

A METHOD WHICH GENERATES SHARP ESTIMATIONS FOR BIG FACTORIALS

CRISTINEL MORTICI

ABSTRACT. We develop in this paper the idea to replace two approximations of a quantity by one of their mean to obtain a more accurate estimation. We use the Stirling's formula and the Burnside's formula to establish new, stronger approximations formula for big factorials. Our best formula we state here is the following:

$$n! \approx \frac{\sqrt{2\pi}}{e^{n+3/8}} \left(n^2 + \frac{3}{4}n + \frac{1}{12} \right)^{\frac{n}{2} + \frac{1}{4}}$$

being stronger than the Burnside's formula, which is considered one of the most accurate until now.

1. INTRODUCTION

The famous Stirling's formula and its different generalizations have a wide class of applications in science as statistical physics or probability theory. In consequence, it has been deeply studied by a large number of authors due to its practical importance. The Stirling's formula:

$$n! \approx \left(\frac{n}{e} \right)^n \sqrt{2\pi n} = \sigma_n, \quad (1.1)$$

in the sense that

$$\lim_{n \rightarrow \infty} \frac{n!}{\left(\frac{n}{e} \right)^n \sqrt{2\pi n}} = 1,$$

is a good approximation for big factorials. In fact, the formula (1.1) was discovered by the French mathematician Abraham de Moivre (1667-1754) in the form

$$n! \approx \text{constant} \cdot \left(\frac{n}{e} \right)^n \sqrt{n},$$

while the Scottish mathematician James Stirling (1692-1770) discovered the constant $\sqrt{2\pi}$ in the previous formula. For details, [1,3] can be consulted.

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2. MAIN RESULT

The next step in this direction is to define more and more accurate approximations for $n!$. Such a result is the following formula introduced by W. Burnside in [2], then rediscovered by Y. Weissman in [4]:

$$n! \approx \sqrt{2\pi} \left(\frac{n + \frac{1}{2}}{e} \right)^{n + \frac{1}{2}} = \beta_n. \quad (2.1)$$

The Burnside's formula has great superiority over the Stirling's formula, in fact, the formula (2.1) is one of the most performant ever known. Moreover, the following inequalities hold true:

$$\sigma_n < n! < \beta_n \quad (2.2)$$

and the advantage of the formula (2.1) is described by the relation:

$$\frac{\beta_n}{n!} < \frac{n!}{\sigma_n}.$$

For proofs and other details, see [1,3].

We develop here the idea to define new approximations of the form

$$n! \approx \mathcal{M}(\sigma_n, \beta_n),$$

where $\mathcal{M}(\sigma_n, \beta_n)$ is a certain mean of σ_n and β_n . If we think to the fact that products are involved in the expressions of σ_n, β_n , then it seems natural to consider geometric means.

On the other side, remark that it is enough to have the ordering

$$\sigma_n < n! < \mathcal{M}(\sigma_n, \beta_n) < \beta_n$$

to obtain a better approximation than the Burnside's formula. We tried with the means $\sqrt{\sigma_n \beta_n}$ and $\sqrt[3]{\sigma_n \beta_n^2}$, but we arrived to the situation:

$$\sigma_n < \sqrt{\sigma_n \beta_n} < \sqrt[3]{\sigma_n \beta_n^2} < n! < \beta_n.$$

After some numerical computations, we deduced that

$$\sigma_n < n! < \sqrt[4]{\sigma_n \beta_n^3} < \beta_n,$$

so one approximation we are talking about is the following:

$$n! \approx \sqrt[4]{\sigma_n \beta_n^3} = \frac{\sqrt{2\pi}}{e^{n+3/8}} \left(n \left(n + \frac{1}{2} \right)^3 \right)^{\frac{n}{4} + \frac{1}{8}} = \tau_n. \quad (2.3)$$

Some comparative values of this approximation with Stirling’s formula and Burnside’s formula are given in the following table:

n	σ_n	$n!$	τ_n	β_n
5	118.02	120	120.18	120.91
7	4980.4	5040	5046.0	5068.0
10	3.5987×10^6	3628800	3.632×10^6	3.6422×10^6
15	1.3004×10^{12}	1307674368000	1.3085×10^{12}	1.3112×10^{12}
17	3.5395×10^{14}	355687428096000	3.5589×10^{14}	3.5654×10^{14}
20	2.4228×10^{18}	2432902008176640000	2.4341×10^{18}	2.4379×10^{18}

Now we are concentrate to simplify the expression of the approximations τ_n and moreover, to give a new stronger estimation. More precisely, we are looking for an estimation μ_n which remains greater than $n!$, but less than τ_n . This can be possible for example, by replacing the polynomial from (2.3)

$$n \left(n + \frac{1}{2} \right)^3 = n^4 + \frac{3}{2}n^3 + \frac{3}{4}n^2 + \frac{1}{8}n$$

by a smaller one, denoted by Q , such that

$$\lim_{n \rightarrow \infty} \left(\frac{n \left(n + \frac{1}{2} \right)^3}{Q(n)} \right)^{\frac{n}{4} + \frac{1}{8}} = 1. \tag{2.4}$$

After some simple computations, we can deduce that (2.4) is fulfilled as soon as

$$\deg \left(Q(n) - n \left(n + \frac{1}{2} \right)^3 \right) \leq 2.$$

We are also careful to obtain a new estimation which is greater than $n!$, in order to be sure that it is stronger than all previous estimations. We arrive at the conclusion that if we use the polynomial

$$Q(n) = \left(n^2 + \frac{3}{4}n \right)^2,$$

then add the free term $1/12$, the approximation (2.3) becomes:

$$n! \approx \frac{\sqrt{2\pi}}{e^{n+3/8}} \left(n^2 + \frac{3}{4}n + \frac{1}{12} \right)^{\frac{n}{2} + \frac{1}{4}} = \mu_n,$$

which is most accurate, as we can see from the next table.

n	$n!$	μ_n	τ_n
5	120	120.08	120.18
7	5040	5042.7	5046.0
10	3628800	3.6303×10^6	3.632×10^6
15	1307674368000	1.3081×10^{12}	1.3085×10^{12}
17	355687428096000	3.5578×10^{14}	3.5589×10^{14}
20	2432902008176640000	2.4335×10^{18}	2.4341×10^{18}

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Valahia University of Târgoviște
Department of Mathematics
Bd. Unirii 18, 130082, Târgoviște, Romania
E-mail address: cmortici@valahia.ro