

THE POINT SPECTRA OF GENERALIZED HAUSDORFF MATRICES

B. E. RHOADES

ABSTRACT. In this paper we obtain the point spectra of generalized Hausdorff matrices and their adjoints, when considered as bounded operators on ℓ_p spaces for $1 \leq p \leq \infty$. We also obtain the spectrum of the Gamma generalized Hausdorff matrices.

Hausdorff matrices were shown by Hurwitz and Silverman [5] to be the class of triangular matrices that commute with C , the Cesàro matrix of order one. Hausdorff [3] reexamined this class, in the process of solving the moment problem over a finite interval, and developed many of the properties of the matrices that now bear his name.

Several generalizations of Hausdorff matrices have been made. In this paper we shall be concerned with the generalized Hausdorff matrices as defined by Hausdorff [3] and Jakimovski [6].

Let $\{\lambda_n\}$ be a sequence satisfying

$$0 \leq \lambda_0 < \lambda_1 < \dots < \lambda_n \dots$$

such that

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n} = \infty. \quad (1.1)$$

For any sequence $\{\mu_n\}$, a generalized Hausdorff matrix is a lower triangular matrix with entries

$$h_{nk} = \lambda_{k+1} \cdots \lambda_n [\mu_k, \dots, \mu_n],$$

where the divided difference is defined by $[\mu_n] = \mu_n$,

$$[\mu_k, \dots, \mu_n] = \frac{[\mu_k, \dots, \mu_{n-1}] - [\mu_{k+1}, \dots, \mu_n]}{\lambda_n - \lambda_k}.$$

Received: March 12, 2009.

2000 Mathematics Subject Classification: Primary: 47G05; Secondary: 47A10.

Key words and phrases: Point spectrum, factorable matrices, generalized Hausdorff matrices.

and where it is understood that $\lambda_{k+1} \cdots \lambda_n = 1$ when $k = n$.

Hausdorff [4] made this definition for $\lambda_0 = 0$, and Jakimovski extended it to the cases in which $\lambda_0 > 0$. If $\lambda_n = n$, then the definition reduces to that of an ordinary Hausdorff matrix. If $\lambda_n = n + \alpha$, then the definition reduces to the E-J generalization developed independently by Endl and Jakimovski [1], [2] and [6].

We shall call such a matrix H an H-J generalized Hausdorff matrix, and will determine the point spectrum of H and its adjoint, H^* , considered as bounded linear operators over the ℓ^p spaces, for $1 \leq p < \infty$.

It is of interest to note that, if $\lambda_0 = 0$, then each row sum of H is μ_0 . A proof of this fact appears in [4]. We now supply a different proof.

Lemma 1.1. *Let $0 \leq j \leq n + 1$. Then, for $\lambda_0 = 0$,*

$$\sum_{k=0}^j h_{n+1,k} = \sum_{k=0}^j h_{nk} - \lambda_{j+1} \cdots \lambda_n [\mu_{j+1}, \dots, \mu_{n+1}].$$

Proof. For $j = 0$, since $\lambda_0 = 0$,

$$\begin{aligned} h_{n+1,0} &= \lambda_1 \cdots \lambda_{n+1} [\mu_0, \dots, \mu_{n+1}] \\ &= \lambda_1 \cdots \lambda_n ([\mu_0, \dots, \mu_n] - [\mu_1, \dots, \mu_{n+1}]) \\ &= h_{n0} - \lambda_1 \cdots \lambda_n [\mu_1, \dots, \mu_{n+1}]. \end{aligned}$$

Assume the finite induction hypothesis. Then

$$\begin{aligned} \sum_{k=0}^{j+1} h_{n+1,k} &= \sum_{k=0}^j h_{n+1,k} + \lambda_{j+2} \cdots \lambda_{n+1} [\mu_{j+1}, \dots, \lambda_{n+1}] \\ &= \sum_{k=0}^j h_{nk} - \lambda_{j+1} \cdots \lambda_n [\mu_{j+1}, \dots, \lambda_{n+1}] + \lambda_{j+2} \cdots \lambda_{n+1} [\mu_{j+1}, \dots, \lambda_{n+1}] \\ &= \sum_{k=0}^j h_{nk} + \lambda_{j+2} \cdots \lambda_n \times (\lambda_{n+1} - \lambda_{j+1}) \frac{([\mu_{j+1}, \dots, \lambda_n] - [\mu_{j+2}, \dots, \lambda_{n+1}])}{\lambda_{n+1} - \lambda_{j+1}} \\ &= \sum_{k=0}^j h_{nk} + \lambda_{j+2} \cdots \lambda_n ([\mu_{j+1}, \dots, \lambda_n] - [\mu_{j+2}, \dots, \lambda_{n+1}]) \\ &= \sum_{k=0}^{j+1} h_{nk} - \lambda_{j+2} \cdots \lambda_n [\mu_{j+2}, \dots, \lambda_{n+1}]. \end{aligned}$$

