{F}^3$ and $\left\{\frac{\partial}{\partial v}, n, b\right\}$ be the Frenet frame on $\mathbb{F}^3$ along C. If k and τ are positive constants along C, then C is called *a circular helix with respect to the Frenet frame.*

Definition 2.2. Let C be a curve of a Finsler manifold $\mathbb{F}^3$. C is a general helix with respect to Frenet frame $\left\{\frac{\partial}{\partial v}, n, b\right\}$ be the Frenet frame on $\mathbb{F}^3$ along C. A curve C such that

$$\frac{k}{\tau} = \text{const.}$$

is called *a general helix with respect to Frenet frame.*

Theorem 2.1. *Let C be a curve of a Finsler manifold $\mathbb{F}^3$. C is a general helix with respect to the Frenet frame $\left\{\frac{\partial}{\partial v}, n, b\right\}$ if and only if*

$$\nabla^*_{\frac{\partial}{\partial s}} \nabla^*_{\frac{\partial}{\partial s}} \nabla^*_{\frac{\partial}{\partial s}} \frac{\partial}{\partial v} - K_1 \nabla^*_{\frac{\partial}{\partial s}} \frac{\partial}{\partial v} = 3k' \nabla^*_{\frac{\partial}{\partial s}} n, \tag{2.1}$$

where $K_1 = \frac{k''}{k} + \tau^2 - k^2$.

Proof. Suppose that C is general helix with respect to the Frenet frame $\left\{\frac{\partial}{\partial v}, n, b\right\}$. Then from (1.1) , we have

$$\nabla^*_{\frac{\partial}{\partial s}} \nabla^*_{\frac{\partial}{\partial s}} \nabla^*_{\frac{\partial}{\partial s}} \frac{\partial}{\partial v} = -3kk' \frac{\partial}{\partial v} + \left(k'' + k\tau^2 - k^3\right) n + \left(3k'\tau\right) b. \tag{2.2}$$

Now, since C is general helix with respect to the Frenet frame then by

$$\frac{k}{\tau} = \text{const.}$$

and this upon the derivation gives rise to

$$k'\tau = k\tau'. \tag{2.3}$$

If we substitute the equations (2.3),

$$n = \frac{1}{k} \nabla^*_{\frac{\partial}{\partial s}} \frac{\partial}{\partial v}, \tag{2.4}$$

and

$$b = \frac{1}{\tau}\nabla^*_{\frac{\partial}{\partial s}} n + \frac{k}{\tau}\frac{\partial}{\partial v} \tag{2.5}$$

in (2.2), we obtain (2.1).

Conversely let us assume that the equation (2.1) holds. We show that the curve C is a general helix. Differentiating covariantly (2.4) we obtain

$$\nabla^*_{\frac{\partial}{\partial s}} n = -\frac{k'}{k^2}\nabla^*_{\frac{\partial}{\partial s}}\frac{\partial}{\partial v} + \frac{1}{k}\nabla^*_{\frac{\partial}{\partial s}}\nabla^*_{\frac{\partial}{\partial s}}\frac{\partial}{\partial v} \tag{2.6}$$

and so

$$\nabla^*_{\frac{\partial}{\partial s}}\nabla^*_{\frac{\partial}{\partial s}} n = \left(-\frac{k'}{k^2}\right)'\nabla^*_{\frac{\partial}{\partial s}}\frac{\partial}{\partial v} - 2\frac{k'}{k^2}\nabla^*_{\frac{\partial}{\partial s}}\nabla^*_{\frac{\partial}{\partial s}}\frac{\partial}{\partial v} + \frac{1}{k}\nabla^*_{\frac{\partial}{\partial s}}\nabla^*_{\frac{\partial}{\partial s}}\nabla^*_{\frac{\partial}{\partial s}}\frac{\partial}{\partial v}. \tag{2.7}$$

If we use (2.1) in (2.7) and making some calculations, we have

$$\nabla^*_{\frac{\partial}{\partial s}}\nabla^*_{\frac{\partial}{\partial s}} n = \left[\left(-\frac{k'}{k^2}\right)' + \frac{K_1}{k}\right]\nabla^*_{\frac{\partial}{\partial s}}\frac{\partial}{\partial v} - k^{'}\frac{\partial}{\partial v} - 2\frac{(k')^2}{k^2}n + \frac{k'\tau}{k}b. \tag{2.8}$$

Also we obtain

$$\nabla^*_{\frac{\partial}{\partial s}}\nabla^*_{\frac{\partial}{\partial s}} n = -k^{'}\frac{\partial}{\partial v} - \left(k^2 - \tau^2\right)n + \tau' b \tag{2.9}$$

since (2.8) and (2.9) are equal, routine calculations show that C is a general helix. □

Theorem 2.2. *Let C be a curve of Finsler manifold $\mathbb{F}^3$. C is a general helix with respect to the Frenet frame $\left\{\frac{\partial}{\partial v}, n, b\right\}$, if and only if*

$$\nabla^*_{\frac{\partial}{\partial s}}\nabla^*_{\frac{\partial}{\partial s}}\nabla^*_{\frac{\partial}{\partial s}}\frac{\partial}{\partial v} - K_1\nabla^*_{\frac{\partial}{\partial s}}\frac{\partial}{\partial v} - K_2\frac{\partial}{\partial v} = 3\lambda\tau'\nabla^*_{\frac{\partial}{\partial s}} n \tag{2.10}$$

where $K_2 = 3k^2\frac{\tau^{'}}{\tau} - 3kk^{'}$ *and* $\lambda = \frac{k}{\tau} = \text{const}.$