□

Theorem 1.1. *Let H be an H-J matrix with $\lambda_0 = 0$. Then, for each n ,*

$$\sum_{k=0}^n h_{nk} = \mu_0.$$

Proof. The result is trivially true for $n = 0$. Assume the induction hypothesis. Then,

from Lemma 1.1, with $j = n + 1$, $\sum_{k=0}^{n+1} h_{n+1,k} = \sum_{k=0}^{n+1} h_{nk} - 0 = \sum_{k=0}^n h_{nk} = \mu_0. \quad \square$

An infinite matrix A is called *conservative* if it is invariant over c , the space of convergence sequences. The necessary and sufficient conditions for this to happen are the well-known Silverman-Toeplitz conditions:

- (i) $\|A\| : \sup_n \sum_u |a_{nk}| < \infty$,
- (ii) $a_k := \lim a_{nk}$ exists for each k ,
- (iii) $t := \lim \sum_n a_{nk}$ exists.

A conservative matrix is called *multiplicative* if each $a_k = 0$. For a space X , $B(X)$ will denote the set of bounded linear operators on X . We shall denote the point spectrum of an operator A by $\sigma_p(A)$.

Theorem 1.2. (a) *Let $H \in B(c)$ be a multiplicative H-J generalized Hausdorff matrix with the μ_n distinct. If $\lambda_0 > 0$, then $\sigma_p(H) = \emptyset$. If $\lambda_0 = 0$, then $\sigma_p(H) = \{\mu_0\}$.*

(b) *If $H \in B(\ell^p), 1 \leq p < \infty$ with the μ_n distinct, then $\sigma_p(H) = \emptyset$.*

Proof. From a result of Jakimovski and Nietze [7], we may write H in the form $H = T^{-1} \text{diag } \mu T$, where T is a lower triangular matrix with entries

$$t_{nk} = (-1)^k \frac{\prod_{\nu=k+1}^n \lambda_\nu}{n-1 \prod_{\nu=k} (\lambda_n - \lambda_\nu)},$$

and $\text{diag } \mu$ is the diagonal matrix with diagonal entries $\{\mu_n\}$. Let $\lambda_0 > 0$. Suppose that $Hx = \xi x$. This equation can be written in the form

$$(T^{-1} \text{diag } \mu T - \xi I)x = 0,$$

or

$$T^{-1}(\text{diag } \mu - \xi I)Tx = 0,$$

or

$$(\text{diag } \mu - \xi I)Tx = 0. \tag{1.2}$$

Let N denote the smallest integer for which $\mu_n \neq 0$. Then, for $n = N$ we have $\mu_N = \xi$. Since the μ_n are distinct, this leads to $Tx = 0$ for $n > N$.

For $n = N + 1$ we have

$$t_{N+1,N}x_N + t_{N+1,N+1}x_{N+1} = 0,$$

i.e.,

$$(-1)^{N+1}x_{N+1} = -\frac{(-1)^N\lambda_{N+1}x_N}{\lambda_{N+1} - \lambda_N},$$

which implies that

$$x_{N+1} = \frac{\lambda_{N+1}x_N}{\lambda_{N+1} - \lambda_N}.$$

It can be shown by induction that

$$\begin{aligned} x_{N+m} &= \frac{\prod_{\nu=N+1}^{N+m} \lambda_\nu}{\prod_{\nu=N+1}^{N+m} (\lambda_\nu - \lambda_N)} = x_N \prod_{\nu=N+1}^{N+m} \left(\frac{\lambda_\nu}{\lambda_\nu - \lambda_N} \right) \\ &= x_N \prod_{\nu=N+1}^{N+m} \left(1 + \frac{\lambda_\nu}{\lambda_\nu - \lambda_N} \right). \end{aligned}$$

Since $\sum \frac{1}{\lambda_n}$ diverges, the product diverges to $+\infty$ and $\{x_n\} \notin c$.