Proof. Let us suppose that C is a general helix with respect to the Frenet frame $\left\{\frac{\partial}{\partial v}, n, b\right\}$. Using (2.2) we write

$$\nabla^*_{\frac{\partial}{\partial s}}\nabla^*_{\frac{\partial}{\partial s}}\nabla^*_{\frac{\partial}{\partial s}}\frac{\partial}{\partial v} = -3kk^{'}\frac{\partial}{\partial v} + \left(k^{''} + k\tau^2 - k^3\right)n + \left(3k^{'}\tau\right)b$$

Considering (2.4) and (2.5) in the above equation we obtain (2.10).

Conversely assume that (2.10) holds. Using (2.6) and (2.7) in (2.10) then making some routine calculations as in Theorem 2.1 we conclude that

$$\frac{\kappa}{\tau} = \text{const}.$$

This means that C is a general helix. □

According to the theorem above we have the following corollary.

Corollary 2.1. *Let C be a curve of Finsler manifold $\mathbb{F}^3$. C is a circular helix with respect to the Frenet frame $\left\{\frac{\partial}{\partial v}, n, b\right\}$ if and only if*

$$\nabla^*_{\frac{\partial}{\partial s}} \nabla^*_{\frac{\partial}{\partial s}} \nabla^*_{\frac{\partial}{\partial s}} \frac{\partial}{\partial v} = \left(\tau^2 - k^2\right) \nabla^*_{\frac{\partial}{\partial s}} \frac{\partial}{\partial v}. \tag{2.11}$$

Proof. From the hypothesis of corollary 2.1 and since C is a circular helix, we can show (2.11) easily. □

Theorem 2.3. *If C is a curve of Finsler manifold $\mathbb{F}^3$. C is a general helix with respect to the Frenet frame $\left\{\frac{\partial}{\partial v}, n, b\right\}$, then*

$$\nabla^*_{\frac{\partial}{\partial s}} \nabla^*_{\frac{\partial}{\partial s}} \nabla^*_{\frac{\partial}{\partial s}} \frac{\partial}{\partial v} - \tilde{K} \nabla^*_{\frac{\partial}{\partial s}} b = 3k' \nabla^*_{\frac{\partial}{\partial s}} n, \tag{2.12}$$

where $\tilde{K}(u) = \frac{k''}{\tau} + k\tau - \frac{k^3}{\tau}$.

Proof. Suppose that C is a general helix with respect to the Frenet frame $\left\{\frac{\partial}{\partial v}, n, b\right\}$. Then from (2.2) and (2.3)

$$\nabla^*_{\frac{\partial}{\partial s}} \nabla^*_{\frac{\partial}{\partial s}} \nabla^*_{\frac{\partial}{\partial s}} \frac{\partial}{\partial v} = -3kk' \frac{\partial}{\partial v} + \left(k'' + k\tau^2 - k^3\right) n + 3k'\tau b. \tag{2.13}$$

If we substitute the equations

$$n = \frac{1}{\tau} \nabla^*_{\frac{\partial}{\partial s}} b, \quad b = \frac{\nabla^*_{\frac{\partial}{\partial s}} n + k \frac{\partial}{\partial v}}{\tau} \tag{2.14}$$

and (2.5) in (2.13), we obtain (2.12). □

Theorem 2.4. *If C is a curve of Finsler manifold $\mathbb{F}^3$. C is a general helix with respect to the Frenet Frame $\left\{\frac{\partial}{\partial v}, n, b\right\}$, then*

$$\nabla^*_{\frac{\partial}{\partial s}} \nabla^*_{\frac{\partial}{\partial s}} \nabla^*_{\frac{\partial}{\partial s}} \frac{\partial}{\partial v} - \tilde{K} \nabla^*_{\frac{\partial}{\partial s}} b = 3k\left(3\lambda\tau' - k'\right) \frac{\partial}{\partial v} + 3\lambda\tau' \nabla^*_{\frac{\partial}{\partial s}} n \tag{2.15}$$

where $\tilde{K} = \frac{k''}{\tau} + k\tau - \frac{k^3}{\tau}$ *and* $\lambda = \frac{k}{\tau} = \text{const}$.

Proof. It is similar to the proof of Theorem 2.3. □

Corollary 2.2. *Let C be a curve of Finsler manifold $\mathbb{F}^3$. C is a circular helix with respect to the Frenet frame $\left\{\frac{\partial}{\partial v}, n, b\right\}$ if and only if*

$$\nabla^*_{\frac{\partial}{\partial s}} \nabla^*_{\frac{\partial}{\partial s}} \nabla^*_{\frac{\partial}{\partial s}} \frac{\partial}{\partial v} = \left(k\tau - \frac{k^3}{\tau}\right) \nabla^*_{\frac{\partial}{\partial s}} b. \tag{2.16}$$

Proof. From the hypothesis of corollary 2.2 and since C is a circular helix, we can show easily (2.16). ☐

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Firat University
Department of Mathematics
23119 *Elazig, Turkey*
E-mail address: `myildirim@firat.edu.tr`

Firat University
Department of Mathematics
23119 *Elazig, Turkey*
E-mail address: `mbektas@firat.edu.tr`