If $\lambda_0 = 0$, then every row sum of H is μ_0 . Thus e is a characteristic vector with characteristic value μ_0 . \square

It is necessary to assume that H is multiplicative, since the matrix generated by $\mu = \{1, 0, 0, \dots\}$ for $n > 0$ with $\lambda_0 = 0$ is conservative, but has point spectrum $\{0, 1\}$.

Special cases of Theorem 1.2 appear in [10]. The special case of Theorem 1.2(b) for ordinary Hausdorff matrices appears as Theorem 2 in [8], and, for the E-J matrices, as Theorem 2 in [9].

Associated with each conservative matrix A is a number $\chi(A)$ defined by $\chi(A) = t - \sum a_k$.

Theorem 1.3. *Let H be a conservative H-J Hausdorff matrix. Then H^* has a total set of eigenvalues.*

Proof. Suppose that $H \in B(c)$. Then H^* takes the form

$$H^* = \begin{pmatrix} \chi(H) & 0 & 0 & \cdots \\ a_0 & H^T & & \\ 0 & \cdots & & \end{pmatrix}$$

Consider the set $\{e_0, \beta_0, \beta_1, \dots\}$, where $e_0 = \{1, 0, 0, \dots\}$, and

$$\beta_n = \{0, t_{n0}, \dots, t_{nn}, 0, \dots\}.$$

Clearly $H^*e_0 = \chi(H)e_0$.

$$\begin{aligned} ((H^*\beta_n)(m+1)) &= \sum_{k=m+1}^{\infty} h_{m+1,k}^* \beta_n(k) = \sum_{k=m+1}^{n+1} h_{m,k-1}^T t_{n,k-1} = \sum_{j=m}^n h_{jm} t_{nj} \\ &= \sum_{j=m}^n \lambda_{m+1} \cdots \lambda_j [\mu_m, \dots, \mu_j] \frac{(-1)^j \prod_{\nu=j+1}^n \lambda_\nu}{\prod_{\nu=j}^{n-1} (\lambda_n - \lambda_\nu)} \\ &= \frac{(-1)^m \lambda_{m+1} \cdots \lambda_n}{\prod_{\nu=m}^{n-1} (\lambda_n - \lambda_\nu)} \times \sum_{j=m}^n (-1)^{m-j} [\mu_m, \dots, \mu_j] \prod_{\nu=m}^{n-1} (\lambda_n - \lambda_j). \end{aligned}$$

□

Lemma 1.2. For each $0 \leq m \leq n$, $\sum_{j=m}^n (-1)^{m-j} [\mu_m, \dots, \mu_j] \prod_{\nu=m}^{j-1} (\lambda_n - \lambda_\nu) = \mu_n$.

Proof. The result is trivially true for $m = n$. Assume that it is true for m . Then

$$\begin{aligned} &\sum_{j=m-1}^n (-1)^{m-1-j} [\mu_{m-1}, \dots, \mu_j] \prod_{\nu=m-1}^{j-1} (\lambda_n - \lambda_j) \\ &= \sum_{j=m-1}^n (-1)^{m-1-j} [\mu_{m-1}, \dots, \mu_j] \prod_{\nu=m}^{j-1} (\lambda_n - \lambda_j) \times \\ &\quad \times (\lambda_n - \lambda_j + \lambda_j - \lambda_{m-1}) \\ &= \sum_{j=m-1}^n (-1)^{m-1-j} [\mu_{m-1}, \dots, \mu_j] \prod_{\nu=m}^j (\lambda_n - \lambda_j) \end{aligned}$$

$$\begin{aligned}
& + \sum_{j=m-1}^n (-1)^{m-1-j} ([\mu_{m-1}, \dots, \mu_{j-1}] - [\mu_m, \dots, \mu_j]) \times \prod_{\nu=m}^{j-1} (\lambda_n - \lambda_j) \\
& = \sum_{j=m-1}^n (-1)^{m-1-j} [\mu_{m-1}, \dots, \mu_j] \prod_{\nu=m}^j (\lambda_n - \lambda_j) \\
& \quad + \sum_{i=m-2}^{n-1} (-1)^{m-i} [\mu_{m-1}, \dots, \mu_i] \prod_{\nu=m}^i (\lambda_n - \lambda_\nu) \\
& \quad - \sum_{j=m-1}^n (-1)^{m-1-j} [\mu_m, \dots, \mu_j] \prod_{\nu=m}^{j-1} (\lambda_n - \lambda_\nu) \\
& = \lambda_n.
\end{aligned}$$

Therefore $H^* \beta_n = \mu_n \beta_n$. \square

The set of eigenvalues is total, since each coordinate sequence can be written as a finite linear combination of the β_n and e_0 .

Theorem 1.3 for the E-J matrices appears in [11].

Corollary 1.1. *Let $H \in B(\ell^p)$, $1 \leq p < \infty$. Then H^* has a total set of eigenvalues.*

Proof. For each of the ℓ^p spaces, $H^* = H^t$. Therefore a simple modification of the proof of Theorem 1.2 yields the result. \square

The special case of Corollary 1.1 for Hausdorff matrices appears as Theorem 3 of [8], and, for the E-J matrices, in [11].

Let Γ_1^a denote the H-J matrix with mass function $\chi(t) = t^a$. Then Γ_1^a is the H-J version of the ordinary Hausdorff gamma matrices defined by Hausdorff [3].

Let $\sigma(A)$ denote the spectrum of a matrix A , considered as a bounded linear operator on c .

Theorem 1.4.

$$\sigma(\Gamma_1^a) = \left\{ \xi : \left| \xi - \frac{1}{2} \right| \leq \frac{1}{2} \right\}.$$

Proof. Clearly $\Gamma_1^a \in B(c)$.

$$\mu_n = \int_0^1 t^{\lambda_n} a t^{a-1} dt = \frac{1}{\lambda_n + a}.$$

By finite induction it can be shown that

$$[\mu_k, \dots, \mu_n] = \frac{1}{\prod_{\nu=k}^n (\lambda_\nu + a)}.$$

Therefore

$$h_{nk} = \frac{\lambda_{k+1} \cdots \lambda_n}{\prod_{i=k}^n (\lambda_i + a)} = a_n b_k,$$

where

$$a_n = \frac{\prod_{\nu=1}^n \lambda_\nu}{\prod_{\nu=0}^n (\lambda_\nu + a)}, \quad b_k = \frac{\prod_{\nu=0}^{k-1} (\lambda_\nu + a)}{\prod_{\nu=1}^k \lambda_\nu},$$

and H is factorable.

A lower triangular matrix A is called factorable if its entries can be written in the form $a_{nk} = a_n b_k$, where a_n depends only on n and b_k depends only on k . The result now follows from Theorem 1 of [12]. □

Let C denote the H-J version of $(C, 1)$, the Cesàro matrix of order one.

Corollary 1.2.

$$\sigma(C) = \left\{ \xi : \left| \xi - \frac{1}{2} \right| \leq \frac{1}{2} \right\}.$$

Proof. Set $a = 1$ in Corollary 1.1. □

REFERENCES

- [1] K. Endl: *Abstracts of short communications and scientific program*, Int. Congress of Math., **73**(1958), 46.
- [2] K. Endl: *Untersuchungen über Momentprobleme bei Verfahren vom Hausdorffschen Typus*, Math. Anal., **139**(1960), 403-432.
- [3] F. Hausdorff: *Summationmethoden und Momentfolgen, I*, Math. Z., **9**(1921), 74-109.
- [4] F. Hausdorff: *Summationmethoden und Momentfolgen, II*, Math. Z., **9**(1921), 280-299.
- [5] W. A. Hurwitz and L. L. Silverman: *On the consistency and equivalence of certain definitions of summability*, Trans. Amer. Math. Soc., **138**(1917), 1-20.
- [6] A. Jakimovski: *The product of summability methods; new classes of transformations and their properties*, Tech. Note, Contract No. AF61(1959)(052)-187.
- [7] Amnon Jakimovski and Hubert Tietz: *Eigenvalues for generalized Hausdorff matrices*, Integral Equations, **14**(1991), 754-759.
- [8] B. E. Rhoades: *Spectra of some Hausdorff operators*, Acta Sci. Math., **32**(1971), 91-100.
- [9] B. E. Rhoades: *The point spectra for generalized Hausdorff operators*, Acta Sci. Math., **53**(1989), 111-118.

- [10] B. E. Rhoades: *The point spectra for certain classes of operators in $B(\ell^p)$* , Int. Eq. and Operator Th., **14**(1991), 747-753.
- [11] B. E. Rhoades: *Spectra of Hausdorff matrices and other transformations*, Abstract, AMS, 1030-40-55.
- [12] B. E. Rhoades and M. Yildirim: *Spectra and fine spectra for factorable matrices*, Int. Eq. and Operator Th., **53**(2005), 127-144.

Indiana University
Department of Mathematics
Bloomington, IN 47405-7106
E-mail address: rhoades@indiana.edu